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Integrity of the symmetric algebra of modules of projective dimension two (**)

1 - Introduction

Let R denote a commutative Noetherian ring and E denote a finitely generated module with a presentation

$$R^m \xrightarrow{\phi} R^n \xrightarrow{\varphi} E \to 0 . \tag{1}$$

If R is an integral domain and E is a torsion free module of projective dimension one, the integrity of the symmetric algebra $S(E)$ of E over R is studied in [1].

Now, let

$$0 \to R^s \xrightarrow{\psi} R^m \xrightarrow{\phi} R^n \xrightarrow{\varphi} E \to 0 \tag{2}$$

be a projective resolution of E.

If the second Betti number s is equal to 1, R is a Cohen-Macaulay domain, E is a torsion-free R-module, $E^* = \mathrm{Hom}_R(E, R)$ is a 3-syzygy module, in [12] the acyclicity of the $Z(E)$-complex, the approximation complex of E, is proved by using syzygetic properties of the ideal $I_1(\psi)$ generated by the entries of a matrix representation of the inclusion $0 \to R \xrightarrow{\psi} R^m$.

When the second Betti number is two and the rank of the first syzygy module N of E is odd, sufficient conditions for E to admit an acyclic $Z(E)$-complex are established in [9] in terms of theoretic properties of the ideal $I_s(\psi)$ generated by the largest sized minors of a matrix representation of the inclusion $0 \to R^s \xrightarrow{\psi} R^m$ (i.e. the ideal $I_2(\psi)$). Moreover in that paper some relationships between the ideal $J(\phi)$ of relations of $S(E)$ and $I_2(\psi)$ are described.

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We study the case rank N even and $s = 2$. We intend in this case too, to connect the syzygetic properties of $S(E)$ with the syzygetic properties of $I_2(\psi)$. If R is a regular ring, a positive result was obtained in [**11**]. However, if we don't suppose that all modules have a finite projective resolution, we obtain a similar result when R is only a CM-ring.

More precisely, in Section 2 we state *sufficient conditions for the acyclicity of* $Z(E)$*-complex* and prove that $J(\phi)$ is a *Cohen-Macaulay prime ideal* of $S(R^n)$, by using notions of duality and counting depths of the modules, which appear in the complex $Z(E)$.

In Section 3 at last we study some modules E of projective dimension two for which $Z(E)$ is acyclic, requiring higher depths of the modules $Z_i(E)$. We point out the connection among the syzygetic properties of Coker $\Lambda^s \varphi$, where s is the second Betti number, and those of $\mathfrak{I}_s$.

2 - The main theorem

Let R be a commutative Noetherian ring and let E be a finitely generated R-module with a presentation (1) where $\phi = (a_{ij})$ is a matrix representation of a map between R^m and R^n, $a_{ij} \in R$.

Let $S(E)$ be the symmetric algebra of E, with the ideal-theoretic presentation $S(E) = R[T_1, \dots, T_n]/J$ where $R[T_1, \dots, T_n] = S_R(R^n)$ and J is the ideal of relations of $S(E)$ generated by the 1-forms

$$f_j = \sum_i^n a_{ij} T_i \in S(R_n) \qquad 1 \leqslant j \leqslant m .$$

We assume that E has rank e, i.e. $E \otimes K = K^e$, where K is the total quotient ring of R.

We consider some conditions on the sizes of Fitting ideals of E.

For any integer $t \geqslant 1$ we denote $I_t(\phi)$ the ideal generated by the $t \times t$ minors of ϕ, i.e. $I_t(\phi)$ is the $(n-t)$-th *Fitting ideal* of E and we consider the *condition*

$$F_k : \operatorname{ht}(I_t(\phi)) \geqslant \operatorname{rank} \phi - t + 1 + k \qquad 1 \leqslant t \leqslant \operatorname{rank} \phi \qquad k \geqslant 0$$

where rank $\phi = \sup\{t \mid I_t(\phi) \neq 0\}$.

Condition F_k can be given in terms of the local number of generators of E

F_k : for each prime ideal $\wp$ of R, if $E_\wp$ is not a free $R_\wp$-module, then

$$\nu(E_\wp) \leqslant \operatorname{depth} \wp + \operatorname{rank} E - k$$

where $\nu = \nu(E_\wp)$ is the minimal number of generators of $E_\wp$.

Hence, if E is F_k and $L_\wp \neq 0$ then rank $L_\wp \leqslant \operatorname{ht} \wp - k$ where $L_\wp$ is such that the sequence $0 \to L_\wp \to R_\wp^v \to E_\wp \to 0$ is exact [12], [13].

We want to study when the symmetric algebra of a module of projective dimension two over a Cohen-Macaulay domain is a Cohen-Macaulay domain, too.

The approximation complex $Z(E)$ of E gives useful information about the theoretic properties of $S(E)$ [3].

If E has the presentation (1), then the $Z(E)$-complex is a complex of graded $S = S(R^n)$-modules

$$Z(E): 0 \to Z_n \otimes S[-n] \xrightarrow{\partial} \dots \to Z_1 \otimes S[-1] \to S \to S(E) \to 0$$

where:

$$Z_i = Z_i(E) = \ker(\Lambda^i R^n \xrightarrow{\partial'} \Lambda^{i-1} R^n \otimes E) \qquad S[-r]_t = S_{t-r}$$

$$\partial'(a_1 \wedge \dots \wedge a_i) = \sum_1^i (-1)^j (a_1 \wedge \dots \wedge \hat{a}_j \wedge \dots \wedge a_i) \otimes \varphi(a_j)$$

$$\partial(e_{i_1} \wedge \dots \wedge e_{i_r} \otimes p(\boldsymbol{e})) = \sum_{j=1}^r (-1)^{r-j} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_r} \otimes e_{i_j} p(\boldsymbol{e})$$

where $\boldsymbol{e} = e_1, \dots, e_n$ is a standard basis of R^n and $\hat{}$ means omission.

If E has rank e, $Z_i = 0$ for $i > n - e$ and the homology of $Z(E)$ does not depend on the chosen presentation of E.

Let us briefly remember that an ideal $\mathfrak{I}$ of a Cohen-Macaulay ring R is said *strongly Cohen-Macaulay*, SCM for short, if

$$\operatorname{depth} Z_i(\mathfrak{I}) \geqslant \min\{d, d - g + 2\}$$

where $g = \operatorname{ht} \mathfrak{I}$ and Z_i is the module of cycles of the Koszul complex associated to a system of generators of $\mathfrak{I}$ [4].

Moreover we will use the acyclicity lemma of Peskine and Szpiro ([8]) and the criterion of Buchsbaum-Eisenbud-Northcott, formulated again by Matsumura ([7], Th. 3.3.2). The following theorem is crucial to state the results of this section.

Theorem 1. *Let R be a Noetherian ring, I be an ideal of R and*

$$0 \to F_k \to \dots \to F_1 \to M \to 0$$

be an exact complex of finitely generated R-modules ($k \geqslant 1$) such that there exists $n \geqslant k$ with $\operatorname{depth}_I F_i \geqslant n$ *for* $i = 1, \dots, k$.

Then $\operatorname{depth}_I M \geqslant n + 1 - k$.

Proof. See [5], Lemma 3.3.

Now we consider a module E of projective dimension two, with the free resolution (2).

In [9] e [10] basic result on the syzygies of E and some consequences about some modules were obtained. More explicitly, let us denote by N the first syzygy module of E, by Q the Coker $\Lambda^s \psi$, by $Z_1(\mathfrak{I}_s)$ the first syzygy module of $\mathfrak{I}_s$, where $\mathfrak{I}_s$ is the ideal generated by the $s \times s$ minors of a matrix presentation of ψ, and by $Z_i(\mathfrak{I}_s)$ the i-th module of cycles of the Koszul complex associated with a system of generators of $\mathfrak{I}_s$.

If E is a torsion-free R-module, the maps concerning these modules are:

a: $(\Lambda^l N)^{**} \simeq Z_l$ $\qquad l \leqslant \text{rank } N$

b: $(\Lambda^t Q)^{**} \to Z_{st}(E)$ $\qquad t \geqslant 1$

c: $(\Lambda^l N)^{**} \simeq (\Lambda^{\sigma - l} N)^*$ $\qquad \sigma = \text{rank } N$.

Moreover, if R is a Cohen-Macaulay domain containing a field k, then:

d: $Q \simeq (Z_1(\mathfrak{I}_s))^*$

e: $(\Lambda^t Q)^{**} \simeq (\Lambda^{r-t} Z_1(\mathfrak{I}_s))^{**} \simeq Z_{r-t}(\mathfrak{I}_s)$ $\qquad r = \text{rank } Q$

f: $(\Lambda^t Q)^{**} \to (\Lambda^{r-t} Q)^*$ and f is an isomorphism, if Q is a reflexive module.

See [9] for a description of these maps.

The conditions under which the symmetric algebra $S(E)$ is an *integral domain* have been a source of interest. We study now when a module E of projective dimension two and $s = 2$ has acyclic $Z(E)$-complex and for which the symmetric algebra $S(E)$ is an integral domain. If rank $N = m - 2$ is odd we have the results of [9].

Thanks to ***b***, it exists a map $(\Lambda^t Q)^{**} \to Z_{2t}(E)$, $t \geqslant 1$, hence the even terms of the $Z(E)$-complex can be connected with the exterior power $(\Lambda^t Q)^{**}$ and consequently with the homology modules of $\mathfrak{I}_2$.

Moreover, for the odd terms $Z_l(E)$, since we have

$$Z_l(E) \simeq (\Lambda^l N)^{**} \simeq (\Lambda^{\text{rank } N - l} N)^* \tag{3}$$

and rank $N - l$ is still even, it is possible to connect such powers with the exterior powers $(\Lambda^{\frac{\text{rank } N - l}{2}} Q)^*$, by the *dual map* $(\Lambda^{\text{rank } N - l} N)^* \to (\Lambda^{\frac{\text{rank } N - l}{2}} Q)^*$.

The case $m-2$ even is more complicated. In fact it is not possible to connect the odd terms of the complex with the homology modules of $\Im_2$. In fact, for l odd we have again (3) but rank $N-l$ is still odd. Consequently the use of the dual module is ineffectual, in general. However it is possible to connect the module $(\Lambda^l N)^{**}$ with $(\Lambda^{\frac{l-1}{2}} Q)^* \otimes N$, and of this latter module it is possible to evaluate the depth. Such reasonings are contained in

Theorem 2. *Let R be a Cohen-Macaulay integral domain containing a field and let E be a torsion-free module of projective dimension two with the free resolution* (2), *with $s=2$. Let $N=Z_1(E)$ be the first syzygy module of E and Q be* $\operatorname{Coker}(\Lambda^2 \psi)$.
Suppose that:

i. *E satisfies F_1*

ii. rank $N = m-2$ *is even*

iii. $(\Lambda^t Q)^{**} \otimes N$ *is reflexive for any t such that* $2t+1 <$ rank N

iv. $\operatorname{depth}(\Lambda^{2t+\varrho} N)^{**}_{\wp} \geqslant \operatorname{depth}((\Lambda^t Q)^{**} \otimes \Lambda^{\varrho} N)^{**}_{\wp} \quad \varrho = 0, 1$

for any t such that $2t+\varrho <$ rank N, *and for any* $\wp \in \operatorname{Spec} R$, $\wp \supset \Im_2$

v. *$\Im_2$ is a strongly Cohen-Macaulay ideal of height* 3.

Then $Z(E)$ is acyclic and $J = J(\phi)$ is a Cohen-Macaulay prime ideal of $S(R^n) = R[T_1, \dots, T_n]$, therefore $S(E)$ is an integral domain.

Proof. Let $\wp$ be a maximal ideal of R containing $\Im_2$. We may assume that $(R, \wp)$ is a local ring and $\dim R = d$.
Thanks to **a** if rank $N = m-2 = 2\nu$, $Z(E)$ is the complex

$$0 \to R \otimes S[-2\nu] \to (\Lambda^{2\nu-1} N)^{**} \otimes S[-2\nu+1] \to (\Lambda^{2\nu-2} N)^{**} \otimes S[-2\nu+2]$$

$$\to \dots \to (\Lambda^3 N)^{**} \otimes S[-3] \to (\Lambda^2 N)^{**} \otimes S[-2] \to N \otimes S[-1] \to S \to S(E) \to 0$$

From hypothesis **iv** and from **e** it follows

$$\operatorname{depth}(\Lambda^{2t} N)^{**} \geqslant \operatorname{depth}(\Lambda^t Q)^{**} = \operatorname{depth}(\Lambda^{r-t} Q^*) = \operatorname{depth} Z_{r-t}(\Im_2)$$

with $r = \text{rank}\, Q$. Moreover $(\Lambda^t Q)^{**} \simeq (\Lambda^t Z_1(\Im_2))^{**} \simeq Z_t(\Im_2)$. From **v** we derive depth $Z_i(\Im_2) \geq d-1$. Moreover, from **i** we get rank $N \leq d-1$, with $d-1 \geq m-2$.

For even terms in the $Z(E)$-complex we have depth $(\Lambda^{2t}N)^* \geq m-2$. For odd terms we consider the following steps:

Step 1. Consider the epimorphisms:

$$\Lambda^{2t}N \otimes N \to \Lambda^{2t+1}N \to 0 \qquad \Lambda^t Q \to \Lambda^{2t}N \to 0 \tag{4}$$

and the sequences obtained by double dualization of $(4)_1$ and tensorizing $(4)_2$ by N:

$$(\Lambda^{2t}N \otimes N)^{**} \to (\Lambda^{2t+1}N)^{**} \to 0 \qquad \Lambda^t Q \otimes N \to \Lambda^{2t}N \otimes N \to 0\,.$$

Moreover we have

$$(\Lambda^t Q \otimes N)^{**} \to (\Lambda^{2t}N \otimes N)^{**} \to 0 \quad \text{hence} \quad (\Lambda^t Q \otimes N)^{**} \to (\Lambda^{2t+1}N)^{**} \to 0\,.$$

Then we can consider the homomorphism

$$(\Lambda^t Q \otimes N)^{**} \to ((\Lambda^t Q)^{**} \otimes N) \to 0$$

which is an isomorphism for all prime ideals $\wp$ of R such that $\text{ht}(\wp) \leq 1$, then it is an isomorphism for all prime ideals of R.

Then, we obtain the following exact sequence

$$((\Lambda^t Q)^{**} \otimes N)^{**} \to (\Lambda^{2t+1}N)^{**} \to 0\,.$$

Step 2. We have the exact sequences (2) where $s = 2$ and

$$0 \to R^2 \xrightarrow{\psi} R^m \to N \to 0\,. \tag{5}$$

Since E is torsion free, by Buchsbaum-Eisenbud criterion of exactness, if follows depth $\Im_2(\psi) \geq 3$. By tensoring (5) with $(\Lambda^t Q)^{**}$, we obtain

$$R^2 \otimes (\Lambda^t Q)^{**} \to R^m \otimes (\Lambda^t Q)^{**} \to N \otimes (\Lambda^t Q)^{**} \to 0\,. \tag{6}$$

The sufficient conditions in order that (6) is exact are verified.

In fact depth$(\mathfrak{I}_2(\psi), (\Lambda^t Q)^{**}) \geq 2$ (since $(\Lambda^t Q)^{**}$ is a reflexive module). Then from Theorem 1 we can conclude that depth$((\Lambda^t Q)^{**} \otimes N) \geq d-2$. Since E is F_1, then $d-2 \geq m-3$. From **iv** we have

$$\operatorname{depth}(\Lambda^{2t+1} N)^{**} \geq \operatorname{depth}((\Lambda^t Q)^{**} \otimes N)^{**} \geq d-2 \geq m-3 .$$

So the $Z(E)$-complex is *exact* and depth $S(E) > 0$.

The acyclicity of the $Z(E)$-complex implies that each symmetric power $S_t(E)$ is a torsion-free module, then $S(E)$ is an *integral domain* and J is a *prime ideal.*

Moreover E is F_0, but in this situation dim $S(E) = \dim R + \operatorname{rank} E = d + e$, the acyclicity of $Z(E)$ implies depth $S(E) \geq d + e$, hence $S(E)$ is *Cohen-Macaulay.*

Example 1. Let R be a Cohen-Macaulay integral domain containing a field and let E be a finitely generated R-module, of projective dimension two, the second Betti number $s = 2$, rank $N = 4$. In this case since $(\Lambda^3 N)^{**} \simeq N^*$ thanks to **c**, the $Z(E)$-complex is

$$0 \to R \otimes S[-4] \to N^* \otimes S[-3] \to (\Lambda^2 N)^{**} \otimes S[-2] \to N \to S \to S(E) \to 0 .$$

If we define the morphism $(\Lambda^2 N)^{**} \to Q \to 0$, if depth$(\Lambda^2 N)^{**} \geq$ depth $Q \geq d-1$, $d = \dim R$, then the acyclicity of $Z(E)$ comes from depth N^*. But if N is self-dual, then $N = N^*$ and depth $N = d-1$; hence $Z(E)$ is exact.

Moreover, in the general case N is not necessarily self-dual, we have

$$(\Lambda^2 N)^{**} = (\Lambda^2 N)^* = ((\Lambda^2 N)^{**})^*$$

and the reflexive power $(\Lambda^2 N)^{**}$ is self-dual.

If $\Lambda^2 N$ is reflexive, then it is self-dual, too.

Example 2. Let E be a finitely generated R-module of projective dimension two, $s = 2$, rank $N = 6$. In this case the $Z(E)$-complex is

$$0 \to R \otimes S[-6] \to N^* \otimes S[-5] \to (\Lambda^4 N)^{**} \otimes S[-4] \to (\Lambda^3 N)^{**} \otimes S[-3]$$

$$\to (\Lambda^2 N)^{**} \otimes S[-2] \to N \otimes S[-1] \to S \to S(E) \to 0 .$$

Consider now the morphism $(\Lambda^{2t}N)^{**} \to (\Lambda^t Q)^{**} \to 0$, $2t < \operatorname{rank} N$. If $\operatorname{depth}(\Lambda^{2t}) \geq \operatorname{depth}(\Lambda^t Q)^{**}$, we have to evaluate depth N^* and $\operatorname{depth}(\Lambda^3 N)^{**}$.

We have $(\Lambda^3 N)^{**} = (\Lambda^3 N)^*$ and the third reflexive power $(\Lambda^3 N)^{**}$ is self-dual. If $N = N^*$, we have to evaluate only $\operatorname{depth}(\Lambda^3 N)^{**}$ or alternatively $\operatorname{depth}(\Lambda^3 N)^*$.

Example 3. Let E be a finitely generated R-module of projective dimension two, $s = 2$, rank $N = 8$. The $Z(E)$-complex is

$$0 \to R \otimes S[-8] \to N^* \otimes S[-7] \to (\Lambda^6 N)^{**} \otimes S[-6] \to (\Lambda^5 N)^{**} \otimes S[-5]$$

$$\to (\Lambda^4 N)^{**} \otimes S[-4] \to (\Lambda^3 N)^{**} \otimes S[-3] \to (\Lambda^2 N)^{**} \otimes S[-2]$$

$$\to N \otimes S[-1] \to S \to S(E) \to 0 .$$

It results that $(\Lambda^3 N)^*$ and $(\Lambda^3 N)^{**}$ are not isomorphic and if we suppose that $\operatorname{depth}(\Lambda^{2t} N)^{**} \geq \operatorname{depth}(\Lambda^t Q)^{**}$, then we have to evaluate depth N^*, $\operatorname{depth}(\Lambda^3 N)^{**}$ and $\operatorname{depth}(\Lambda^3 N)^*$.

3 - Further results

Let E be a R-module, having projective dimension two and the free resolution (2) with $s = 1$.

In the following we need some conditions, whose the first one was introduced in [3].

More explicitly, let R be a Cohen-Macaulay ring and E be a torsion-free module. The *first condition* is

$$F_t^*\colon \nu(E_{\wp}) \leq \frac{1}{2}(\operatorname{ht} \wp - t) + \operatorname{rank} E \qquad \forall \wp \in \operatorname{Spec} R .$$

Note that, if E satisfies F_t^*, then E satisfies also F_t.

Now, let R be a local Cohen-Macaulay ring and let E be rank e, finitely generated R-module with the presentation

$$0 \to N \to R^n \to E \to 0 . \tag{7}$$

The *second condition* (*sliding depth condition*) is

$$SD_k : \text{depth } Z_i(E) \geqslant d - n + 1 + k \qquad \forall i \leqslant n - e$$

where $d = \dim R$ and k is a fixed integer.

Theorem 3. *Let R be a Cohen-Macaulay ring, let E be a R-module of rank e having the free resolution (2) with $s = 1$. Let $\Im_1 = \Im = (a_1, \ldots, a_m)$ be the ideal defined by the entries of a matrix presentation of ψ. Then we have:*

1. *If E satisfies F_t^*, $t \in \{0, 1\}$, then $\Im_1$ satisfies F_t^*.*

2. *If $\Im_1$ satisfies F_t^*, $t \in \{0, 1\}$, E is torsion-free and $E^* = \mathrm{Hom}_R(E, R)$ is a 3-syzygy module, then E satisfies F_t^*.*

Proof. Let $\wp \in \mathrm{Spec}\, R$. We may assume that $(R, \wp)$ is a local ring and the resolution above is minimal.

Since E satisfies F_t^*, $t \in \{0, 1\}$, from the exact sequences (7) and

$$0 \to R \to R^m \to N \to 0$$

we have $\mathrm{rank}\, N \leqslant \frac{1}{2}(\mathrm{ht}\ \wp - t)$, $t \in \{0, 1\}$. Then $m - 1 \leqslant \frac{1}{2}(\mathrm{ht}\ \wp - t)$. Since $\nu(\Im_1) \leqslant m$, we get $\nu(\Im_1) \leqslant \frac{1}{2}(\mathrm{ht}\ \wp - t) + 1$. Therefore $\Im_1$ satisfies F_t^*.

Since E is torsion-free, N is a reflexive module. We shall prove that E satisfies F_t^*, $t \in \{0, 1\}$, then $m - 1 \leqslant \frac{1}{2}(\mathrm{ht}\ \wp - t)$.

We consider the exact sequence (7) and its dual sequence. Then $\phi : R^m \to R^n$ is the composite of the maps:

$$R^m \to N \qquad N \to N^{**} \qquad N^{**} \xrightarrow{\alpha^*} R^n .$$

If $\nu(\Im_1) = r \leqslant m$, N has a free summand of rank $m - r$ and $\phi = \begin{pmatrix} 1 & 0 \\ 0 & \phi' \end{pmatrix}$ where 1 is the identity matrix of size $m - r$ and ϕ' has all of its entries in $\wp$.

This contradicts the minimality hypothesis, thus we have $\nu(\Im_1) = m$, $m - 1 \leqslant \frac{1}{2}(\mathrm{ht}\ \wp - t)$, and E satisfies F_t^*, $t \in \{0, 1)$.

Corollary 1. *Let R be an integral Cohen-Macaulay domain and let E be a torsion-free module with the free resolution* (2) *with* $s=1$. *If $\Im_1$ satisfies F_1^* and E^* is a 3-syzygy module, then $Z(E)$ is acyclic and $S(E)$ is a domain.*

Proof. By Theorem 3, E satisfies F_1^* and we can conclude as in Example 4.6 of [3].

Theorem 4. *Let R be an integral Cohen-Macaulay ring of dimension d and E be a torsion-free R-module with a resolution* (2), $s=1$. *Then:*

1. *If $\Im_1$ satisfies SD_m, then E satisfies SD_e and $Z(E)$ is acyclic.*

2. *If E satisfies $SD_{e+2(m-1)}$, then $\Im_1$ satisfies SD_m.*

Proof. The assumption about $\Im_1$ implies depth $Z_i(\Im_1) \geqslant d+i$. It follows

$$\operatorname{depth}(\Lambda^r L)^{**} = \operatorname{depth} Z_{m-1-r}(\Im_1) \geqslant d+m-1-r \geqslant d-n+r+e\,.$$

But $(\Lambda^r L)^{**} \simeq Z_r(E)$ and we can conclude that E has SD_e.

Put $k=e+2(m-1)$. If E satisfies SD_k, we have:

$$\operatorname{depth}(Z_r(E)) \geqslant d-n+r+k\,.$$

But
$$\operatorname{depth} Z_r(E) = \operatorname{depth}(\Lambda^r L)^{**} = \operatorname{depth} Z_{m-1-r}(\Im_1)$$

and
$$\operatorname{depth} Z_t(\Im_1) \geqslant d-n+m-1+k \geqslant d+t\,.$$

Theorem 5. *Let R be a Cohen-Macaulay ring, E a module having projective dimension two with the free resolution* (2), $P = \operatorname{Coker} \Lambda^s \phi$. *Let $\Im_s$ be the ideal defined by the entries of a matrix representation of ψ. Then:*

1. *If P satisfies F_t^*,* $t=\operatorname{rank}(L)$, *with L such that*

$$0 \to L \to \Lambda^s R^m \xrightarrow{\Lambda^s\phi} \Lambda^s R^n \to P \to 0$$

is an exact sequence, then $\Im_s$ satisfies F_{2-t}^*, $t \in \{0,1,2\}$.

2. *If P is a torsion-free module, $\forall \wp \in \operatorname{Spec} R$, $\wp \supset \Im_s$ for which the presentation*

$$\Lambda^s R_\wp^m \xrightarrow{\Lambda^s \phi} \Lambda^s R_\wp^n \to P_\wp \to 0$$

is minimal, if P^* is a 3-syzygy module, then if $\Im_s$ satisfies F^*_{2-t}, P satisfies F^*_t, $t \in \{0, 1, 2\}$.

Proof. If $\wp$ is a prime ideal, $\wp \supset \Im_s$, we may assume that $(R, \wp)$ is a local ring and that the resolution

$$0 \to L \to \Lambda^s R^m \to \Lambda^s R^n \to P \to 0$$

is minimal.

Since P satisfies the condition F^*_t, it follows rank $\Lambda^s \phi \leqslant \frac{1}{2}(\operatorname{ht} \wp - t)$, that is $\binom{m}{s} \leqslant \frac{1}{2}(\operatorname{ht} \wp + t)$.

Therefore we have $\nu(\Im_s) \leqslant \frac{1}{2}(\operatorname{ht} \wp + t - 2) + 1$ and **1** is proved.

Let us consider the exact sequence

$$0 \to M \to \Lambda^s R^n \to P \to 0$$

with $M = \operatorname{Im} \Lambda^s \phi$. By dualizing and remembering that M is a reflexive module and P^* is a 3-syzygy module, we have the exact sequence

$$0 \to P^* \to (\Lambda^s R^n)^* \to M^* \to 0\,.$$

$\Lambda^s \phi$ is the composite of the following maps:

$$\Lambda^s R^m \to M \qquad M \to M^{**} \qquad M^{**} \to (\Lambda^s R^n)^{**}\,.$$

We want to prove that $\nu(\Im_s) = \binom{m}{s}$.

If $\nu(\Im_s) = r < \binom{m}{s}$, M would have a free summand of rank $\binom{m}{s} - r$ and $\Lambda^s \phi$ could be written as $\begin{pmatrix} 1 & 0 \\ 0 & \phi' \end{pmatrix}$ where 1 is an identity matrix of size $\binom{m}{s} - r$ and ϕ' has all of its entries in $\wp$, contradicting the minimality hypothesis of the resolution.

Therefore it is $\nu(\Im_s) = \binom{m}{s} = \operatorname{rank}(\Lambda^s \phi) + t$ and since $\Im_s$ has the property F^*_{2-t}, we have

$$\operatorname{rank}(\Lambda^s \phi) + t \leqslant \tfrac{1}{2}(\operatorname{ht} \wp - t) \quad \text{that is} \quad \operatorname{rank}(\Lambda^s \phi) \leqslant \tfrac{1}{2}(\operatorname{ht} \wp - t)\,,$$

and **2** is proved.

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Sommario

Si stabiliscono risultati sull'aciclicità del complesso di approssimazione di un R-modulo E di dimensione proiettiva 2. Se R è un dominio Cohen-Macaulay, il secondo numero di Betti è 2 ed il primo modulo di sizigie di E ha rango pari, si prova l'integrità dell'algebra simmetrica di E, modulo proprietà sizigetiche del modulo E.

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