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Regularity for weak solutions to the Dirichlet problem in Morrey space (**)

1 - Introduction

In this paper we consider the *Dirichlet problem* for the equation

$$Lu + b_i u_{x_i} - (d_i u)_{x_i} + cu = (f_i)_{x_i} \tag{1.1}$$

in an open bounded domain $\Omega \subset \boldsymbol{R}^n$ with $n \geqslant 3$, where L is an *elliptic operator in divergence form* defined by

$$L = -\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial}{\partial x_i})$$

and a_{ij} are measurable bounded functions in Ω while b_i, d_i, c, f_i belong to certain Morrey spaces. Regularity results for local solutions to (1.1) or to the related Dirichlet problem, with hypoteses which do not imply high integrability of the coeffients or of f_i have been proved in recent times by many authors in particular cases, for example for Poisson and Schrödinger equations (see e.g. [1], [2], [4], [5], ...) who generalized classical results by [8], [11], [10].

Aizenman and Simon in [1] studied continuity properties of solutions to equations of the form $-\Delta u + Vu = f$ where V, f belong to the Stummel-Kato class. This assumption does not require high integrability ($V, f \in L^p(\Omega)$, $p > \frac{n}{2}$) as in previous works (see e.g. [8], [11]). We recall that the Stummel-

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Kato class $S(\Omega)$ is the set of the locally integrable functions f such that

$$\lim_{\varepsilon\to 0}\sup_{x\in\Omega}\int_{\{y\in\Omega:\,|x-y|<\varepsilon\}} f(y)\,|x-y|^{2-n}\,dy = 0.$$

This work has been extended by [2], who replaced Δ with A, where A is a second order elliptic operator in divergence form.

In the paper [4] it was shown that the Stummel-Kato class may be considered as a limiting case of the scale of Morrey spaces $L^{1,\lambda}(\Omega)$, $0<\lambda<n$, precisely

$$L^{1,\lambda}(\Omega)\subset S(\Omega)\subset L^{1,\mu}(\Omega) \qquad 0\leqslant\mu\leqslant n-2<\lambda<n.$$

In [4], [5] Di Fazio studied various kinds of regularity of solutions to special cases of equation (1.1).

In this paper we assume that the coefficients b_i^2, d_i^2, c belong to $L^{1,\gamma}(\Omega)$ with $\gamma\in]n-2,n[$, proving various regularity results (L^p, BMO, ...) for the solutions of the Dirichlet problem, assuming $f_i\in L^{2,\lambda}(\Omega)$, $0<\lambda<n$. We remark that we must require $\gamma>n-2$ because we want to give a weak statement to the problem (see Lemma 1) and because if $\gamma\leqslant n-2$ we do not have regularity results. In fact for example

$$u(x)=\frac{1}{|x|^{\frac{n}{2^*}}\log|x|}+k|x|^2 \qquad \left(2^*=\frac{2n}{n-2}\right)$$

in $B_{\frac{1}{2}}(0)\equiv B$, for convenient k belongs to $H_0^1(B)$ and is a weak solution to the Dirichlet problem

$$\int_B (u_{x_i}v_{x_i}+d_i u v_{x_i})\,dx = -\int_B f_i v_{x_i}\,dx$$

with $d_i=\frac{1}{|x|}\left(\frac{n}{2^*}+\frac{1}{\log|x|}\right)\frac{x_i}{|x|}\in L^{2,n-2}(B)$ while $f_i=-kx_i\left(\frac{1}{\log|x|}+2+\frac{n}{2^*}\right)$ are regular functions but neither $u\in L^p(B)$ for $p>2^*$ nor $u\in L^{2^*,\lambda}(B)$ if $\lambda>0$.

The same problem has been discussed in [6] by Di Fazio, who assumed $b_i=c=0$.

We have been able to remove this restriction refining an iteration argument he used by means of various technical lemmas we collected in Sec. 3.

We wish to thank Prof. E. B. Fabes for some helpful conversation on these topics.

2 - Preliminaries

Let Ω be an open bounded domain in $\boldsymbol{R}^n (n \geq 3)$ such that

$$|\Omega(x, r)| = |\{y \in \Omega : |x-y| < r)| \geq A r^n$$

for $r \in]0, \delta]$ where δ is the diameter of Ω and A is a positive constant independent of $x \in \overline{\Omega}$ and r. Here and in the sequel we denote by $|E|$ the Lebesgue measure of a measurable set $E \subseteq \boldsymbol{R}^n$.

For $p \in [1, +\infty[$ and $\lambda \in [0, n[$ we set

$$\|f\|_{p,\lambda}^p = \sup_{x \in \Omega, \rho > 0} \frac{1}{\rho^\lambda} \int_{B_\rho(x) \cap \Omega} |f(y)|^p \, dy$$

where $B_\rho(x) = \{y \in \boldsymbol{R}^n : |y-x| < \rho\}$. The subset of those functions of $L^p(\Omega)$ satisfying $\|f\|_{p,\lambda} < +\infty$ will be called the *Morrey space* $L^{p,\lambda}(\Omega)$, while the set of those measurable functions f such that

$$\sup_{t>0} t^p |\{y \in \Omega : |f(y)| > t\} \cap B_\rho(x)| \leq C \rho^\lambda$$

for some $C > 0$ independent of $\rho > 0$ and $x \in \Omega$ will be called the *weak Morrey space* $L_w^{p,\lambda}(\Omega)$.

Obviously we have $L^{p,\lambda}(\Omega) \subseteq L_w^{p,\lambda}(\Omega) \subseteq L^{q,\lambda}(\Omega)$ for $1 \leq p < q < +\infty$ and $0 < \lambda < n$.

For every $f \in L^1_{loc}(\Omega)$ we set

$$\|f\|_{*,\Omega} = \sup_{B_\rho(x) \subset\subset \Omega} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |f(y) - f_B| \, dy .$$

If f is an integrable function in E we put $f_E = \dfrac{1}{|E|} \int_E f(y) \, dy$.

We will call $BMO(\Omega)$ (see e.g. [7]) the subset of $L^1_{loc}(\Omega)$ such that $\|f\|_{*,\Omega} < +\infty$.

Let us now consider functions a_{ij}, b_i, c, f_i $(i, j = 1, 2, \dots, n)$ defined and measurable in Ω with the hypotheses

I$_1$. $a_{ij} = a_{ji}$ with $i, j = 1, 2, \dots, n$ and $\mu^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \mu|\xi|^2$ for every $\xi \in \boldsymbol{R}^n$ a.e. in Ω; μ is a real positive constant.

I$_2$. b_i^2, d_i^2, $c \in L^{1,\gamma}(\Omega)$ for $i, j = 1, 2, \dots, n$ with $\gamma \in]n-2, n[$.

I$_3$. $f_i \in L^{2,\lambda}(\Omega)$ for $i, j = 1, 2, \dots, n$ with $\lambda \in [0, n[$.

The purpose of our work is to study the regularity of the *weak solutions* of (1.1), i.e. the solutions $u \in H_0^1(\Omega)$ to the equation

$$\int_\Omega (a_{ij} u_{x_j} v_{x_i} + b_i u_{x_i} v + \mathrm{d}_i u v_{x_i} + cuv)\,\mathrm{d}x = -\int_\Omega f_i v_{x_i}\,\mathrm{d}x, \qquad \forall v \in H_0^1(\Omega). \tag{2.1}$$

We observe that (2.1) makes sense by virtue of $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ (see also the following Lemma 1).

3 - Preliminary lemmas

We will now give some technical lemmas.

Lemma 1. *Let $f \in L^{2,\gamma}(\Omega)$ with $n-2<\gamma<n$ and $u \in L^{2,\nu+2}(\Omega)$ such that $|\nabla u| \in L^{2,\nu}(\Omega)$ with $0 \leqslant \nu < n-2$; then $fu \in L^{2,\gamma+\nu-n+2}(\Omega)$ and*

$$\|fu\|_{2,\gamma+\nu-n+2} \leqslant C\|f\|_{2,\gamma}(\|\nabla u\|_{2,\nu} + \|u\|_{2,\nu+2}) \tag{3.1}$$

where C is a constant independent of f and u.

For the proof see [6] Lemma 4.1.

Lemma 2. *Let $f \in L^{2,\gamma}(\Omega)$ with $n-2<\gamma<n$ and $u \in L^{2,\nu+2}(\Omega)$ such that $|\nabla u| \in L^{2,\nu}(\Omega)$ with $0 \leqslant \nu < n-2$; then $f^2 u \in L^{1,\gamma+\frac{\nu-n+2}{2}}(\Omega)$ and $fu_{x_i} \in L^{1,\frac{\gamma+\nu}{2}}(\Omega)$.*

Proof. Since

$$\int_{B_\rho \cap \Omega} f^2 |u|\,\mathrm{d}x \leqslant \Big(\frac{1}{\rho^\gamma}\int_{B_\rho\cap\Omega} f^2\,\mathrm{d}x\Big)^{\frac12}\Big(\frac{1}{\rho^{\gamma+\nu-n+2}}\int_{B_\rho\cap\Omega} f^2u^2\,\mathrm{d}x\Big)^{\frac12}\rho^{\gamma+\frac{\nu-n+2}{2}}$$

and
$$\int_{B_\rho\cap\Omega} |fu_{x_i}|\,\mathrm{d}x \leqslant \Big(\frac{1}{\rho^\gamma}\int_{B_\rho\cap\Omega} f^2\,\mathrm{d}x\Big)^{\frac12}\Big(\frac{1}{\rho^\nu}\int_{B_\rho\cap\Omega} u_{x_i}^2\,\mathrm{d}x\Big)^{\frac12}\rho^{\frac{\gamma+\nu}{2}}$$

where B_ρ is any ball centered at $x \in \Omega$, the result follow from Lemma 1.

Lemma 3. *Let $u \in L^{\frac{2n}{n-2},\nu}(\Omega) \cap L^{1,\mu}(\Omega)$ with $0 \leqslant \nu < n$ and $0 \leqslant \mu < n$; then $u \in L^{2,\sigma}(W)$ with $\sigma = \dfrac{4\mu + \nu(n-2)}{n+2}$.*

Proof. Let B_ρ be any ball centered at $x \in \Omega$, then we have

$$\int_{B_\rho \cap \Omega} u^2 \, dx = \int_{B_\rho \cap \Omega} |u|^{\frac{2n}{n+2}} |u|^{\frac{4}{n+2}} \, dx$$

$$\leq \Big(\frac{1}{\rho^\nu} \int_{B_\rho \cap \Omega} |u|^{\frac{2n}{n-2}} \, dx\Big)^{\frac{n-2}{n+2}} \Big(\frac{1}{\rho^\mu} \int_{B_\rho \cap \Omega} |u| \, dx\Big)^{\frac{4}{n+2}} \rho^{\frac{4\mu + \nu(n-2)}{n+2}}$$

$$\leq \|u\|_{\frac{2n}{n-2}, \nu}^{\frac{2n}{n+2}} \|u\|_{1,\mu}^{\frac{4}{n+2}} \rho^{\frac{4\mu + \nu(n-2)}{n+2}} .$$

Lemma 4. *Let $f \in L^{2,\gamma}(\Omega)$ with $n - 2 < \gamma < n$ and $u \in L^{2,\nu+2}(\Omega)$ such that $|\nabla u| \in L^{2,\nu}(\Omega)$ for $0 \leq \nu < 2n - \gamma - 4$. Then*

$$I_2(f^2 u) = \int_\Omega \frac{f^2(y) u(y)}{|x-y|^{n-2}} \, dy \in L_w^{p_\mu, \mu}(\Omega)$$

with $\mu = \gamma + \nu - n + 2$ *and* $\dfrac{1}{p_\mu} = \dfrac{1}{2} - \dfrac{1}{n-\mu}$. *Further*

$$I_2(f u_{x_i}) = \int_\Omega \frac{f(y) u_{x_i}(y)}{|x-y|^{n-2}} \, dy \in L_w^{p_\nu, \nu}(\Omega)$$

with $\dfrac{1}{p_\nu} = \dfrac{1}{2} - \dfrac{1}{n-\nu}$.

Proof. We have

$$|I_2(f^2 u)(x)| \leq \int_\Omega \frac{f^2(y)|u(y)|}{|x-y|^{n-2}} \, dy \leq \Big(\int_\Omega \frac{f^2(y)}{|x-y|^{n-2}} \, dy\Big)^{\frac{1}{2}} \Big(\int_\Omega \frac{f^2(y) u^2(y)}{|x-y|^{n-2}} \, dy\Big)^{\frac{1}{2}}$$

$$= (I_2(f^2)(x))^{\frac{1}{2}} (I_2(f^2 u^2)(x))^{\frac{1}{2}} .$$

From Lemma 1 in [4] we have that $I_2(f^2)$ is a bounded function. Since $\mu < n - 2$, we have, as in the proof of Theorem 2 in [3]

$$(I_2(f^2 u^2)(x))^{\frac{1}{2}} \leq C(M(f^2 u^2)(x))^{\frac{n-\mu-2}{2(n-\mu)}}$$

where $M(f^2 u^2)$ is the Hardy-Littlewood maximal function of $f^2 u^2$.

Since by Lemma 1 $f^2 u^2 \in L^{1,\gamma+\nu-n+2}(\Omega)$, we have $(I_2(f^2 u^2))^{\frac{1}{2}} \in L^{p_\mu, \mu}(\Omega)$ (see e.g. Theorem 1 in [3]).

Similarly for $I_2(f u_{x_i})$.

Lemma 5. *If u is a weak solution of* (1.1) *let u_1 be the weak solution of $Lw = -(d_i u + f_i)_{x_i}$ then $u - u_1 \equiv u_2$ is a weak solution of $Lw = -(b_i u_{x_i} + cu)$ and there exists a constant C such that*

$$\int_{B_\rho \cap \Omega} |\nabla u_1|^2 \, dx \leq C\Big(\frac{1}{\rho^2} \int_{B_{2\rho} \cap \Omega} u_1^2 \, dx + \int_{B_{2\rho} \cap \Omega} d_i^2 u^2 \, dx + \int_{B_{2\rho} \cap \Omega} f_i^2 \, dx\Big) \tag{3.2}$$

and

$$\int_{B_\rho \cap \Omega} |\nabla u_2|^2 \, dx \leq C\Big(\frac{1}{\rho^2} \int_{B_{2\rho} \cap \Omega} u^2 \, dx + \int_{B_{2\rho} \cap \Omega} |\nabla u_1|^2 \, dx + \int_{B_{2\rho} \cap \Omega} b_i^2 u^2 \, dx + \int_{B_{2\rho} \cap \Omega} c u^2 \, dx\Big) \tag{3.3}$$

where $B_\rho(x)$ is any ball centered at x and $B_{2\rho}$ is the concentric ball of radius 2ρ.

The proof of this last lemma follows from standard arguments using cut-off functions η and test functions $v = u_1 \eta^2$ and $v = u\eta^2$ for (3.2) and (3.3) respectively.

4 - Regularity results

Given $f \in L^1(\Omega)$, $f_i \in L^2(\Omega)$ $i = 1, 2, , \ldots, n$, we recall that a function $u \in L^1(\Omega)$ is a *very weak solution* to the Dirichlet problem (see [9])

$$Lu = f + (f_i)_{x_i} \quad \text{in } \Omega \qquad u|_{\partial\Omega} = 0 \tag{4.1}$$

if for every $\phi \in C^0(\overline{\Omega}) \cap H_0^1(\Omega)$ such that $L\phi \in C^0(\overline{\Omega})$ we hawe

$$\int_\Omega u(x) L\phi(x) \, dx = \int_\Omega (f(x)\phi(x) - f_i(x)\phi_{x_i}(x)) \, dx .$$

If we suppose that f, $f_i^2 \in L^{1,\lambda}(\Omega)$ with $0 < \lambda < n$, the problem (4.1) has a unique solution u. Then we can suppose that $u = u_1 + u_2$ with u_1 being the very weak solution of the problem

$$Lu = (f_i)_{x_i} \quad \text{in } \Omega \qquad u|_{\partial\Omega} = 0 \tag{4.1'}$$

and u_2 the very weak solution of the problem

$$Lu = f \quad \text{in } \Omega \qquad u|_{\partial\Omega} = 0 . \tag{4.1''}$$

Theorem 1. *The solution u_1 of* (4.1′) *has the following regularity*

i. *If* $0 < \lambda < n - 2$, *then* $u_1 \in L_w^{p_\lambda, \lambda}(\Omega)$ *where* $\frac{1}{p_\lambda} = \frac{1}{2} - \frac{1}{n - \lambda}$ *and there exists a constant* C *such that*

$$t^{p_\lambda} |\{x \in \Omega \colon |u_1(x)| > t\} \cap B_\rho(x_0)| \leq C\rho^\lambda \|f_i\|_{2,\lambda}^{p_\lambda}$$

for any $t > 0$ *and any ball* $B_\rho(x_0)$ *centered at* $x_0 \in \Omega$

ii. *If* $\lambda = n - 2$, *then* $u_1 \in BMO_{\mathrm{loc}}(\Omega)$ *i.e.* $u_1 \in BMO(\Omega')$ *for every* $\Omega' \subset\subset \Omega$ *and there exists a constant* $C = C(n, \mu, d, \operatorname{diam}\Omega)$ *where* $d = \operatorname{dist}(\Omega', \partial\Omega)$ *such that*

$$\|u_1\|_{*, \Omega'} \leq C \|f_i^2\|_{1, n-2}^{\frac{1}{2}}$$

iii. *If* $n - 2 < \lambda < n$ *then* u_1 *is locally Hölder-condinuous.*

Proof. For the proof of **i** and **iii** see respectively [**6**] Th. 4.3 and [**10**].

Now we prove **ii**. Let $B = B_\rho(x_0)$ be a ball centered at $x_0 \in \Omega'$ with $0 < \rho < \frac{d}{10}$. Put $B^* = B_{4\rho}(x_0)$ and $B^{**} = B_{5\rho}(x)$ for every $x \in B$. Then we have $u_1 = \overline{u} + \overline{\overline{u}}$ where $\overline{u}$ and $\overline{\overline{u}}$ are very weak solutions respectively to

$$Lu = -(f_i \chi_{B^*})_{x_i} \qquad \text{in } \Omega \qquad u|_{\partial\Omega} = 0 \tag{4.2}$$

$$Lu = -(f_i(1 - \chi_{B^*}))_{x_i} \qquad \text{in } \Omega \qquad u|_{\partial\Omega} = 0\,. \tag{4.3}$$

For the solution of (4.2) and (4.3) we have the following representation formulas (see [**6**] Theorem 3.1)

$$\overline{u}(x) = -\int_{B^*} f_i(y) g_{y_i}(x, y)\, dy \qquad \overline{\overline{u}}(x) = -\int_{\Omega \setminus B^*} f_i(y) g_{y_i}(x, y)\, dy$$

where $g(x, y)$ is the Green function for the operator L.

Let
$$B^{**} = \bigcup_{k=1}^{\infty} \{y \in \Omega \colon \frac{5\rho}{2^k} \leq |y - x| < \frac{5\rho}{2^{k-1}}\} = \bigcup_{k=1}^{\infty} R_k$$

and
$$\widetilde{R}_k = \{y \in \Omega \colon \frac{5\rho}{2^{k+1}} \leq |y - x| < \frac{5\rho}{2^{k-2}}\}.$$

Using Caccioppoli's inequality in the annuli we have the following estimate with the Hardy-Littlewood maximal function

$$|\overline{u}(x)| \leqslant \int_{B^*} |g_{y_i}(x, y)| \, |f_i(y)| \, dy \leqslant \int_{B^{**}} |g_{y_i}(x, y)| \, |f_i(y)| \, dy$$

$$= \sum_{k=1}^{\infty} \int_{R_k} |g_{y_i}(x, y)| \, |f_i(y)| \, dy \leqslant \sum_{k=1}^{\infty} \Big(\int_{R_k} g_{y_i}^2(x, y) \, dy \Big)^{\frac{1}{2}} \Big(\int_{R_k} f_i^2(y) \, dy \Big)^{\frac{1}{2}}$$

$$\leqslant C \sum_{k=1}^{\infty} \frac{2^k}{5\rho} \Big(\int_{\tilde{R}_k} g^2(x, y) \, dy \Big)^{\frac{1}{2}} \Big(\Big(\frac{2^{k-1}}{5\rho} \Big)^n \int_{R_k} f_i^2(y) dy \Big)^{\frac{1}{2}} \Big(\frac{5\rho}{2^{k-1}} \Big)^{\frac{n}{2}}$$

$$\leqslant C\rho (M f_i^2)^{\frac{1}{2}} (x)$$

where we have estimated $g(x, y)$ by $|x - y|^{2-n}$ (see e.g. [9] p. 67). Since for $1 < q < 2$

$$\frac{1}{|B|} \int_B |\overline{u} - \overline{u}_B| \, dx \leqslant \frac{2}{|B|} \int_B |\overline{u}| \, dx \leqslant 2 \Big(\frac{1}{|B|} \int_B |\overline{u}|^q \, dx \Big)^{\frac{1}{q}}$$

$$\leqslant c_\rho \Big(\frac{1}{|B|} \int_B (M f_i^2)^{\frac{q}{2}} \, dx \Big)^{\frac{1}{q}}$$

we have to estimate the last integral.

Let $\psi = (M f_i^2)^{\frac{1}{2}}$, we have $\psi \in L_w^{2, n-2}(\Omega)$ and

$$t^2 \, |\{x \in \Omega : \psi(x) > t\} \cap B_\rho(x_0)| = t^2 \, |\{\psi > t\} \cap B| \leqslant C\rho^{n-2} \, \|f_i^2\|_{1, n-2} \qquad t > 0$$

(see [3] Theorem 1).

Then
$$\frac{1}{|B|} \int_B (M f_i^2)^{\frac{q}{2}} \, dx = \frac{Cq}{\rho^n} \int_0^{+\infty} t^{q-1} \, |\{\psi > t\} \cap B| \, dt$$

$$= Cq\rho^{-n} \Big(\int_0^{\varepsilon} t^{q-1} \, |\{\psi > t\} \cap B| \, dt + \int_{\varepsilon}^{+\infty} t^{q-1} \, |\{\psi > t\} \cap B| \, dt \Big)$$

$$\leqslant Cq\rho^{-n} \Big(|B| \int_0^{\varepsilon} t^{q-1} \, dt + C_1 \rho^{n-2} \, \|f_i^2\|_{1, n-2} \int_{\varepsilon}^{+\infty} t^{q-3} \, dt \Big)$$

$$\leqslant Cq \Big(\frac{\varepsilon^q}{q} + C_1 \rho^{-2} \frac{\varepsilon^{q-2}}{2-q} \, \|f_i^2\|_{1, n-2} \Big).$$

Let $\varepsilon = \rho^{-1} \|f_i^2\|_{1, n-2}^{\frac{1}{2}}$ we have $\Big(\frac{1}{|B|} \int_B (M f_i^2)^{\frac{1}{q}} \, dx \Big)^{\frac{1}{q}} \leqslant \frac{C}{\rho} \, \|f_i^2\|_{1, n-2}^{\frac{1}{2}}$.

For the function $\overline{\overline{u}}$ we have

$$\frac{1}{|B|}\int_B |\overline{\overline{u}}(x) - \overline{\overline{u}}(x_0)|\,dx = \frac{1}{|B|}\int_B \Big|\int_{\Omega\setminus B^*} (g_{y_i}(x, y) - g_{y_i}(x_0 y))f_i(y)\,dy\Big|\,dx$$

$$= \frac{1}{|B|}\int_B J(x)\,dx$$

and
$$J(x) \leqslant \int_{\Omega\setminus B^*} |(g(x, y) - g(x_0, y))_{y_i}|\,|f_i(y)|\,dy$$

$$\leqslant \int_{\Omega_d} |(g(x, y) - g(x_0, y))_{y_i}|\,|f_i(y)|\,dy + \int_{\Omega^d} |(g(x, y) - g(x_0, y))_{y_i}|\,|f_i(y)|\,dy$$

where $\Omega_d = \{y \in \Omega\colon 4\rho \leqslant |y - x_0| < d\}$ and $\Omega^d = \{y \in \Omega\colon |y - x_0| \geqslant d\}$.

Let
$$\Omega_{d,k} = \{y \in \Omega_d\colon 2^{k+1}\rho \leqslant |y - x_0| < 2^{k+2}\rho\}$$

$$\tilde{\Omega}_{d,k} = \{y \in \Omega_d\colon 2^{k}\rho \leqslant |y - x_0| < 2^{k+3}\rho\}.$$

We have

$$\int_{\Omega_d} |(g(x, y) - g(x_0, y))_{y_i}|\,|f_i(y)|\,dy$$

$$\leqslant \sum_{k=1}^{\infty}\Big(\int_{\Omega_{d,k}} |g(x, y) - g(x_0, y))_{y_i}|^2\,dy\Big)^{\frac12}\Big(\int_{\Omega_{d,k}} |f_i(y)|^2\,dy\Big)^{\frac12}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{2^k\rho}\Big(\int_{\tilde{\Omega}_{d,k}} |g(x,y)-g(x_0,y)|^2 dy\Big)^{\frac12} \frac{1}{(2^{k+2}\rho)^{n-2}}\Big(\int_{\Omega_{d,k}} |f_i(y)|^2\,dy\Big)^{\frac12}(2^{k+2}\rho)^{\frac{n-2}{2}}$$

$$\leqslant C\|f_i^2\|^{\frac12}_{1,n-2}\sum_{k=1}^{\infty}(2^k\rho)^{\frac{n-4}{2}}\Big(\int_{\tilde{\Omega}_{d,k}} |g(x, y) - g(x_0, y)|^2\,dy\Big)^{\frac12}.$$

To estimate the first integral we use the fact that $g(\cdot, y)$ is a weak solution of $Lu = 0$ in $B_{\frac{3}{4}|x_0 - y|}(x_0)$ and then we can apply De Giorgi-Nash's theorem and Harnack's inequality. Then we have

$$\Big(\int_{\tilde{\Omega}_{d,k}} |g(x, y) - g(x_0, y)|^2 \, dy\Big)^{\frac{1}{2}}$$

$$\leqslant C\Big(\int_{\tilde{\Omega}_{d,k}} \Big[\Big(\frac{1}{|B_{\frac{3}{4}|x_0 - y|}|} \int_{B_{|y - x_0|}(x_0)} g^2(x, y)\, dx\Big)\Big(\frac{2\rho}{|x_0 - y|}\Big)^{2\alpha}\Big] dy\Big)^{\frac{1}{2}}$$

$$\leqslant C\rho^{\alpha}\Big(\int_{\tilde{\Omega}_{d,k}} \frac{1}{|x_0 - y|^{2n - 4 + 2\alpha}}\, dy\Big)^{\frac{1}{2}}$$

$$\leqslant C\rho^{\alpha} \frac{1}{(2^k \rho)^{n - 2 + \alpha}} (2^{k+2}\rho)^{\frac{n}{2}} \leqslant C\rho^{\frac{4 - n}{2}} \Big(\frac{1}{2^k}\Big)^{\frac{2\alpha - 4 + n}{2}}$$

then

$$\int_{\Omega_d} |g(x, y) - g(x_0, y)|_{y_i} |f_i(y)| \, dy \leqslant C\|f_i\|_{2, n-2} \sum_{k=1}^{\infty} \Big(\frac{1}{2^k}\Big)^{\alpha} = C_1 \|f_i\|_{2, n-2} .$$

Similarly we can define

$$\Omega^{d,k} = \{y \in \Omega \colon 2^{k-1} d \leqslant |y - x_0| < 2^k d\}$$
$$\tilde{\Omega}^{d,k} = \{y \in \Omega \colon 2^{k-2} d \leqslant |y - x_0| < 2^{k+1} d\}$$

and we have $$\int_{\Omega^d} |(g(x, y) - g(x_0, y))_{y_i}| \, |f_i(y)| \, dy$$

$$\leqslant \sum_{k=1}^{\infty} \Big(\int_{\Omega^{d,k}} |(g(x, y) - g(x_0, y))_{y_i}|^2 dy\Big)^{\frac{1}{2}} \Big(\int_{\Omega} |f_i(y)|^2 \, dy\Big)^{\frac{1}{2}}$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{2^{k-2} d}\Big(\int_{\tilde{\Omega}^{d,k}} |(g(x, y) - g(x_0, y)|^2 dy\Big)^{\frac{1}{2}} \Big(\int_{\Omega} |f_i(y)|^2 \, dy\Big)^{\frac{1}{2}}$$

$$\leqslant \frac{C}{d^{\frac{n}{2} - 1}} \sum_{k=1}^{\infty} \Big(\frac{1}{2}\Big)^{\frac{n}{2} - 1} \Big(\int_{\Omega} |f_i(y)|^2 \, dy\Big)^{\frac{1}{2}} \leqslant C d^{1 - \frac{n}{2}} \|f_i\|_{2, n-2} .$$

We get the last inequality observing for $y \in \widetilde{\Omega}_{d,k}$, that

$$|g(x, y) - g(x_0, y)| \leqslant g(x, y) + g(x_0, y)$$

$$\leqslant C\Big(\frac{1}{|x, y|^{n-2}} + \frac{1}{|x_0 - y|^{n-2}}\Big) \leqslant Cd^{n-2}\Big(\frac{1}{2^k}\Big)^{n-2}.$$

Now, if $\rho \geqslant \dfrac{d}{10}$ for $\lambda \in [0, n-2[$, from [6] p. 5 we have

$$|u_1(x)| \leqslant C\|f_i\|_{2,\lambda}^{1-\frac{2}{p_\lambda}} (M(f_i^2))^{\frac{1}{p_\lambda}}(x)$$

where $M(f_i^2)$ is the Hardy-Littlewood maximal function of f_i^2.

For $\varepsilon = (\rho^{-2}\|f_i^2\|_{1,n-2})^{\frac{1}{p_\lambda}}$ we have

$$\frac{1}{|B_\rho|}\int_{B_\rho}|u_1 - u_{1B_\rho}|\,dx \leqslant \frac{2}{|B_\rho|}\int_{B_\rho}|u_1|\,dx \leqslant C\|f_i^2\|_{2,\lambda}^{1-\frac{2}{p_\lambda}}\,\frac{2}{|B_\rho|}\int_{B_\rho} M(f_i^2)^{\frac{1}{p_\lambda}}\,dx$$

$$\leqslant \frac{C}{\rho^n}\|f_i^2\|_{2,\lambda}^{1-\frac{2}{p_\lambda}}\Big(\int_0^\varepsilon |\{(Mf_i^2)^{\frac{1}{p_\lambda}} > t\} \cap B_\rho|\,dt + \int_\varepsilon^{+\infty} |\{(Mf_i^2)^{\frac{1}{p_\lambda}} > t\} \cap B_\rho|\,dt\Big)$$

$$\leqslant \frac{C}{\rho^n}\|f_i\|_{2,\lambda}^{1-\frac{2}{p_\lambda}}\Big(\int_0^\varepsilon |B_\rho|\,dt + \rho^{n-2}\|f_i^2\|_{1,n-2}\int_\varepsilon^{+\infty} t^{-p_\lambda}\,dt\Big) \leqslant Cd^{\frac{-2}{p_\lambda}}\|f_i\|_{2,n-2}.$$

Remark 1. The solution u_1 of (4.1′) also belongs to $L^{2\lambda+2}(\Omega)$ if $0<\lambda<n-2$.

In fact, if we let $\varepsilon > 0$, $\Omega_t = \{x \in \Omega: |u_1(x)| > t\}$ and $B_\rho(x)$ any ball centered at $x \in \Omega$ then we have

$$\int_{B_\rho \cap \Omega} u_1^2\,dx = 2\int_0^{+\infty} t|(\Omega_t \cap B_\rho)|\,dt$$

$$= 2\int_0^\varepsilon t|(\Omega_t \cap B_\rho)|\,dt + 2\int_\varepsilon^{+\infty} t|(\Omega_t \cap B_\rho)|\,dt \leqslant C(\varepsilon^2\rho^n + \rho^\lambda\varepsilon^{2-p_\lambda}\|f_i\|_{2,\lambda}^{p_\lambda})$$

and choosing $\varepsilon = \rho^{\frac{\lambda-n+2}{2}}\|f_i\|_{2,\lambda}$ the result follows.

Theorem 2. *The solution u_2 of (4.1″) has the following regularity*

i. *If $0 < \lambda < n-2$ then $u_2 \in L_w^{p_\lambda,\lambda}(\Omega)$ where $\dfrac{1}{2p_\lambda} = \dfrac{1}{2} - \dfrac{1}{n-\lambda}$ and there exists a constant C such that*

$$t^{p_\lambda}|\{x \in \Omega: |u_1(x)| > t\} \cap B_\rho(x_0)| \leqslant C\rho^\lambda\|f\|_{1,\lambda}^{p_\lambda}.$$

ii. *If $\lambda = n-2$ then $u_2 \in BMO_{\mathrm{loc}}(\Omega)$; i.e. $u_2 \in BMO(\Omega')$ for every $\Omega' \subset\subset \Omega$ and there exists a constant $C = C(n, \mu, d, \operatorname{diam}\Omega)$ where $d = \operatorname{dist}(\Omega', \partial\Omega)$ such that $\|u_2\|_{*,\Omega'} \leqslant C\|f\|_{1,n-2}$.*

iii. *If $n-2 < \lambda < n$ then u_2 is locally Hölder-continuous.*

Proof. For the proof of **i** and **iii** see [5]; **ii** is partially studied in [5]. In particular in [5] Theorem 2.2 it is shown that if $x_0 \in \Omega' \subset\subset \Omega$ and $B_\rho(x_0)$ is a ball centered at x_0 with $0 < \rho < \frac{d}{2}$, then

$$\frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} |u_2(x) - u_{2B_\rho(x_0)}| \, dx \leq C\|f\|_{1, n-2}. \tag{4.4}$$

We will now prove (4.4) for $\frac{d}{2} \leq \rho < \operatorname{diam} \Omega$.

Since u_2 is the very weak solution of (4,1″), we have the reprensentation formula (see e.g. [5])

$$u_2(x) = \int_\Omega g(x, y) f(y) \, dy$$

where $g(x, y)$ is the Green's function of L.

As in Theorem 1 we can estimate $g(x, y)$ by $|x - y|^{2-n}$. We have

$$|u_2(x)| \leq C \int_\Omega \frac{|f(y)|}{|x - y|^{n-2}} \, dy = C I_2(f)(x).$$

From a result about Riesz's potentials (see [3] Theorem 2), since $f \in L^{1, n-2}(\Omega)$ for $0 < \lambda < n - 2$ we have $I_2(f) \in L_w^{q, \lambda}(\Omega)$ where $\frac{1}{q} = 1 - \frac{2}{n - \lambda}$ and

$$t^q |\{x \in \Omega \colon |u_2(x)| > t\} \cap B_\rho(x_0)| = t^q |(|u_2| > t) \cap B_\rho| \leq C\rho^\lambda \|f\|_{1,\lambda}^q$$

for any ball $B_\rho(x_0)$ centered at $x_0 \in \Omega'$ and any $t > 0$.

Then from

$$\frac{1}{|B|} \int_{B_\rho} |u_2 - u_{2B}| \, dx \leq \frac{C}{\rho^n} \int_{B_\rho} |u_2| \, dx \leq C d^{-n} \int_{B_\rho} |u_2| dx = C d^{-n} \int_0^\infty |(|u_2| > t) \cap B_\rho| \, dt$$

$$= C d^{-n} \Big(\int_0^\varepsilon |(|u_2| > t) \cap B_\rho| \, dt + \int_\varepsilon^{+\infty} |(|u_2| > t) \cap B_\rho| \, dt \Big)$$

$$\leq C d^{-n} \Big(|B_\rho| \varepsilon + C\rho^\lambda \|f\|_{1,\lambda}^q \int_\varepsilon^{+\infty} t^{-q} dt \Big) \leq C d^{-n} \Big(\rho^n \varepsilon + \rho^\lambda \|f\|_{1,\lambda}^q \frac{\varepsilon^{1-q}}{q - 1} \Big)$$

choosing $\varepsilon = \rho^{\frac{\lambda - n}{q}} \|f\|_{1,\lambda}$ we drow the conclusion.

Remark 2. The solution u_2 of (4.1″) also belongs to $L^{1,\lambda+2}(\Omega)$ if $0 < \lambda < n - 2$. The proof is similar to that of Remark 1.

(Main) Theorem 3. *Let u be a weak solution of* (1.1) *and* $f_i \in L^{2,\lambda}(\Omega)$. *Then*

i. $u \in L_w^{p_\lambda,\lambda}(\Omega)$ *where* $\dfrac{1}{p_\lambda} = \dfrac{1}{2} - \dfrac{1}{n-\lambda}$ *if* $0 < \lambda < n-2$

ii. $u \in BMO_{\mathrm{loc}}(\Omega)$ *if* $\lambda = n-2$

iii. *u is locally Hölder-continous if* $n-2 < \lambda < n$.

Proof. To prove **i** we consider the weak solution u of the Dirichlet problem for the equation (1.1) as $u_1 + u_2$, where u_1 is the weak of the Dirichlet problem for the equation $Lw = -(d_i u + f_i)_{x_i}$ and u_2 is the weak solution of the Dirichlet problem for the equation $Lw = -(b_i u_{x_i} + cu)$.

We will note that u_1 and u_2 are moreover the very weak solutions of the same problems. We now use an iteration process. We will show the first step of this process.

We can suppose I_2 to be satisfied with $\gamma = n-2+\dfrac{\lambda}{h}$, with $h \in N$ such that $\dfrac{1}{h} < \dfrac{n-2}{\lambda} - 1$. With such a choice we get $n-2<\gamma<\min(2n-4-\lambda, n-2+\lambda)$.

As u is a weak solution of (1.1) we have $u \in L^{2,2}(\Omega)$, $|\nabla u| \in L^{2,0}(\Omega)$ and, from Lemma 1, $d_i u \in L^{2,\gamma-n+2}(\Omega)$. From Theorem 1, $u_1 \in L^{p_{\gamma_0},\gamma_0}(\Omega)$, where $\gamma_0 = \gamma - n + 2$ and $\dfrac{1}{p_{\gamma_0}} = \dfrac{1}{2} - \dfrac{1}{n-\gamma_0}$. From Remark 1, $u_1 \in L^{2,\gamma_0+2}(\Omega)$ and from (3.2) $|\nabla u_1| \in L^{2,\gamma_0}(\Omega)$.

From Lemma 2 $b_i u_{x_i} + cu \in L^{1,\frac{\gamma}{2}}(\Omega)$, and from Theorem 2. $u_2 \in L_w^{p_{\frac{\gamma}{2}},\frac{\gamma}{2}}(\Omega)$ where $\dfrac{1}{p_{\frac{\gamma}{2}}} = 1 - \dfrac{2}{n-\frac{\gamma}{2}}$. It follows from Remark 2 that $u_2 \in L^{1,\frac{\gamma}{2}+2}(\Omega)$.

Then applying Lemma 3 $u_2 \in L^{2,\sigma_0+2}(\Omega)$, where $\sigma_0 = \dfrac{2\gamma_0}{n+2}$.

From (3.3) we have

$$\int_{B_\rho \cap \Omega} |\nabla u_2|^2 \, dx \leq C\Big(\frac{1}{\rho^{\gamma_0+2}} \int_{B_{2\rho}\cap\Omega} u_1^2 \, dx + \frac{1}{\rho^{\sigma_0+2}} \int_{B_{2\rho}\cap\Omega} u_2^2 \, dx$$

$$+ \frac{1}{\rho^{\gamma_0}} \int_{B_{2\rho}\cap\Omega} |\nabla u_1|^2 \, dx + \frac{1}{\rho^{\gamma_0}} \int_{B_{2\rho}\cap\Omega} b_i^2 u^2 \, dx + \frac{1}{\rho^{\gamma_0}} \int_{B_{2\rho}\cap\Omega} cu^2 \, dx\Big)\rho^{\sigma_0}.$$

We now observe that $|\nabla u_2| \in L^{2,\sigma_0}(\Omega)$, because $u_1 \in L^{2,\gamma_0+2}(\Omega)$, $b_i u$ and $|c|^{\frac{1}{2}} u$ belong to $L^{2,\gamma^0}(\Omega)$ and $|\nabla u_1| \in L^{2,\gamma_0}(\Omega)$.

So by Lemma 2 $b_i u_{x_i} + cu \in L^{1, \frac{\gamma}{2} + \frac{\sigma_0}{2}}(\Omega)$. Then, from Theorem 2, $u_2 \in L_w^{p_\alpha, \alpha}(\Omega)$, $\alpha = \frac{\gamma}{2} + \frac{\sigma_0}{2}$, $\frac{1}{p_\alpha} = 1 - \frac{2}{n-\alpha}$ and from Remark 2 $u_2 \in L^{1,\alpha+2}(\Omega)$.

From the representation formula we have

$$|u_2(x)| = |\int_\Omega (b_i(y) u_{x_i}(y) + c(y) u(y)) g(x, y)\, dy|$$

$$\leq C(\int_\Omega \frac{|b_i(y) u_{x_i}(y)|}{|x-y|^{n-2}}\, dy + \int_\Omega \frac{|c(y) u(y)|}{|x-y|^{n-2}}\, dy) = C(I_2(b_i u_{x_i})(x) + I_2(cu)(x)).$$

From Lemma 4 $I_2(cu) \in L_w^{p_\mu \mu}(\Omega)$, where $\mu = \gamma_0 + \sigma_0$, $\frac{1}{p_\mu} = \frac{1}{2} - \frac{1}{n-\mu}$ and $I_2(b_i u_{x_i}) \in L_w^{p_{\sigma_0}, \sigma_0}(\Omega)$, where $\frac{1}{p_{\sigma_0}} = \frac{1}{2} - \frac{1}{n-\sigma_0}$. Then $u_2 \in L_w^{p_{\sigma_0}, \sigma_0}(\Omega)$ and, with a proof similar to Remark 1, $u_2 \in L^{2^*, \frac{n\sigma_0}{n-2}}(\Omega)$.

Then applying Lemma 3 we obtain $u_2 \in L^{2, 2\sigma_0 + 2}(\Omega)$ and from (3.3) $|\nabla u_2| \in L^{2, 2\sigma_0}(\Omega)$.

Iterating this process we have u_2 (and then $u) \in L^{2, \gamma_0 + 2}(\Omega)$ and $|\nabla u| \in L^{2, \gamma_0}(\Omega)$.

Now since $u \in L^{2, \gamma_0 + 2}(\Omega)$ and $|\nabla u| \in L^{2, \gamma_0}(\Omega)$ it follows from Lemma 1 $d_i u \in L^{2, 2\gamma_0}(\Omega)$.

We note that $2\gamma_0 \leq \lambda$. As above we obtain $u_1 \in L_w^{p_{2\gamma_0}, 2\gamma_0}(\Omega)$ where $\frac{1}{p_{2\gamma_0}} = \frac{1}{2} - \frac{1}{n - 2\gamma_0}$. As before, we have by iteration u_2 (and then u) $\in L^{2, 2\gamma_0 + 2}(\Omega)$. After h steps we have the statement.

To prove **ii** and **iii** we also observe that $u \in L^{p_\mu, \mu}(\Omega)$ $\forall \mu \in [0, n-2[$, $\frac{1}{p_\mu} = \frac{1}{2} - \frac{1}{n-\mu}$. Then $d_i u \in L^{2, \gamma_0 + \mu}(\Omega)$ because $u \in L^{2, \mu + 2}(\Omega)$ and $|\nabla u| \in L^{2, \mu}(\Omega)$. If μ is chosen such that $2n - 4 - \gamma < \mu < n - 2$ we have $d_i u + f_i \in L^{2, \tau}(\Omega)$ with $n - 2 \leq \tau = \min(\lambda, \gamma_0 + \mu)$. Then from Theorem 1 $u_1 \in BMO_{\mathrm{loc}}(\Omega)$ if $\lambda = n - 2$ and u_1 is locally Hölder-continous if $\lambda > n - 2$.

Moreover, because of $b_i u_{x_i} + cu \in L^{1, \frac{\gamma}{2} + \frac{\mu}{2}}(\Omega)$, if we choose μ as above, we have $\frac{\gamma}{2} + \frac{\mu}{2} \geq n - 2$ and we obtain the desired result for u_2.

Remark 3. If $b_i = c = 0$ the proof of the last theorem gives a simplified version of that of Theorem 4.3. in [6]

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Sommario

Si studia la regolarità delle soluzioni dell'equazione (1.1) *nell'ipotesi che i coefficienti e i termini noti appartengano a convenienti spazi di Morrey.*

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