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Almost Hermitian structures and connections on TM (**)

1 - Introduction

In the paper [6], E. Heil proves that if on a Finsler manifold $F^n = (M, F)$ with the Finsler metric

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

an integrable almost complex Finsler structure $f^i_j(x, y)$, satisfies the condition

$$g_{rs}(x, y) \quad f^r_i(x, y) \quad f^s_j(x, y) = g_{ij}(x, y) \tag{1.1}$$

then the Finsler metric is not anything else than a Riemann metric. At the same result we are led when f is an almost complex structure, that does not depend on the element of support y (cf. with M. Fukui, a result quoted in [10]). In the real field, such structures were studied for the first time by A. Moór [18]. We remark that in Moór's paper $g_{ij}(x, y)$ are not derivatives of F^2 and the simplifications gained by using the homogeneity relation are omitted (see H. Rund [22] too).

The study of Finsler manifolds endowed with an almost complex structure $f^i_j(x)$, starting from the point of view of the almost Hermitian metrics is due to G. B. Rizza (see [19], [20], [21]) by considering on the tangent bundle of a differentiable manifold M the isomorphism

$$\varphi^i_{\theta j} = \delta^i_j \cos\theta + f^i_j(x) \sin\theta \qquad 0 \leqslant \theta \leqslant 2\pi \tag{1.2}$$

with the property

$$g_{rs}(x, y)\, \varphi^r_{\theta i} \varphi^s_{\theta j} = g_{ij}(x, y). \tag{1.3}$$

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A manifold admitting a Finsler metric and an almost complex structure satisfying the condition (1.3) is called *Rizza manifold* and the structure is called the *Rizza structure* (cf. Y. Ichjyō [7]-[10]).

This paper follows the Moór's line concerning the study of almost Hermitian structures and let the possibility of extension of Rizza's geometric point of view.

This study has not been possible without drawing up the whole geometric device concerning the Lagrange geometry, a fact carried out recently by R. Miron [15], [16].

R. Miron's fundamental terms and concepts [14], [15], [16] and M. Matsumoto's basic notions [13] are supposed known (see also [4]).

2 - Almost Hermitian *d*-structures on M

A pair of *d*-tensor fields $(g_{ij}(x, y),\ f^i_j(x, y))$ on M, where $g_{ij}(x, y)$ is a *d*-tensor fields of the type $(0, 2)$ and $f^i_j(x, y)$ is a *d*-tensor field of the type $(1, 1)$, satisfying conditions

$$f^i_j f^k_i = -\delta^k_j \tag{2.1}$$

$$\begin{array}{l} g_{ij}(x, y) = g_{ji}(x, y) \qquad \operatorname{rank}\|g_{ij}(x, y)\| = n \\ g \text{ is a positive definite metric for any } y \neq 0 \end{array} \tag{2.2}$$

$$g_{rs} f^r_i f^s_j = g_{ij} \tag{2.3}$$

is said to be an *almost Hermitian d-structure* on M.

It results that n must be even $(n = 2n')$, $g_{ij}(x, y)$ determines a generalized metric *d*-structure on M [16], and $f^i_j(x, y)$ determines an almost complex *d*-structure on M [2], [3], [12], [19], ...

Let $\overset{\circ}{N}$ be a *non-linear* connection on M. Consider the *d*-tensor fields

$$G^h = g_{ij}(x, y)\, dx^i \otimes dx^j \qquad \overset{*}{F} = f^i_j(x, y)\, \overset{\circ}{\delta}_i \otimes dx^j \tag{2.4}$$

where $\overset{\circ}{\delta}_i = \partial_i - \overset{\circ}{N}{}^j_i(x, y)\, \dot{\partial}_j$. G^h is of the type $(0, 2)$ and symmetric, $\overset{*}{F}$ is of the type $(1, 1)$. Both are of rank $2n'$ and globally defined on TM.

We denote with $\overset{m}{\underset{\circ}{|}}$ and $\overset{m}{|}$ the h and v-covariant derivatives with respect to the $\overset{\circ}{N}$-canonical metrical *d*-connection $F\overset{m}{\Gamma}(\overset{\circ}{N})$ [15]:

$$\overset{m}{F}{}^i_{jk} = \frac{1}{2} g^{ih} (\overset{\circ}{\delta}_j g_{hk} + \overset{\circ}{\delta}_k g_{jh} - \overset{\circ}{\delta}_h g_{jk}) \qquad \overset{m}{C}{}^i_{jk} = \frac{1}{2} \bar{g}^{ih} (\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk})\,. \tag{2.5}$$

Theorem 1.

i. *There exists a d-connection* $\hat{D}$ *on* TM *having the properties*:

$$\hat{D}^h_X G^h = 0 \qquad \hat{D}^v_X G^h = 0 \qquad \hat{D}^h_X \overset{*}{F} = 0 \qquad \hat{D}^v_X \overset{*}{F} = 0 . \tag{2.6}$$

ii. *In the basis* $(\overset{\circ}{\delta}_i, \dot{\partial}_i)$, $\hat{D}^h$ *and* $\hat{D}^v$ *have the coefficients given by*

$$\begin{aligned} \hat{F}^i_{jk} &= \frac{1}{2} g^{ih} (\overset{\circ}{\delta}_j g_{hk} + \overset{\circ}{\delta}_k g_{jh} - \overset{\circ}{\delta}_h g_{jk}) - \frac{1}{2} f^i_r f^r_j \overset{m}{|}_k \\ \hat{C}^i_{jk} &= \frac{1}{2} g^{ih} (\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}) - \frac{1}{2} f^i_r f^r_j \overset{m}{|}_k . \end{aligned} \tag{2.7}$$

iii $\hat{D}$, *given by* (2.7), *depends only on* $\overset{\circ}{N}$, g_{ij} *and* f^i_j.

The proof is immediate. We call $\hat{D}$, given by (2.7), the $\overset{\circ}{N}$-*canonical almost Hermitian d-connection.*

Be $\|g^{ij}(x, y)\| = \|g_{ij}(x, y)\|^{-1}$ and denote with $\underset{1}{A}, \underset{2}{A}$ and $\underset{1}{Q}, \underset{2}{Q}$ the Obata's operators of g_{ij} and f^i_j

$$\underset{1}{A}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j - g_{ij} g^{kh}) \qquad \underset{2}{A}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j + g_{ij} g^{kh}) \tag{2.8}$$

$$\underset{1}{Q}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j - f^k_i f^h_j) \qquad \underset{2}{Q}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j + f^k_i f^h_j) . \tag{2.9}$$

Theorem 2. *Any almost Hermitian d-connection* (N, F, C) *on* M *satisfying* (2.6) *is given by*

$$\begin{aligned} N^i_j &= \overset{\circ}{N}{}^i_j - X^i_j \\ F^i_{jk} &= \overset{m}{F}{}^i_{jk} + \overset{m}{C}{}^i_{jr} X^r_k - \frac{1}{2} f^i_r (f^r_j \overset{m}{|}_k + f^r_j \overset{m}{|}_s X^s_k) + \underset{1}{A}{}^{pi}_{jt} Q^{st}_{pr} Y^r_{sk} \\ C^i_{jk} &= \overset{m}{C}{}^i_{jk} - \frac{1}{2} f^i_r f^r_j \overset{m}{|}_k + \underset{1}{A}{}^{pi}_{jt} \underset{1}{Q}{}^{st}_{pr} Z^r_{sk} \end{aligned} \tag{2.10}$$

where X^i_j, Y^i_{jk}, Z^i_{jk} *are the arbitrary d-tensor fields on* M.

Proof. We look for almost Hermitian d-connections $F\Gamma = (N, F, C)$ of the form (see [**16**])

$$N^i_j = \overset{\circ}{N}{}^i_j - X^i_j \qquad F^i_{jk} = \hat{F}^i_{jk} + \hat{C}^i_{jr} X^r_k + B^i_{jk} \qquad C^i_{jk} = \hat{C}^i_{jk} + D^i_{jk} . \tag{2.11}$$

We obtain for B and D the system of equations

$$\underset{2}{\Lambda} B = 0 \qquad \underset{2}{\Lambda} D = 0 \qquad \underset{2}{Q} B = 0 \qquad \underset{2}{Q} D = 0$$

which is compatible with the solution

$$B^i_{jk} = (\underset{1}{\Lambda}\,\underset{1}{Q})^{si}_{jr} Y^r_{sk} \qquad D^i_{jk} = (\underset{1}{\Lambda}\,\underset{1}{Q})^{si}_{jr} Z^r_{sk} \qquad \forall X^i_j,\, Y^i_{jk},\, Z^i_{jk}.$$

This result with (2.7) and (2.11) leads to (2.10).

Remark 1. Since $g_{ij}(x, y)$ is a generalized metric we have that $M^n = (M, g_{ij}(x, y))$ is a *generalized Lagrange space* [16]. Let $E(x, y)$ be the absolute energy of M^n, i.e. $E(x, y) = g_{ij}(x, y)\, y^i y^j$. It is said [15], that M^n is a space *with weakly regular metric* if (M, E) is a Lagrange space, i.e. rank $\|g^*_{ij}(x, y)\| = n$, where $g^*_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j E$. If $g_{ij}(x, y)$ of M^n $(n = 2n')$ has the properties (2.2) and (2.3) and M^n has a weakly regular metric, then we can take $\overset{o}{N}{}^i_j = \overset{c}{N}{}^i_j$ over all in this paragraph, where ([11], [15])

$$\text{(2.12)} \qquad \overset{c}{N}{}^i_j = \dot{\partial}_j G^i \qquad G^i = \frac{1}{4} g^{*ij} [(\dot{\partial}_j \partial_k E)\, y^k - \partial_j E].$$

Theorem 3. *If $M^{2n'}$ is of local Minkowski type with weakly regular metric and has a d-tensor field $f^i_j(x, y) = f^i_j(y)$ such that relations* (2.1), (2.2) *and* (2.3) *are satisfied, then we have*

1. *The $\overset{c}{N}$-canonical connection* (2.7) *has the form*

$$\text{(2.13)} \qquad \overset{c}{F}{}^i_{jk} = 0 \qquad \overset{c}{C}{}^i_{jk} = \frac{1}{2} g^{ih} (\dot{\partial}_j g_{hk} + \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}) - \frac{1}{2} f^i_s f^s_j \overset{m}{|}_k .$$

2. *The torsion tensor fields of* (2.13) *are given by*

$$\text{(2.14)} \qquad \begin{matrix} \overset{c}{T}{}^i_{jk} = 0 \qquad \overset{c}{R}{}^i_{jk} = 0 \qquad \overset{c}{P}{}^i_{jk} = 0 \\ \overset{c}{C}{}^i_{jk} \qquad \overset{c}{S}{}^i_{jk} = -\frac{1}{2} f^i_s (f^s_j \overset{m}{|}_k - f^s_k \overset{m}{|}_j). \end{matrix}$$

3. *The curvature tensor fields of* (2.13) *are given by*

$$\text{(2.15)} \qquad \overset{c}{R}{}_j{}^i{}_{kl} = 0 \qquad \overset{c}{P}{}_j{}^i{}_{kl} = 0 \qquad \overset{c}{S}{}_j{}^i{}_{kl} = \underset{(k,\, l)}{\mathfrak{A}} \{\dot{\partial}_l \overset{c}{C}{}^i_{jk} + \overset{c}{C}{}^s_{jk} \overset{c}{C}{}^i_{sl}\},$$

where $\mathfrak{A}$ denotes the alternate summation.

4. *The Bianchi identities of* (2.13) *are*

$$(2.16)\qquad \underset{(i,\,j,\,k)}{S}\{\overset{c}{S}{}^h_{ir}\overset{c}{S}{}^r_{jk} + \overset{c}{S}{}^h_{ij}\overset{m}{|}_k - \overset{c}{S}{}^h_{ijk}\} = 0 \qquad \underset{(i,\,j,\,k)}{S}\{\overset{c}{S}{}^h_{tir}\overset{c}{S}{}^r_{jk} + \overset{c}{S}{}^h_{tij}\overset{m}{|}_k\} = 0$$

where S *is the cyclic summation.*

The proof is elementary if we take into account that in this case we have $g_{ij}(x, y) = g_{ij}(y)$ and hence $\overset{c}{N}{}^i_j = 0$.

Evidently, the *deflection tensor field* of the d-connection (2.13), i.e. $\overset{c}{D}{}^i_j = y^k \overset{c}{F}{}^i_{kj} - \overset{c}{N}{}^i_j$ is zero.

Now, we denote with $a_{ij}(x, y)$ the d-tensor field given by

$$(2.17)\qquad a_{ij} = f^r_i g_{rj}$$

which determines an *almost symplectic d-structure* on M and let us consider

$$(2.18)\qquad A^h = \frac{1}{2} a_{ij}(x, y)\, dx^i \wedge dx^j .$$

A^h is a d-tensor field of type (0, 2), anti-symmetric, of rank $n = 2n'$ and globally defined on TM.

If $\|a^{ij}(x, y)\| = \|a_{ij}(x, y)\|^{-1}$ then the Obata's operators of $a_{ij}(x, y)$ are given by

$$(2.19)\qquad \underset{1}{\Phi}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j - a_{ij} a^{kh}) \qquad \underset{2}{\Phi}{}^{kh}_{ij} = \frac{1}{2}(\delta^k_i \delta^h_j + a_{ij} a^{kh}) .$$

Proposition 1. *The following twelve commutativities hold*

$$\underset{\alpha}{A}\,\underset{\beta}{Q} = \underset{\beta}{Q}\,\underset{\alpha}{A} \qquad \underset{\alpha}{Q}\,\underset{\beta}{\Phi} = \underset{\beta}{\Phi}\,\underset{\alpha}{Q} \qquad \underset{\alpha}{\Phi}\,\underset{\beta}{A} = \underset{\beta}{A}\,\underset{\alpha}{\Phi} \qquad (\alpha, \beta = 1, 2).$$

Theorem 4.

i. *An almost Hermitian d-connection D has the property:* $D^h_X A^h = 0$, $D^v_X A^h = 0$, $\forall X \in \mathfrak{X}(\mathrm{M})$.

ii. *The Obata's operators* (2.8), (2.9) *and* (2.19) *are covariantly constant with respect to D.*

iii. *The d-tensor fields* $\underset{2}{A}{}^{hi}_{jr} R_h{}^r{}_{kl}$, $\underset{2}{Q}{}^{hi}_{jr} R_h{}^r{}_{kl}$, $\underset{2}{\Phi}{}^{hi}_{jr} R_h{}^r{}_{kl}$ *(and the analogous fields we obtain by replacing* $R_j{}^i{}_{kl}$ *with* $P_j{}^i{}_{kl}$, $S_j{}^i{}_{kl}$*) and their h- and v-covariant derivatives of every order vanish, for every almost Hermitian d-connection D.*

Proof. **i** and **ii** are evident. For **iii** we apply the Ricci formulas to g_{ij}, f^i_j and a_{ij}. Taking into account of **ii** we get the statement.

Let us consider the transformation $F\Gamma(\overset{\circ}{N}) \to F\overline{\Gamma}(\overset{\circ}{N})$ of almost Hermitian d-connections, which preserve the non-linear connection $\overset{\circ}{N}$. Owing to Theorem 2 they are given by

$$(2.20) \qquad \overline{N}_j^i = N_j^i \qquad \overline{F}_{jk}^i = F_{jk}^i + (\underset{1}{\Lambda}\underset{1}{Q})_{jr}^{si} Y_{sk}^r \qquad \overline{C}_{jk}^i = C_{jk}^i + (\underset{1}{\Lambda}\underset{1}{Q})_{jr}^{si} Z_{sk}^r$$

where Y_{jk}^i, Z_{jk}^i are arbitrarily given d-tensor fields. We have

Theorem 5. *The set of all transformations* (2.20) *with mapping product form an Abelian group* G_{ah}, *which is isomorphic to the additive group of the pairs of d-tensor fields* $(\underset{1}{\Lambda}\underset{1}{Q}Y, \underset{1}{\Lambda}\underset{1}{Q}Z)$.

We shall pay attention to the invariants of the group G_{ah}. By direct calculations we have ([14], [17])

Theorem 6. *The following d-tensor fields are invariants of the group* G_{ah}

$$\overset{1}{R}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} R_{rs}^t$$

$$(2.21) \qquad \overset{1}{T}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} T_{rs}^t \qquad \overset{1}{C}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} C_{rs}^t$$

$$\overset{1}{P}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} P_{rs}^t \qquad \overset{1}{S}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} S_{rs}^t$$

$$\overset{2}{R}(f)_{jk}^i = \underset{1}{Q}_{tm}^{ir}\underset{1}{Q}_{jk}^{ms} R_{rs}^t$$

$$(2.22) \qquad \overset{2}{P}(f)_{jk}^i = \underset{1}{Q}_{tm}^{ir}\underset{2}{Q}_{jk}^{ms} P_{rs}^t \qquad \overset{2}{C}(f)_{jk}^i = \underset{2}{Q}_{tm}^{ir}\underset{2}{Q}_{jk}^{ms} C_{rs}^t$$

$$\overset{3}{T}(f)_{jk}^i = T_{jk}^i - f_m^i (f_j^r P_{kr}^m - f_k^r P_{jr}^m)$$

$$(2.23) \qquad \overset{3}{R}(f)_{jk}^i = R_{jk}^i - S_{rs}^i f_j^r f_k^s - f_m^i (f_j^r C_{kr}^m - f_k^r C_{jr}^m)$$

$$\overset{*}{T}_{ijk} = \underset{(i,j,k)}{\mathcal{S}} \{a_{im} T_{jk}^m\} \qquad \overset{*}{S}_{ijk} = \underset{(i,j,k)}{\mathcal{S}} \{a_{im} S_{jk}^m\}$$

$$(2.24) \qquad \nu_{ijk} = \underset{(j,k)}{\mathcal{A}} \{a_{km} P_{ij}^m\} \qquad \kappa_{ijk} = \underset{(i,j)}{\mathcal{A}} \{a_{im} C_{jk}^m\}$$

$$\overset{*}{R}_{ijk} = \underset{(i,j,k)}{\mathcal{S}} \{a_{im} R_{jk}^m\}, \qquad a_{km} T_{ij}^m + \nu_{ijk}, \qquad a_{im} S_{jk}^m + \kappa_{ijk}.$$

Theorem 7.

i. *The d-tensor fields* $\overset{1}{T}(f)$, $\overset{1}{S}(f)$ *vanish if and only if there exists a semi-symmetric almost Hermitian d-connection* $F\Gamma(\overset{\circ}{N})$.

ii. *The invariant* $\overset{*}{T}_{ijk}$ *(resp.* $\overset{*}{S}_{ijk}$*) vanish if and only if there exists an almost Hermitian d-connection* $F\Gamma(\overset{\circ}{N})$ *with* $T^i_{jk} = 0$ *(resp.* $S^i_{jk} = 0$*).*

iii. *The induced almost symplectic d-structure* a_{ij} *does not depend on the support element* y*, if and only if* $\kappa_{ijk} = 0$.

Proof. The statement **i** results from

$$T^i_{jk} = \sigma_j \delta^i_k - \sigma_k \delta^i_j \qquad S^i_{jk} = \tau_j \delta^i_k - \tau_k \delta^i_j \qquad \sigma, \tau \in \mathfrak{X}^*(M).$$

Because $\underset{1}{\Lambda}\underset{1}{Q} = \underset{1}{\Phi}\underset{1}{Q}$ if we put $Q^{sm}_{pr} Y^r_{sk} = \alpha T^m_{pk}$ in (2.20), where α is a real number, we have $\overline{\overset{1}{T}}{}^i_{jk} = (1 + \frac{3}{2}\alpha)\overset{1}{T}{}^i_{jk} - \frac{\alpha}{2} a^{ip} \overset{*}{T}_{pjk}$. Taking $\alpha = -\frac{2}{3}$, $\overset{*}{T}_{ijk} = 0$, implies $\overline{T}^i_{jk} = 0$. The converse is evident. The statement about $\overset{*}{S}_{ijk}$ is proved in the same way and then we get **ii.** To prove **iii** pay attention to $a_{ij}|_k = \dot{\partial}_k a_{ij} - \kappa_{ijk} = 0$.

3 - Almost Hermitian structures on tangent bundle

The existence of the non-linear connection $\overset{\circ}{N}$ on TM (for example $\overset{\circ}{N} = \overset{c}{N}$ given by (2.12)), allows us to consider the d-tensor fields

$$G^v = g_{ij}(x, y)\, \overset{\circ}{\delta} y^i \otimes \overset{\circ}{\delta} y^j \tag{3.1}$$

$$\overset{**}{F} = f^i_j(x, y)\, \dot{\partial}_i \otimes \overset{\circ}{\delta} y^j \qquad \overset{2}{F} = f^i_j(x, y)\, \overset{\circ}{\delta}_i \otimes \overset{\circ}{\delta} y^j \qquad \overset{3}{F} = f^i_j(x, y)\, \dot{\partial}_i \otimes dx^j \tag{3.2}$$

$$A^v = \frac{1}{2} a_{ij}(x, y)\, \overset{\circ}{\delta} y^i \wedge \overset{\circ}{\delta} y^j \tag{3.3}$$

which are globally defined on TM, of rank $2n$' on TM and of the type $(0, 2)$, $(1, 1)$ and $(0, 2)$, respectively. Then

$$G = G^h + G^v \qquad \tilde{G} = G^h - G^v \tag{3.4}$$

are *Riemann structures* on TM,

$$F^I = \overset{*}{F} + \overset{**}{F} \qquad F^{II} = \overset{*}{F} - \overset{**}{F} \qquad F^{III} = \overset{2}{F} + \overset{3}{F} \tag{3.5}$$

are *almost complex structures* on TM, and

(3.6) $A = A^h + A^v, \quad \tilde{A} = A^h - A^v$ and $\hat{A} = a_{ij}(x, y)\, dx^i \wedge \overset{\circ}{\delta} y^j$

are *almost symplectic structures* on TM, each of rank $4n'$.

Theorem 8. *The pairs of d-tensor fields* (G, F^{I}) $(\tilde{G}, F^{\mathrm{I}})$, (G, F^{II}), $(\tilde{G}, F^{\mathrm{II}})$, (G, F^{III}) *are almost Hermitian structures on* TM, *with the induced almost symplectic structures* $A, \tilde{A}, A, \tilde{A}$, and $\hat{A}$, *respectively.*

Theorem 9.

i. *If the invariants* $\overset{3}{T}(f), \overset{3}{R}(f)$ *of the group* G_{ah} *vanish, then there exists a linear connection* D *on* TM *with the properties:* $(D_X Y^v)^h = 0$, $D_X G = 0$, $D_X \tilde{G} = 0$, $D_X F^{\mathrm{III}} = 0$, $D_X F^{\mathrm{I}} = 0$, $D_X F^{\mathrm{II}} = 0$, $\forall X, Y \in \mathfrak{X}(\mathrm{TM})$.

ii. *The connection* D *is given by the* $\overset{\circ}{N}$*-canonical almost Hermitian d-connection* $\bar{D}$, (2.7).

From Theorem 6 and (2.7) we easily get

Theorem 10. *The almost Hermitian structures* (G, F^{I}), $(\tilde{G}, F^{\mathrm{I}})$, $(\tilde{G}, F^{\mathrm{II}})$, (G, F^{III}) *are almost Kähler structures, if and only if there exists an almost Hermitian d-connection that, respectively, satisfies the conditions*:

$$(G, F^{\mathrm{I}}) \text{ and } (\tilde{G}, F^{\mathrm{II}}): \quad T^i_{jk} = 0, \; S^i_{jk} = 0, \; \kappa_{ijk} + a_{km} R^m_{ij} = 0, \; \nu_{ijk} = 0$$

$$(G, F^{\mathrm{II}}) \text{ and } (\tilde{G}, F^{\mathrm{I}}): \quad T^i_{jk} = 0, \; S_{ijk} = 0, \; \kappa_{ijk} - a_{km} R^m_{ij} = 0, \; \nu_{ijk} = 0$$

$$(G, F^{\mathrm{III}}): \quad \overset{*}{R}_{ijk} = 0, \; \nu_{ijk} + a_{km} T^m_{ij} = 0$$

where, in the last case, the a_{ij}*'s do not depend on the support element* y.

Theorem 11.

i. *The almost Kähler structures* (G, F^{I}) *and* $(\tilde{G}, F^{\mathrm{I}})$ *are Kähler structures, if and only if the invariants of the group* G_{ah} *fulfill the conditions*

(3.7) $\overset{1}{T}(f) = 0 \quad \overset{1}{R}(f) = 0 \quad \overset{1}{C}(f) = 0 \quad \overset{1}{P}(f) = 0 \quad \overset{1}{S}(f) = 0\,.$

ii. *The almost Kähler structures* (G, F^{II}) *and* $(\tilde{G}, F^{\mathrm{II}})$ *are Kähler structures, if and only if the invariants of the group* G_{ah} *fulfill the conditions*

$$\overset{1}{T}(f) = 0 \quad \overset{2}{R}(f) = 0 \quad \overset{2}{C}(f) = 0 \quad \overset{2}{P}(f) = 0 \quad \overset{1}{S}(f) = 0. \tag{3.8}$$

iii. *The almost Kähler structure* (G, F^{III}) *is a Kähler structure, if and only if the invariants of the group* G_{ah} *fulfill the conditions*

$$\overset{3}{T}(f) = 0 \quad \overset{3}{R}(f) = 0\,. \tag{3.9}$$

Proof. The almost Kähler structure (G, F^{I}) is a Kähler structure, if and only if the Nijenhuis's d-tensor field attached to F^{I}, i.e. $\tilde{N}(F^{\mathrm{I}})(X, Y)$, is zero. But $\tilde{N}(F^{\mathrm{I}})(X, Y) = 0$, $\forall X, Y \in \mathfrak{X}(\mathrm{TM})$ is equivalent to the equations

$$\tilde{N}(F^{\mathrm{I}})(\overset{\circ}{\delta}_j, \overset{\circ}{\delta}_k) = 0 \quad \tilde{N}(F^{\mathrm{I}})(\overset{\circ}{\delta}_j, \dot{\partial}_k) = 0 \quad \tilde{N}(F^{\mathrm{I}})(\dot{\partial}_j, \dot{\partial}_k) = 0$$

which are equivalent with (3.7). The proof of **ii** and **iii** follow the same pattern.

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Sommario

Nel presente lavoro sono studiate le d-strutture quasi hermitiane su una varietà differenziabile M *e le strutture quasi hermitiane sul fibrato tangente* TM.

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