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Moves on coloured spines (**)

1 - Preliminaries

Throughout this paper we shall work in the PL category, for which we refer to [14] and [7]; all manifolds will be closed and connected, unless otherwise stated. For graph theory see [15].

A *singular n-manifold* is a compact connected n-dimensional polyhedron, admitting a triangulation K as a closed n-pseudomanifold such that for each vertex v of K, the link $\mathrm{lk}(v, K)$ is a closed connected $(n-1)$-manifold. A vertex v of K whose link $\mathrm{lk}(v, K)$ is (resp. is not) the $(n-1)$-sphere is called *regular* (resp. *singular*).

Note also that if $N = |K|$ is a singular n-manifold, for each h-simplex σ^h of K, with $h \geq 1$, the link $\mathrm{lk}(\sigma^h, K)$ is always an $(n-h-1)$-sphere.

From now on the term *graph* will be used instead of *multigraph* (i.e. including multiple edges between two distinct vertices, but not loops), whereas *pseudograph* will denote that both loops and multiple edges are allowed.

A *coloured graph* is a pair (Γ, γ), where $\Gamma = (V(\Gamma), E(\Gamma))$ is a (pseudo)graph and $\gamma: E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}$ is a map; Δ_n is called *colour-set* and γ a *generalized edge-colouring* on Γ. For each $B \subseteq \Delta_n$, a *B-residue* of (Γ, γ) is a connected component of the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. In the following, for each subset $\{c_1, \ldots, c_h\}$ of the colour-set, we denote by $(\hat{c}_1, \ldots, \hat{c}_h)$ its complement; if $v \in V(\Gamma)$, $\Gamma_B(v)$ will denote the B-residue of Γ containing v. Moreover we shall write Γ_c and Γ_{cd} instead of $\Gamma_{\{c\}}$ and $\Gamma_{\{c, d\}}$.

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An $(n+1)$-*crystallized structure* is a coloured graph (Γ, γ) such that, for each $c \in \Delta_n$, the $\{c\}$-residues are cliques (i.e. complete graphs). In particular, if all cliques have length two, (Γ, γ) is called an $(n+1)$-*coloured graph.*

An $(n+1)$-*pondered structure* is a triple $\mathscr{P} = (\overline{\Gamma}, \overline{\gamma}, \omega)$ where $\overline{\Gamma}$ is an oriented graph, regular of degree $2(n+1)$, $\overline{\gamma}$ is a generalized edge-colouring on $\overline{\Gamma}$, with colour-set Δ_n, and ω: $E(\overline{\Gamma}) \to \{0, 1, 2\}$ is a map, called *weight*, on $\overline{\Gamma}$ such that:

1. for each $c \in \Delta_n$, the components of $\overline{\gamma}^{-1}(c)$ are elementary (generally not oriented) cycles
2. if $c \in \Delta_n - \{0\}$, then for each edge $e \in \overline{\gamma}^{-1}(c)$, $\omega(e) = 1$
3. let e and f be 0-coloured adjacent (oriented) edges:

if $e(1) = f(0)$, then we have the following five possibilities:

$$\omega(e) = \omega(f) = 1, \quad \omega(e) = 1 \quad \omega(f) = 0, \quad \omega(e) = 2 \quad \omega(f) = 0,$$

$$\omega(e) = 0 \quad \omega(f) = 2, \quad \omega(e) = 2 \quad \omega(f) = 1$$

if $e(1) = f(1)$, then $\omega(e) = 0$, $\omega(f) = 1$ or $\omega(e) = 0$, $\omega(f) = 2$
if $e(0) = f(0)$, then $\omega(e) = 0$, $\omega(f) = 2$ or $\omega(e) = 1$, $\omega(f) = 2$ (see Figure 1).

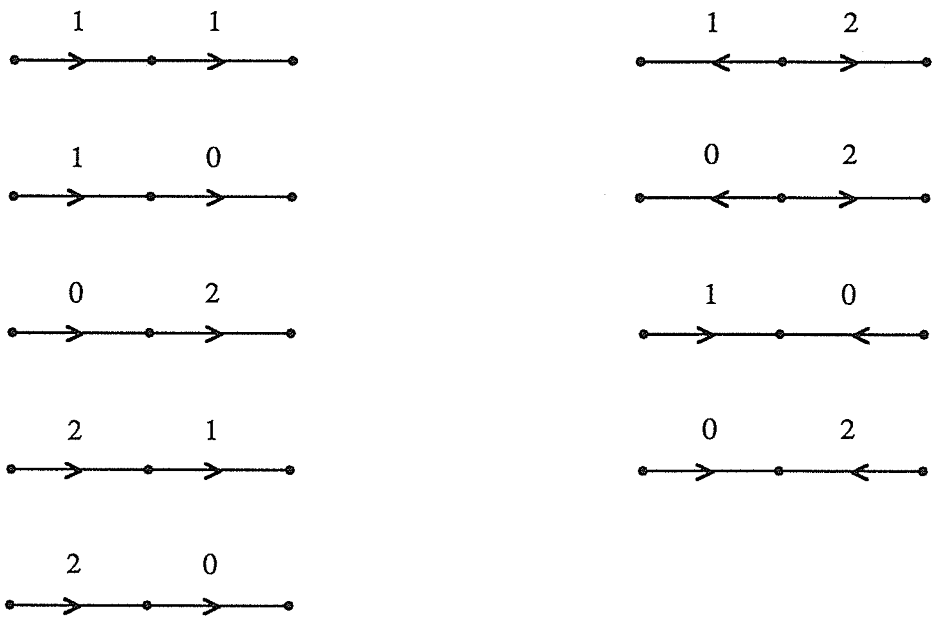

Fig. 1.

Remark 1. If Θ is a component of $\overline{\gamma}^{-1}(d)$, d being any colour, such that $\omega(e)=1$ for each edge e in Θ, then Θ is an elementary oriented cycle. Hence a pondered structure whose edges have all weight 1 is an oriented structure in the sense of [3] and conversely.

Given an $(n+1)$-pondered structure $\mathcal{P}=(\overline{\Gamma}, \overline{\gamma}, \omega)$, we can always construct a unique $(n+1)$-crystallized structure (Γ, γ) associated to $\mathcal{P}$ in the following way:

delete all loops of $E(\overline{\Gamma})$

for every $c \in \Delta_n$, replace each component of $\overline{\gamma}^{-1}(c)$ by the clique over the same set of vertices, colouring c all its edges.

The above construction can be reversed, but obviously not in a unique way; therefore a given crystallized structure can produce many pondered structures (see [1]).

If K is an n-dimensional pseudocomplex [8], the *disjoint star* $\mathrm{std}(s, K)$ of a simplex s in K is the disjoint union of the n-simplexes containing s, with re-indentification of the $(n-1)$-simplexes containing s and of all their faces; the *disjoint link* of s in K is the subcomplex $\mathrm{lkd}(s, K)=\{\tau \in \mathrm{std}(s, K) \mid s \cap \tau = \emptyset\}$.

A *vertex-coloration* on K is a map which associates a colour $c \in \Delta_n$ to each vertex of K and is injective on every simplex of K. If K is homogeneous, the pair (K, ξ) is called a *coloured n-complex.*

Let (Γ, γ) be an $(n+1)$-crystallized structure; we can construct a coloured n-complex $(K(\Gamma), \xi(\Gamma))$ in the following way:

take an n-simplex $\sigma(v)$ for each $v \in V(\Gamma)$ and label its vertices by Δ_n

for each $c \in \Delta_n$ and each pair v, w of c-adjacent vertices in Γ, identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertices labelled c, so that equally labelled vertices coincide.

The above construction can be easily reversed in order to associate an $(n+1)$-crystallized structure $(\Gamma(K), \gamma(K))$ to each coloured n-complex (K, ξ).

Note that, by construction, each $(\hat{c}_0, \dots, \hat{c}_h)$-residue Ξ of (Γ, γ) corresponds to a unique h-simplex s of $K(\Gamma)$, whose vertices are labelled by $\{c_0, \dots, c_h\}$ and conversely; moreover $K(\Xi)=\mathrm{lkd}(s, K(\Gamma))$.

It is easy to see that $(\Gamma(K(\Gamma)), \gamma(K(\Gamma)))=(\Gamma, \gamma)$; conversely $(K(\Gamma(K)), \xi(\Gamma(K)))=(K, \xi)$ iff the disjoint star of every simplex in K is strongly connected. In this case (K, ξ) is said to be *represented* by (Γ, γ). Moreover (Γ, γ) is an $(n+1)$-coloured graph iff $|K(\Gamma)|$ is a closed pseudomanifold; if $|K(\Gamma)|$ is a (singular) manifold N we say that N is *represented* by (Γ, γ).

For a general survey on edge-coloured graphs representing manifolds, see [5].

Let $\mathcal{P} = (\overline{\Gamma}, \overline{\gamma}, \omega)$ be a n-pondered structure (with colour-set Δ_{n-1}); we can consider an $(n+1)$-coloured graph $(B(\mathcal{P}), \beta)$ defined in the following way:

i. $V(B) = V(\overline{\Gamma}) \times \{0, 1\}$

ii. for each $v \in V(\overline{\Gamma})$, join $(v, 0)$ and $(v, 1)$ by an n-coloured edge

iii. let $v, w \in V(\overline{\Gamma})$ be adjacent vertices of $\overline{\Gamma}$, such that the edge e between them is directed from v to w; then join (v, h) and (w, k) $(h, k \in \Delta_1)$ by an edge coloured $\overline{\gamma}(e)$ iff $h \leq k$ and $\omega(e) = h + k$.

$(B(\mathcal{P}), \beta)$ is called the *pluri-bijoin associated to the pondered structure* $\mathcal{P}$.

If $(B(\mathcal{P}), \beta)$ is a crystallization of a closed, connected n-manifold M, then the n-crystallized structure associated to $\mathcal{P}$ represents a spine of M (see [1]). Note that the pluri-bijoin associated to a pondered structure $\mathcal{P}$ doesn't depend on the orientations of the edges of weight 0 or 2; therefore, to make the correspondence between $\mathcal{P}$ and $B(\mathcal{P})$ clear, we drop the orientations on these edges considering only those on the edges of weight 1.

Let $\mathcal{P} = (\overline{\Gamma}, \overline{\gamma}, \omega)$ be a $(n+1)$-pondered structure; a *generalized weak cycle* μ of $\overline{\Gamma}_{i,j}$ $(i, j \in \Delta_n)$ is a cycle of $\overline{\Gamma}$, whose edges are alternatively coloured i and j, such that for each pair e, f of adjacent edges of μ we have:

a. if both e and f are not 0-coloured then either $e(0) = f(0)$ or $e(1) = f(1)$ (note that by condition (2) in the definition of pondered structure we always have $\omega(e) = \omega(f) = 1$);

b. suppose that one of the edges, e say, is 0-coloured, (i.e. $\omega(f) = 1$) then one of the following conditions must hold:

if $f(1)$ is an endpoint of e, then either $\omega(e) = 2$ or $\omega(e) = 1$ and $e(1) = f(1)$

if $f(0)$ is an endpoint of e, then either $\omega(e) = 0$ or $\omega(e) = 1$ and $e(0) = f(0)$.

2 - Dipoles in singular n-manifolds

Let us recall from [4] and [6] that, given an $(n+1)$-coloured graph (Γ, γ) with $\#V(\Gamma) > 2$, a *dipole of type* h (or simply h-*dipole*) *involving colours* $c_1, \ldots, c_h$ $(1 \leq h \leq n)$ is a subgraph Θ of Γ formed by two vertices x and y joined by h edges $e_1, \ldots, e_h$ with $\gamma(e_i) = c_i$, $1 \leq i \leq h$, such that x and y

belong to different $(\hat{c}_1, \ldots, \hat{c}_h)$-residues of Γ. We shall denote such a dipole by $\Theta = (x, y)$.

By *cancelling the dipole* Θ *from* Γ we mean performing the following operations on Γ:

1. delete the vertices x and y, the edges $e_1, \ldots, e_h$ and the resulting *hanging* edges from Γ

2. for each $c \in (\hat{c}_1, \ldots, \hat{c}_h)$, if v (resp. w) is the vertex of Γ c-adjacent to x (resp. to y) then join v and w by an edge coloured c.

The inverse procedure is called *adding the dipole* Θ.

If $(\tilde{\Gamma}, \tilde{\gamma})$ is an $(n+1)$-coloured graph obtained from (Γ, γ) by cancelling the dipole Θ, then Θ is called *proper*, iff $|K(\Gamma)|$ and $|K(\tilde{\Gamma})|$ are homeomorphic.

In [6] the following sufficient condition for Θ to be proper is proved:

Proposition 1. *Let* (Γ, γ) *be an* $(n+1)$*-coloured graph and* $\Theta = (x, y)$ *an* h*-dipole of* Γ *involving colours* $c_1, \ldots, c_h$; *if either* $\Gamma_{(\hat{c}_1 \ldots \hat{c}_h)}(x)$ *or* $\Gamma_{(\hat{c}_1 \ldots \hat{c}_h)}(y)$ *represents an* $(n-h)$*-sphere, then* Θ *is proper.*

If $N = |K|$ is any singular n-manifold, from now on *we shall always suppose that all singular vertices of K have the same colour, say the «last» colour n*; otherwise we can always perform suitable subdivisions on K, in order to obtain a triangulation satisfying the above property.

As a consequence of Proposition 1, we have:

Corollary. *Let* (Γ, γ) *be an* $(n+1)$*-coloured graph representing a singular n-manifold. If* Θ *is an h-dipole of* Γ*, then:*

a. *if either* $h > 1$ *or* Θ *doesn't involve colour n, then* Θ *is proper*

b. *if* $h = 1$ *and* Θ *involves colour n, then* Θ *is proper iff at least one of the corresponding vertices of* $K(\Gamma)$ *is non-singular (i.e. if either* $\Gamma_{\hat{n}}(x)$ *or* $\Gamma_{\hat{n}}(y)$ *is an* $(n-1)$*-sphere).*

Remark 2. Note that all $\hat{c}$-residues $(c \in \Delta_n)$ of Γ, not containing x and y, remain unaltered, whereas if Ξ is a $\hat{c}$-residue $(c \in \Delta_{n-1})$ containing x or y, then, by deleting Θ from Γ, we cancel a proper dipole from Ξ. Therefore we never affect the disjoint links of the regular vertices of $K(\Gamma)$.

3 - Dipoles in pondered structures

Let $\mathscr{P} = (\bar{\Gamma}, \bar{\gamma}, \omega)$ be a n-pondered structure.

Definition 1. By a *dipole of type* h_A *involving colours* $c_1, \dots, c_h \in \Delta_{n-1}$ $(1 \leq h \leq n-1)$, we mean a subgraph $\bar{\Theta}$ of $\bar{\Gamma}$ formed by a pair (a, b) of vertices, joined by h edges $e_1, \dots, e_h$ directed from a to b, such that:

i. for $1 \leq i \leq h$, $\omega(e_i) = 1$ and $\bar{\gamma}(e_i) = c_i$

ii. $\bar{\Gamma}_{(\hat{c}_1, \dots, \hat{c}_h)}(a) \neq \bar{\Gamma}_{\hat{c}_1, \dots, \hat{c}_h)}(b)$.

We write $\bar{\Theta} = (a, b)$ to denote the dipole.

Definition 2. Two edges e and f of $\bar{\Gamma}$ are *GW-equivalent with respect to a subset* C *of* Δ_{n-1}, write $e \sim_C f$, iff $\bar{\gamma}(e), \bar{\gamma}(f) \in C$ and there is a finite sequence $e = \varepsilon_0, \varepsilon_1, \dots, \varepsilon_r = f$ of edges of $\bar{\Gamma}_C$ such that, for each $i \in \{0, \dots, r-1\}$, ε_i and ε_{i+1} belong to the same generalized weak cycle of $\bar{\Gamma}_C$.

Obviously «$\sim_C$» is an equivalence. For each edge $e \in E(\bar{\Gamma}_C)$, we shall denote by $E^{(C)}(e)$ the class of e with respect to the GW-equivalence relative to the subset C, whereas the symbols $E_1^{(C)}, \dots, E_r^{(C)}$ denote the elements of $E(\bar{\Gamma}_C)/\sim_C$.

Definition 3. By a *dipole of type* 1_B, we mean a subgraph $\bar{\theta}$ of $\bar{\Gamma}$ formed by a vertex x such that there exist $p, q \in \{1, \dots, r\}$ with $p \neq q$, such that for each $j \in \Delta_{n-1}$ and for each pair f_j', f''_j of j-coloured edges of $\bar{\Gamma}$ incident with x, $f_j' \in E_p^{\Delta_{n-1}}$ and $f''_j \in E_q^{\Delta_{n-1}}$.

Definition 3′. By a *dipole of type* h_B *involving colours* $c_1, \dots, c_{h-1} \in \Delta_{n-1}$ $(1 < h \leq n)$, we mean a subgraph $\bar{\theta}$ of $\bar{\Gamma}$ formed by a vertex x and $h-1$ loops $e_1, \dots, e_{h-1}$ in x such that:

i. for $1 \leq i \leq h-1$, $\bar{\gamma}(e_i) = c_i$

ii. there exist $p, q \in \{1, \dots, r\}$ with $p \neq q$, such that for each $j \in (\hat{c}_1, \dots, \hat{c}_{h-1})$ and for each pair f_j', f''_j of j-coloured edges of $\bar{\Gamma}_{(\hat{c}_1, \dots, \hat{c}_{h-1})}$ incident with x, $f_j' \in E_p^{(C)}$ and $f''_j \in E_q^{(C)}$, where $C = (\hat{c}_1, \dots, \hat{c}_{h-1})$.

From now on, unless otherwise stated, by a *dipole of type* h_B we shall mean also the case $h = 1$. As in Definition 1 we use the notation $\bar{\theta} = (x)$.

In the following, given a dipole $\bar{\Theta} = (a, b)$ of type h_A involving colours $c_1, \dots, c_h$, if $v \in \{a, b\}$ we denote by v_+^j (resp. by v_-^j), $j \in \Delta_n$, the vertex, diffe-

2_A - dipole involving colours $\{0, 1\}$ (cases I and II)

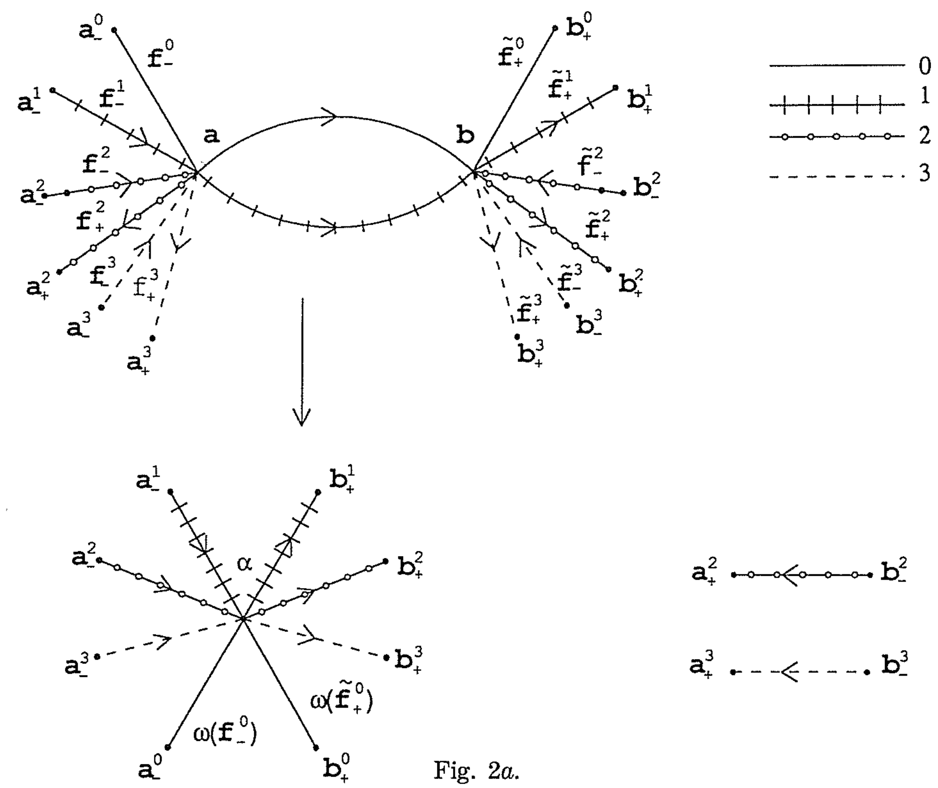

Fig. 2*a*.

rent from a and b, j-adjacent to v by an edge either having weight 1 and direction from v to v^j_+ (resp. from v^j_- to v) or having weight 0 (resp. 2). Moreover, for each $j \in \Delta_{n-1}$, call f^j_ε (resp. $\tilde{f}^j_\varepsilon$), $\varepsilon \in \{+, -\}$, the j-coloured edges joining a^j_ε and a (resp. b^j_ε and b).

If $\bar{\theta} = (x)$ is a dipole of type h_B involving colours $c_1, \ldots, c_{h-1}$, then label by x^j_+ (resp. by x^j_-), $j \in (\hat{c}_1, \ldots, \hat{c}_{h-1})$, the vertex j-adjacent to x, either having weight 1 and oriented from x to x^j_+ (resp. from x^j_- to x) or having weight 0 (resp. weight 2). Moreover, for each $j \in (\hat{c}_1, \ldots, \hat{c}_{h-1})$, call f^j_ε, $\varepsilon \in \{+, -\}$, the j-coloured edges joining x^j_ε and x (see Figures 2 and 3 for the above notations).

Note that, if $\bar{\Theta} = (a, b)$ is a dipole of type h_A of $\bar{\Gamma}$ involving colours $c_1, \ldots, c_h$, then we can distinguish the following cases:

case I. all edges incident with a and b have weight 1

case II. some 0-coloured edges, incident with a and b, have weight diffe-

2_A - dipole involving colours $\{1, 2\}$ (cases III)

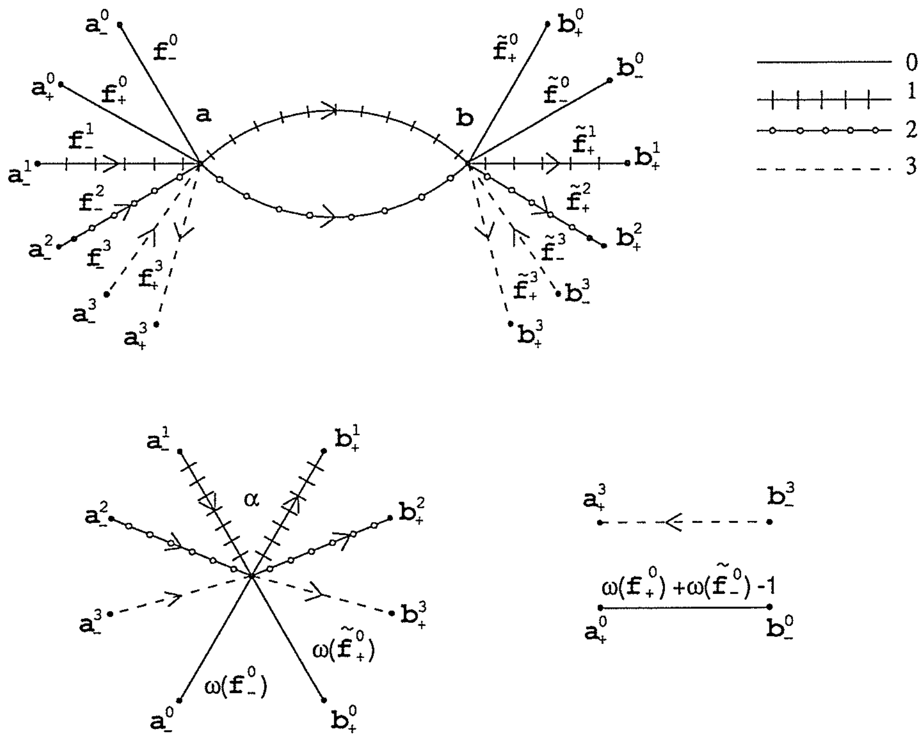

Fig. 2*b*.

rent from 1 and the colour 0 is involved in the dipole; in this case we have the following possibilities:

case IIa. $\omega(f_-^0) = 2 \quad \omega(\tilde{f}_+^0) = 1$

case IIb. $\omega(f_-^0) = 1 \quad \omega(\tilde{f}_+^0) = 0$ or $\omega(f_-^0) = 2 \quad \omega(\tilde{f}_+^0) = 0$

case III. some 0-coloured edges, incident with a and b, have weight different from 1 and the colour 0 is not involved in the dipole; we can have:

$$\omega(f_+^0) = 0 \quad \omega(f_-^0) = 2 \quad \omega(\tilde{f}_+^0) = 1 \quad \omega(\tilde{f}_-^0) = 1$$

or
$$\omega(f_-^0) = \omega(\tilde{f}_-^0) = 2, \quad \omega(f_+^0) = \omega(\tilde{f}_+^0) = 0$$

or
$$\omega(f_+^0) = \omega(f_-^0) = 1, \quad \omega(\tilde{f}_-^0) = 2, \quad \omega(\tilde{f}_+^0) = 0\,.$$

Definition 4. If $(\overline{\Gamma}, \overline{\gamma}, \omega)$ is an n-pondered structure and $\overline{\Theta} = (a, b)$ is a dipole of type h_A of $\overline{\Gamma}$ involving colours $c_1, \dots, c_h$, then the n-pondered structure $(\overline{\Gamma}', \overline{\gamma}', \omega')$ is said to be *obtained from* $(\overline{\Gamma}, \overline{\gamma}, \omega)$ *by deleting* Θ, iff it is constructed as follows:

i. delete from $\overline{\Gamma}$ the vertices a and b and all the edges incident with them (including $e_1, \dots, e_h$)

ii. add a new vertex α

iii. for each $j \in \Delta_{n-1} - \{0\}$, join a_-^j (resp. b_+^j) with α by a j-coloured edge of weight 1 directed from a_-^j to α (resp. from α to b_+^j)

iv. for each $j \in (\hat{c}_1, \dots, \hat{c}_h)$, join a_+^j with b_-^j by a j-coloured edge. In **case I**, **II** and in **case III** for $j \neq 0$, the new edge has weight 1 and is directed from b_-^j to a_+^j; in case **III** for $j = 0$ the new edge has weight $\omega(f_+^0) + \omega(\bar{f}_-^0) - 1$ and, if such a weight is 1, the direction is from a_+^0 to b_-^0

v. join a_-^0 (resp. b_+^0) and α by a 0-coloured edge of weight $\omega(f_-^0)$ (resp. $\omega(\bar{f}_+^0)$); if the weight is 1, the new edge is directed from a_-^0 to α (resp. from α to b_+^0).

All the edges of $\overline{\Gamma}$ not incident with a or b remain unchanged (see Figure 2).

Remark 3. If $\overline{\Theta} = (a, b)$ is a dipole of type h_A of $\overline{\Gamma}$ involving colours $c_1, \dots, c_h$, let $i_k \in (\hat{c}_1, \dots, \hat{c}_h)$, $k \in \{1, \dots, r\}$, be some colours such that $b_+^{i_k} = b_-^{i_k} = b$. Then, deleting the dipole, join $a_+^{i_k}$ (resp. $a_-^{i_k}$) and α by means of an i_k-coloured edge of weight $\omega(f_+^{i_k})$ (resp. $\omega(f_-^{i_k})$); if such a weight is 1, orient the edge from α to $a_+^{i_k}$ (resp. from $a_-^{i_k}$ to α).

Definition 5. The n-pondered structure $(\overline{\Gamma}', \overline{\gamma}', \omega')$ is said to be obtained *by deleting a dipole* $\overline{\theta} = (x)$ *of type* 1_B from $\overline{\Gamma}$, iff it is constructed as follows:

i. delete from $\overline{\Gamma}$ the vertex x

ii. for each $j \in \Delta_{n-1}$, join x_-^j and x_+^j by a j-coloured edge of weight $\omega(f_+^j) + \omega(f_-^j) - 1$. If the weight is 1 the edge is directed from x_-^j to x_+^j or from x_+^j to x_-^j according to $\omega(f_+^j)$ being different or equal to zero.

All the edges of $\overline{\Gamma}$ not incident with x remain unchanged.

Definition 5′. The n-pondered structure $(\overline{\Gamma}', \overline{\gamma}', \omega')$ is said to be obtained *by deleting a* $\overline{\theta} = (x)$ *dipole of type* h_B $(1 < h \leq n)$ from $\overline{\Gamma}$, iff it is constructed as follows:

3_B - dipole involving colours $\{1, 3\}$

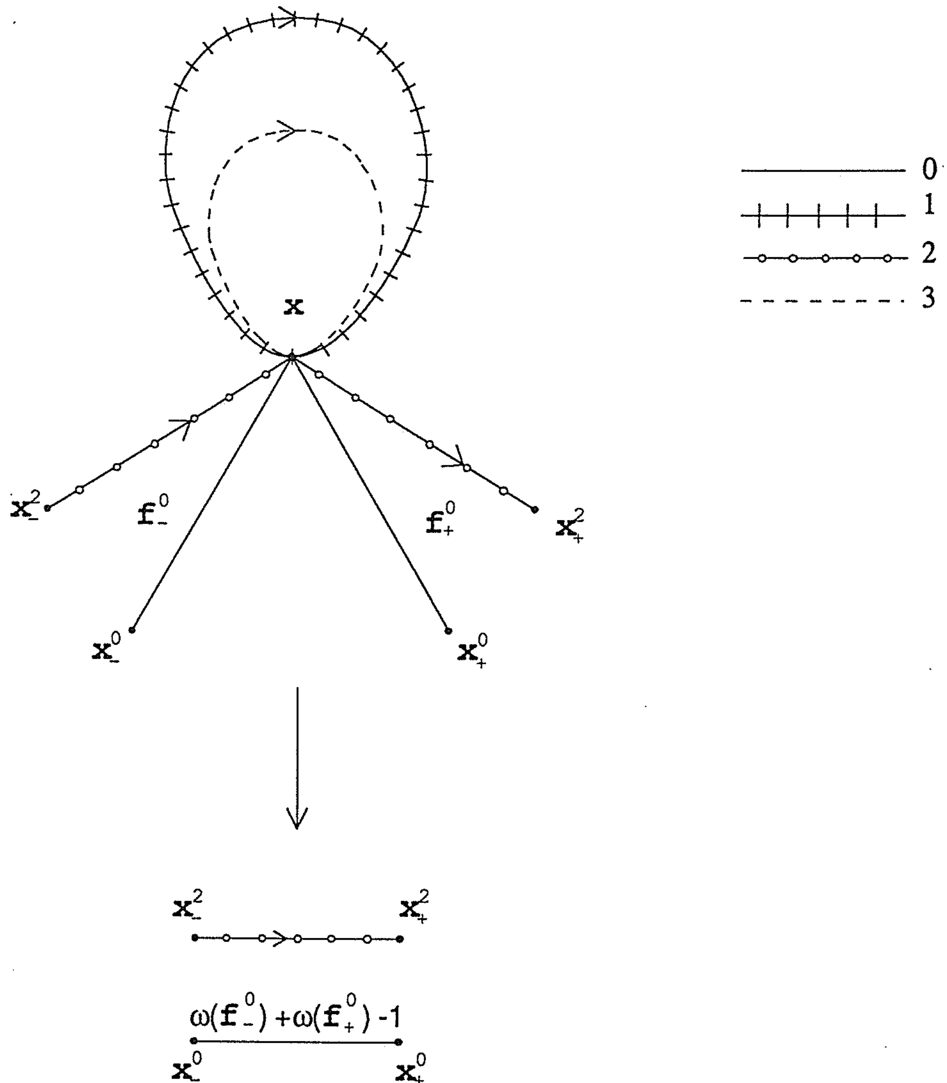

Fig. 3.

i. delete from $\overline{\Gamma}$ the vertex x and all the loops and edges incident with it

ii. for each $j \in (\hat{c}_1, \ldots, \hat{c}_{h-1})$, join x^j_- and x^j_+ by a j-coloured edge of weight $\omega(f^j_+) + \omega(f^j_-) - 1$. If the weight is 1 the edge is directed from x^j_- to x^j_+ or from x^j_+ to x^j_- according to $\omega(f^j_+)$ being different or equal to zero.

All the edges of $\overline{\Gamma}$ not incident with x remain unchanged (see Figure 3).

Remark 4. Suppose there exists a colour j such that $x_-^j = x_+^j$, then by deleting $\overline{\theta}$, the j-coloured edge of **ii** in Definition 5 and in Definition 5′ is a loop with $x_-^j = x_+^j$ as base-point.

Remark 5. If $(\overline{\Gamma}, \overline{\gamma}, \omega)$ is an oriented structure, then we can replace statements **iii**, **iv** and **v** in Definition 4 by:

iii′. for each $j \in \Delta_{n-1}$, join a_-^j (resp. b_+^j) with α by a j-coloured edge of weight 1 directed from a_-^j to α (resp. from α to b_+^j);

iv′. for each $j \in (\hat{c}_1, \ldots, \hat{c}_h)$, join a_+^j with b_-^j by a j-coloured edge of weight 1 directed from b_-^j to a_+^j.

Note that the new pondered structure, obtained by deleting a dipole of type h_A or h_B is still an oriented structure.

Definition 6. A dipole $\overline{\Theta} = (a, b)$ (resp. $\overline{\theta} = (x)$) of $(\overline{\Gamma}, \overline{\gamma}, \omega)$ of type h_A (resp. h_B) is said *proper* iff $|K(B(\overline{\Gamma}'))| \simeq |K(B(\overline{\Gamma}))|$, $(\overline{\Gamma}', \overline{\gamma}', \omega')$ being the n-pondered structure obtained from $\overline{\Gamma}$ by deleting $\overline{\Theta}$ (resp. $\overline{\theta}$).

Proposition 2. *With the above notations, if $\Theta = (a, b)$ (resp. $\overline{\theta} = (x)$) is a dipole of type h_A (resp. h_B) involving colours $c_1, \ldots, c_h$ (resp. $c_1, \ldots, c_{h-1}$), then $\Theta = ((a, 0), (b, 1))$ (resp. $\theta = ((x, 0), (x, 1))$) is a dipole of $B(\overline{\Gamma})$ of type h, involving colours $c_1, \ldots, c_h$ (resp. $c_1, \ldots, c_{h-1}, n$).*

Proof. Via bijoin-construction, in $B(\overline{\Gamma})$ the vertices $(a, 0)$ and $(b, 1)$ (resp. $(x, 0)$ and $(x, 1)$) are joined by h edges $e_1, \ldots, e_h$ (resp. $\varepsilon_1, \ldots, \varepsilon_h$), with $\beta(e_i) = c_i$, $i \in \{1, \ldots, h\}$ (resp. $\beta(\varepsilon_i) = c_i$, $i \in \{1, \ldots, h-1\}$ and $\beta(\varepsilon_h) = n$). Moreover from **iii** of Definition 1 (resp. **ii** of Definition 3), it follows:

$$(B(\overline{\Gamma})_{(\hat{c}_1, \ldots, \hat{c}_h)})(a, 0) \neq (B(\overline{\Gamma})_{(\hat{c}_1, \ldots, \hat{c}_h)})(b, 1)$$

$$(\text{resp. } (B(\overline{\Gamma})_{(\hat{c}_1, \ldots, \hat{c}_{h-1}, \hat{n})})(x, 0) \neq (B(\overline{\Gamma})_{(\hat{c}_1, \ldots, \hat{c}_{h-1}, \hat{n})})(x, 1)).$$

Proposition 3. *If $(\overline{\Gamma}, \overline{\gamma}, \omega)$ is an n-pondered structure, $\overline{\Theta} = (a, b)$ (resp. $\overline{\theta} = (x)$) a dipole of type h_A (resp. h_B) of $\overline{\Gamma}$, $\Theta = ((a, 0), (b, 1))$ (resp. $\theta = ((x, 0), (x, 1))$) the corresponding h-dipole of $B(\overline{\Gamma})$ and $(\overline{\Gamma}', \overline{\gamma}', \omega')$ the n-pondered structure obtained by deleting $\overline{\Theta}$ (resp. $\overline{\theta}$) from $\overline{\Gamma}$, then $B(\overline{\Gamma}')$ is the $(n+1)$-coloured graph obtained from $B(\overline{\Gamma})$ by deleting Θ (resp. θ).*

Proof. With the above notations, to delete Θ (resp. θ) from $B(\overline{\Gamma})$, we cancel the vertices $(a, 0)$ and $(b, 1)$ (resp. $(x, 0)$ and $(x, 1)$) and join $(a, 1)$ and $(b, 0)$ by an n-coloured edge. Moreover, if $v_j, w_j, j \in (\widehat{c}_1, \dots, \widehat{c}_h)$, are the vertices j-adjacent to $(a, 0)$ and $(b, 1)$ respectively (resp. $v_j, w_j, j \in (\widehat{c}_1, \dots, \widehat{c}_{h-1}, \widehat{n})$, are the vertices j-adjacent to $(x, 0)$ and $(x, 1)$ respectively) then we must join v_j and w_j by a j-coloured edge.

Set now $(a, 0) = (\alpha, 0)$, $(b, 1) = (\alpha, 1)$ and $v_j = a^j_+$ (resp. x^j_+), $w_j = b^j_-$ (resp. x^j_-). Obviously, by *shrinking* the colour n in the so obtained $(n+1)$-coloured graph, we obtain $\overline{\Gamma}'$.

Remark 6. If $|K(B(\overline{\Gamma}))|$ is a (closed) n-manifold, then all dipoles of type h_A and h_B in $\overline{\Gamma}$ are proper.

Remark 7. If $|K(B(\overline{\Gamma}))|$ is a singular n-manifold, then the corollary of Proposition 1 assures that every dipole of $\overline{\Gamma}$ of type h_A (resp. h_B), with $h \geqslant 1$ (resp. $h > 1$) is proper. If $|K(B(\overline{\Gamma}))|$ is a singular 3-manifold, $\overline{\theta} = (x)$ a dipole of type 1_B in $\overline{\Gamma}$ and $E^{(C)}_p$, $E^{(C)}_q$, $(C = (\widehat{c}_1, \dots, \widehat{c}_{h-1}))$ are the two equivalence classes of Definition 3, then $\overline{\theta}$ is proper, iff the following equality holds:

$$(*) \qquad \sum_{i,j \in \Delta_2} \overline{g}_{ij}(E^{(C)}_s) = 2 + \# V(E^{(C)}_s)$$

either for $s = p$ or for $s = q$, $\overline{g}_{ij}(E^{(C)}_s)$ being the number of generalized weak cycles of the subgraph $E^{(C)}_s$, with $i, j \in \Delta_2$. In fact, if $(*)$ holds, an easy calculation on Euler characteristic assures that $E^{(C)}_s$ represents S^2.

Remark 8. Matveev and Piergallini defined a complete system of moves on special (*standard*) spines of 3-manifolds (see **[11]**, **[12]** and **[13]**). Lins, in **[9]**, studied the relation between Matveev-Piergallini moves on special spines and moves (dipoles) on *3-gems* **[10]**, i.e. 4-coloured graphs representing 3-manifolds. Therefore Lins' work, together with Proposition 3, gives the link between dipoles on coloured spines and Matveev-Piergallini moves.

In **[2]** a crystallized structure $\widetilde{\Gamma}$ is defined, which is associated to an alternate (balanced) presentation of a group, such that all possible bijoin B on $\widetilde{\Gamma}$ are non contracted graphs, since $B_{\hat{2}}$ is not connected. Deleting a suitable finite sequence of dipoles of type 1_A in $\widetilde{\Gamma}$, we obtain a new crystallized structure $\widetilde{\Gamma}'$ such that:

1. $\widetilde{\Gamma}$ is a spine of a closed 3-manifold M^3 iff $\widetilde{\Gamma}'$ is a spine of M^3;
2. if $\widetilde{\Gamma}$ (and consequently $\widetilde{\Gamma}'$) is a spine of a closed 3-manifold M^3, there

exists a pondered structure $\overline{\Gamma}'$, associated to $\widetilde{\Gamma}$, such that $B(\overline{\Gamma}')$ is a (seminormal) crystallization of M^3.

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Sommario

In questo lavoro definiamo movimenti combinatori su grafi colorati che rappresentano spine di varietà PL e studiamo gli effetti di tali movimenti sui complessi corrispondenti.

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