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Almost contact homogeneous manifolds (**)

1 - Introduction

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. More precisely, M is a $\mathcal{C}^\infty$ differentiable manifold of dimension $2n+1$, φ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on M such that

$$(1.1) \qquad \varphi^2 = -I + \eta \otimes \xi \qquad \eta(\xi) = 1$$

where I denotes the identity transformation. The Riemannian metric g is compatible with φ, i.e.

$$(1.2) \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y)$$

for all vector fields X, Y. For more details we refer, for example, to [2].

An almost contact metric manifold is said to be an *almost contact homogeneous manifold* if a Lie group G of isometries acts transitively and effectively on M and φ is invariant under the action of the group G. Applying a result of Kiričenko [9] on homogeneous Riemannian spaces with invariant tensor structures, we have the following infinitesimal characterization of almost contact homogeneous manifolds (see also [4]).

Theorem 1. *A connected, simply connected and complete almost contact manifold $(M, \varphi, \xi, \eta, g)$ is homogeneous if and only if there exists a tensor field*

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T of type $(0, 3)$ *such that*

$$\tilde{\nabla} g = 0 \qquad \tilde{\nabla} R = 0 \qquad \tilde{\nabla} T = 0 \qquad \tilde{\nabla}_X \varphi = 0 \tag{1.3}$$

for every vector field X on M. Here $\tilde{\nabla} = \nabla - \tilde{T}$, ∇ *is the Levi-Civita connection of M, R is the Riemannian curvature tensor and* $\tilde{T}$ *is the tensor field T in the form* $(1, 2)$.

Such a tensor T will be called an *almost contact homogeneous structure*. The compact Lie group $U(n) \times 1$ acts in a natural way on the vector space of tensors with the same symmetries of the almost contact homogeneous structures. In [**6**], D. Chinea, C. Gonzalez and E. Padron decomposed such a vector space into eighteen invariant and irreducible subspaces. Due to the presence of so-called *isotypic components* (i.e. isomorphic subspaces), many different decompositions are possible. Of course, a particular choice must be motivated by geometrical reasons: in [8] a new decomposition, which fits nicely with the classes of Riemannian homogeneous structures of F. Tricerri and L. Vanhecke [**11**], is given.

The main purpose of this note is to use this decomposition in order to obtain some geometrical results about almost contact homogeneous manifolds. More precisely, in Section **2**, after a short review of the decompositions described in [8] and [**6**], we compare them with the classes of Riemannian and almost contact homogeneous structures found in [**11**] and in [3], respectively. In this way, we get the complete classification of naturally reductive almost contact manifolds divided into 2^6 classes. Among other results, it follows that a naturally reductive almost cosymplectic manifold is cosymplectic; a nearly k-cosymplectic manifold of type $\mathcal{T}_2$ is cosymplectic; a nearly trans-Sasakian manifold of type $\mathcal{T}_2$ is trans-Sasakian.

In the last Section we discuss some examples which show that the inclusions between some classes are strict.

2 - Geometrical results

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold of dimension $2n + 1$, $n \geq 3$. If ϕ denotes the fundamental 2-form of M defined by

$$\phi(X, Y) = g(\varphi X, Y) \tag{2.1}$$

for all $X, Y \in \mathfrak{X}(M)$ (Lie algebra of $\mathcal{C}^\infty$ vector fields), it is well known that the covariant derivative $\nabla\phi$ verifies

$$
\begin{aligned}
(2.2)\qquad (\nabla_X\phi)(Y, Z) &= -(\nabla_X\phi)(Z, Y)\\
&= -(\nabla_X\phi)(\varphi Y, \varphi Z) + \eta(Y)(\nabla_X\phi)(\xi, Z) + \eta(Z)(\nabla_X\phi)(Y, \xi)
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In [3] Chinea and Gonzalez decomposed the vector space $\mathcal{C}$ of all tensors of type $(0, 3)$, enjoying the same symmetries as $\nabla_x\phi$ in (2.2), into twelve irreducible subspaces $\mathcal{C}_i$, invariant under the action of the group $U(n)$ (regarded as a subgroup of $U(n) \times 1$). In this way, they characterized some classes of almost contact manifods in terms of invariant subspaces of $\mathcal{C}$.

We give here a new description of the classes $\mathcal{C}_i$, which will be useful later. For every $p \in M$, $(T_pM, \varphi_p, \xi_p, \eta_p, g_p)$ is an almost contact vector space. Having fixed an adapted orthonormal basis $(e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi)$ of T_pM, there is a standard representation of $U(n) \times 1$ on $T_pM = V$. As well known we have $V = \bar{V} \oplus \boldsymbol{R}\xi$ where $\bar{V} = \{X \in V \mid g_p(X, \xi) = 0\}$. Moreover (φ, g_p) defines on $\bar{V}$ an almost Hermitian structure and $\bar{V}$ behaves like the tangent space to an almost complex manifold. It follows from Theorem 3.1 of [8] that

$$
(2.3)\qquad \mathcal{C} \simeq 2[\![\lambda^{2,0}]\!] \oplus [\![\sigma^{2,0}]\!] \oplus 2[\lambda_0^{1,1}] \oplus 2\boldsymbol{R} \oplus 2[\![\lambda^{1,0}]\!] \oplus [\![A]\!] \oplus [\![\lambda^{3,0}]\!] \oplus [\![\lambda_0^{2,1}]\!].
$$

We refer to [8] for more details and to [7] and [10] for the notation adopted. An inspection of their dimensions as a function of n shows that there are the following isomorphisms with the classes of [3]:

$$
(2.4)\qquad
\begin{array}{lll}
\mathcal{C}_1 \simeq [\![\lambda^{3,0}]\!] & \mathcal{C}_2 \simeq [\![A]\!] & \mathcal{C}_3 \simeq [\![\lambda_0^{2,1}]\!]\\
\mathcal{C}_4 \simeq \mathcal{C}_{12} \simeq [\![\lambda^{1,0}]\!] & \mathcal{C}_5 \simeq \mathcal{C}_6 \simeq \boldsymbol{R} & \mathcal{C}_7 \simeq \mathcal{C}_8 \simeq [\lambda_0^{1,1}]\\
\mathcal{C}_9 \simeq [\![\sigma^{2,0}]\!] & \mathcal{C}_{10} \simeq \mathcal{C}_{11} \simeq [\![\lambda^{2,0}]\!]. &
\end{array}
$$

As it is shown in [9], a connected m-dimensional homogeneous Riemannian manifold (M, g) admits a *Riemannian homogeneous structure* T, i.e., a tensor fields which satisfies the first three conditions of Theorem 1. In [11] Tricerri and Vanhecke considered the vector space $\mathcal{T}$ of all tensors with the three mentioned simmetries of T and decomposed it into three irreducible components $\mathcal{T}_i$, invariant under the action of the orthogonal group $O(m)$. In this way, they obtained a complete classification of the Riemannian homogeneous structures divided into eight classes. For example, the class $\mathcal{T}_3$ characterizes the naturally reductive homogeneous manifolds. Moreover, when M is an almost contact homogeneous manifold, there is an induced action of $U(n) \times 1 \subset O(2n+1)$ on $\mathcal{T}$, which further decomposes into eighteen invariant subspaces $\mathcal{A}_i$ ([8], Section 2). The connection between the two decompositions is the following

$$\mathcal{T}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \qquad \mathcal{T}_2 = \mathcal{C}_9 \oplus \ldots \oplus \mathcal{C}_{18} \qquad \mathcal{T}_3(V) = \mathcal{C}_3 \oplus \ldots \oplus \mathcal{C}_8 \tag{2.5}$$

where

$$\begin{array}{lll} \mathcal{C}_1 \simeq \mathcal{C}_5 \simeq \mathcal{C}_{11} \simeq \mathcal{C}_{18} \simeq [\![\lambda^{1,0}]\!] & \mathcal{C}_2 \simeq \mathcal{C}_8 \simeq \mathcal{C}_9 \simeq \boldsymbol{R} & \mathcal{C}_3 \simeq [\![\lambda^{3,0}]\!] \\ \mathcal{C}_4 \simeq \mathcal{C}_{13} \simeq [\![\lambda_0^{2,1}]\!] & \mathcal{C}_6 \simeq \mathcal{C}_{17} \simeq [\![\lambda^{2,0}]\!] & \mathcal{C}_7 \simeq \mathcal{C}_{15} \simeq \mathcal{C}_{16} \simeq [\lambda_0^{1,1}] \\ \mathcal{C}_{10} \simeq [\![B]\!] & \mathcal{C}_{12} \simeq [\![A]\!] & \mathcal{C}_{14} \simeq [\![\sigma^{2,0}]\!]\,. \end{array} \tag{2.6}$$

Thus, the connected, simply connected, almost contact naturally reductive manifolds of dimension $2n+1$, $n > 2$, are classified into 2^6 classes given by all the invariant subspaces of the decomposition of $\mathcal{T}_3$.

Using the methods of [8], Section 2, one can get an explicit description of the various subspaces $\mathcal{C}_i$. Here, we list only the classes needed for the examples of Section 3.

$$\mathcal{C}_2 = \{T \in \mathcal{T} \mid T_{XYZ} = \langle X, Y\rangle \psi(Z) - \langle X, Z\rangle \psi(Y),\ \psi \in V^*,\ \psi \circ \varphi = 0\}$$

$$\mathcal{C}_7 = \{T \in \mathcal{T} \mid T_{XYZ} = \eta(X)\, T_{\xi\varphi Y\varphi Z} + \eta(Y)\, T_{\varphi X\xi\varphi Z} + \eta(Z)\, T_{\varphi X\varphi Y\xi},\ \bar{c}_{23}(T)(\xi) = 0\}$$

$$\mathcal{C}_8 = \{T \in \mathcal{T} \mid T_{XYZ} = \frac{1}{2n}\,(\eta(X)\langle \varphi Y, Z\rangle + \eta(Y)\langle X, \varphi Z\rangle + \eta(Z)\langle \varphi X, Y\rangle),\ \bar{c}_{23}(T)(\xi) = 0\}$$

$$\mathcal{C}_{14} = \{T \in \mathcal{T} \mid T_{XYZ} = \eta(Y)\, T_{\varphi X\xi\varphi Z} + \eta(Z)\, T_{\varphi X\varphi Y\xi} = \eta(Y)\, T_{X\xi Z} + \eta(Z)\, T_{XY\xi},$$
$$\underset{X,Y,Z}{\mathfrak{S}} T_{XYZ} = 0, \quad \bar{c}_{12}(T)(\xi) = 0, \quad c_{12}(T) = 0\}$$

for all $X, Y, Z \in V$. The traces $\bar{c}_{12}$ such that $\bar{c}_{23}$ are defined by

$$\bar{c}_{12}(T)(X) = \sum_{i=1}^{2n+1} T_{e_i\varphi e_i X} \qquad \bar{c}_{23}(T)(X) = \sum_{i=1}^{2n} T_{Xe_i\varphi e_i} \tag{2.7}$$

where $(e_1, \ldots, e_{2n+1})$ is an orthonormal basis of V and $(e_1, \ldots, e_{2n})$ is an orthonormal basis of $\bar{V}$.

The last condition of Theorem 1 can be written as

$$(\nabla_X \phi)(Y, Z) = -T_{X\varphi YZ} - T_{XY\varphi Z} \qquad X, Y, Z \in \mathfrak{X}(M) \tag{2.8}$$

and this leads one to consider the homomorphism $h: \mathcal{T} \mapsto \mathcal{C}$ of $U(n) \times 1$-modules defined by

$$h(T)_{XYZ} = T_{X\varphi YZ} + T_{XY\varphi Z}\,, \tag{2.9}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Let $\mathcal{A}$ be a subspace of $\mathcal{T}$, invariant with respect to the representation of $U(n) \times 1$. If T is a homogeneous almost contact structure on M and $T \in \mathcal{A}$ for any $p \in M$, we say that M *is of type* $\mathcal{A}$ and write $M \in \mathcal{A}$.

Now we give some geometrical results. For the definitions of the classes of almost contact manifolds which appear in the theorems, we refer to [3].

Theorem 2.

i. *If $M \in \mathcal{A}_2$, then M is a trans-Sasakian manifold of type $\mathcal{T}_1$.*

ii. *If $M \in \mathcal{A}_3$, then M is a nearly k-cosymplectic naturally reductive manifold*

iii. *If $M \in \mathcal{A}_8$, then M is a trans-Sasakian naturally reductive manifold.*

iv. *If $M \in \mathcal{A}_4 \oplus \mathcal{A}_7$, then M is a semi-cosymplectic and normal naturally reductive manifold.*

v. *If $M \in \mathcal{A}_9$, then M is a trans-Sasakian manifold of type $\mathcal{T}_2$.*

vi. *If $M \in \mathcal{A}_{10}$, then M is a cosymplectic manifold of type $\mathcal{T}_2$.*

vii. *If $M \in \mathcal{A}_{13} \oplus \mathcal{A}_{15} \oplus \mathcal{A}_{16}$, then M is a homogeneous semi-cosymplectic normal manifold of type $\mathcal{T}_2$.*

Proof. Since the proofs of the above results are very similar, we give the details only for **i**. If $T \in \mathcal{A}_2 \simeq \boldsymbol{R}$, then $h(T) \in 2\boldsymbol{R} \simeq \mathcal{C}_5 \oplus \mathcal{C}_6$ and, from the classification of almost contact manifolds given in [3], we see that $\mathcal{C}_5 \oplus \mathcal{C}_6$ is the class of the trans-Sasakian manifolds.

Theorem 3. *A naturally reductive almost cosymplectic manifold is cosymplectic.*

Proof If M is naturally reductive, then $M \in \mathcal{T}_3$. From (2.5) and (2.6) it follows that $T \in [\![\lambda^{3,0}]\!] \oplus [\![\lambda_0^{2,1}]\!] \oplus [\![\lambda^{1,0}]\!] \oplus [\![\lambda^{2,0}]\!] \oplus [\lambda_0^{1,1}] \oplus \boldsymbol{R}$. Since M is an almost cosymplectic manifold, in [3] it is shown that $h(T) = \nabla\phi \in [\![A]\!] \oplus [\![\sigma^{2,0}]\!]$, then $\nabla\phi = 0$ and M is cosymplectic.

In a similar way, we get

Theorem 4. *A nearly k-cosymplectic homogeneous manifold of type $\mathcal{T}_2$ is cosymplectic. A nearly trans-Sasakian homogeneous manifold of type $\mathcal{T}_2$ is a trans-Sasakian manifold.*

Theorem 5. *An almost contact homogeneous manifold is cosymplectic if and only if it belongs to the subspace*

$$\mathcal{T}_+ = \{T \in \mathcal{T} \mid T_{XYZ} = T_{X\varphi Y\varphi Z} + \eta(Y)\, T_{X\xi Z} + \eta(Z)\, T_{XY\xi}, \quad X, Y, Z \in V\}. \tag{2.10}$$

Proof. Since $\mathcal{T}_+ = \operatorname{Ker} h$, from (2.8) it follows that $M \in \mathcal{T}_+$ if and only if $\nabla\varphi = 0$.

As stated in the Introduction, Chinea, Gonzales and Padron gave another decomposition of $\mathcal{T}$ under the action of $U(n) \times 1$ (see [6], Prop. 2). They first noted that $\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$ where

$$\mathcal{T}_- = \{T \in \mathcal{T} \mid T_{XYZ} = -T_{X\varphi Y\varphi Z} + \eta(Y)\, T_{X\xi Z} + \eta(Z)\, T_{XY\xi}\} \tag{2.11}$$

and then proved that

$$\mathcal{T}_+ = \mathcal{H}_{13} \oplus \ldots \oplus \mathcal{H}_{18} \qquad \mathcal{T}_- = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_{12} \tag{2.12}$$

each $\mathcal{H}_i$ being an invariant and irreducible subspace. From (2.11) it is easy to see that $\mathcal{T}_- \simeq \mathcal{C}$. This fact and Theorem 3.1 of [8] imply that there are the following isomorphisms

$$\begin{array}{ll} \mathcal{H}_i \simeq \mathcal{C}_i & i = 1, 2, 3, 4, 9, 10, 11, 12 \\ \mathcal{H}_5 \simeq \mathcal{C}_6 & \mathcal{H}_6 \simeq \mathcal{C}_5 \quad \mathcal{H}_7 \simeq \mathcal{C}_8 \quad \mathcal{H}_8 \simeq \mathcal{C}_7 . \end{array} \tag{2.13}$$

This allows us to find geometrical meaning of some of the classes H_i.

Theorem 6.

i. *If $M \in \mathcal{H}_1$, then M is a nearly k-cosymplectic homogeneous manifold.*

ii. *If $M \in \mathcal{H}_2 \oplus \mathcal{H}_9$, then M is an almost cosymplectic homogeneous manifold.*

iii. *If $M \in \mathcal{H}_5 \oplus \mathcal{H}_6$, then M is a trans-Sasakian homogeneous manifold.*

iv. *If $M \in \mathcal{H}_5 \oplus \mathcal{H}_8$, then M is a quasi-Sasakian homogeneous manifold.*

v. *If* $M \in \mathcal{K}_3 \oplus \mathcal{K}_7 \oplus \mathcal{K}_8$, *then M is a semi-cosymplectic normal homogeneous manifold.*

vi. *If* $M \in \mathcal{K}_1 \oplus \mathcal{K}_5 \oplus \mathcal{K}_6$, *then M is a nearly trans-Sasakian homogeneous manifold.*

vii. *If* $M \in \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_9 \oplus \mathcal{K}_{10}$, *then M is a quasi-k-cosymplectic homogeneous manifold.*

viii. *If* $M \in \mathcal{K}_3 \oplus \ldots \oplus \mathcal{K}_8$, *then M is a normal homogeneous manifold.*

ix. *If* $M \in \mathcal{K}_1 \oplus \ldots \oplus \mathcal{K}_{10}$, *then M is an almost k-contact homogeneous manifold.*

x. *If* $M \in \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \mathcal{K}_7 \oplus \ldots \oplus \mathcal{K}_{12}$, *then M is an semi-cosymplectic homogeneous manifold.*

Proof. In all cases we use the following remark: if $T \in \mathcal{K}_i \subseteq \mathcal{T}_-$, then $\nabla\phi \in \mathcal{C}_j$, with i and j such that $\mathcal{K}_i \simeq \mathcal{C}_j$ (see (2.12)).

3 - Examples

A. The following is an example of an almost contact homogeneous structure of type $\mathcal{T}_1$ which belongs to $\mathcal{C}_2$. Let $H^{2n+1} = \{(y_1, \ldots, y_{2n+1}) \in \mathbf{R}^{2n+1} \mid y_1 > 0\}$ be the $2n+1$-dimensional *hyperbolic space* endowed with the Riemannian metric

$$g = (cy_1)^{-2} \sum_{i=1}^{2n+1} (dy_i)^2 \qquad c \in \mathbf{R} \quad c > 0. \tag{3.1}$$

The vector fields $E_i = cy_i\left(\frac{\partial}{\partial y_i}\right)$, $i = 1, \ldots, 2n+1$, form an orthonormal basis for (H^{2n+1}, g). Put $\xi = E_1$, then the tensor T given by

$$T_{XYZ} = g(X, Y)g(\xi, Z) - g(X, Z)g(\xi, Y) \qquad X, Y \in \mathfrak{X}(H^{2n+1}) \tag{3.2}$$

is a Riemannian homogeneous structure (see [11]). If we define φ by

$$\varphi E_h = \sum_{k=1}^{2n+1} \varphi_h^k E_k \qquad \varphi_h^k \in \mathcal{C}^\infty(H^{2n+1}) \qquad h = 1, \ldots, 2n+1 \tag{3.3}$$

then φ satisfies the last condition of Theorem 1 if and only if the components φ_h^k are constant on H^{2n+1}. An explicit computation shows that T is an homogeneous contact metric structure which belongs to $\mathcal{C}_2$.

B. Let $G(k)$ be the connected simply-connected 3-dimensional Lie group of matrices

$$(3.4) \qquad \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbf{R}$ and $k \neq 0$ is a fixed real number. $G(k)$ is an example of an almost contact manifold which admits an almost contact homogeneous structure which belongs to $\mathcal{H}_5 \oplus \mathcal{H}_6$. To see this, let us consider the linearly independent left invariant 1-forms on $G(k)$

$$(3.5) \qquad \alpha = e^{-kz}\, dz \qquad \beta = e^{kz}\, dy \qquad \eta = dz\,.$$

The corresponding dual basis of left invariant vector fields is formed by

$$(3.6) \qquad X_1 = e^{kz} \frac{\partial}{\partial x} \qquad X_2 = e^{-kz} \frac{\partial}{\partial y} \qquad X_3 = \frac{\partial}{\partial z}\,.$$

Since $[X_1, X_3] = -kX_1$, $[X_2, X_3] = kX_2$ and the other brackets are zero, $G(k)$ is a solvable non-nilpotent Lie group and $g = \alpha \otimes \alpha + \beta \otimes \beta + \eta \otimes \eta$ is a left invariant metric. If we put

$$(3.7) \qquad \varphi X_1 = X_2 \qquad \varphi X_2 = X_1 \qquad X_3 = \xi$$

the tensor field T given by

$$(3.8) \qquad 2g(T_X Y, Z) = g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)$$

$X, Y, Z \in \mathfrak{g}$ (Lie algebra of $G(k)$), is an almost contact homogeneous structure which satisfies

$$(3.9) \qquad T_{XX_2\xi} = -k\beta(X) \qquad T_{XX_1X_2} = 0 \qquad T_{XX_1\xi} = k\alpha(X)\,.$$

One can check that $G(k)$ belongs to the class $\mathcal{H}_5 \oplus \mathcal{H}_6 \subset \mathcal{T}_-$.

C. Let G be the Lie group of real matrices of the form,

$$(3.10) \qquad \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix}$$

endowed with the left invariant metric

$$g = \alpha^1 \otimes \alpha^1 + \alpha^2 \otimes \alpha^2 + \alpha^3 \otimes \alpha^3 \qquad \lambda > 0 \tag{3.11}$$

where $\alpha^1 = e^z dx$, $\alpha^2 = e^{-z} dy$, $\alpha^3 = \lambda dz$. We shall see that G is an almost contact homogeneous manifold which belongs to the class $\mathcal{H}_6$. The dual basis of invariant vector fields

$$Y_1 = e^{-z} \frac{\partial}{\partial x} \qquad Y_2 = e^z \frac{\partial}{\partial y} \qquad Y_3 = \frac{1}{\lambda} \frac{\partial}{\partial z} \tag{3.12}$$

satisfies

$$[Y_3, Y_1] = \frac{1}{\lambda} Y_1 \qquad [Y_3, Y_2] = \frac{1}{\lambda} Y_2 \tag{3.13}$$

and the other brackets are zero. If we define

$$\varphi Y_1 = Y_2 \qquad \varphi Y_2 = -Y_1 \qquad Y_3 = \xi \tag{3.14}$$

we get an almost contact structure on G such that the tensor T of (3.8) satisfies

$$T_{XY_1Y_2} = 0 \qquad T_{X\xi Y_1} = -\frac{1}{\lambda} \alpha_1(X) \qquad T_{X\xi Y_2} = -\frac{1}{\lambda} \alpha_2(X) \tag{3.15}$$

where $X \in \mathfrak{g}$ and $\mathfrak{g}$ is the Lie algebra of G. One can check that G belongs to the class $\mathcal{H}_6 \subset \mathcal{T}_-$. We also note that (G, g) is a 4-symmetric space (see [3]) which is isomorphic to the semi-direct product of $\boldsymbol{R}$ and $\boldsymbol{R}^2$, both with the additive group structure.

D. The following example shows that the inclusion $\mathcal{C}_7$ (or $\mathcal{C}_8) \subset \mathcal{C}_7 \oplus \mathcal{C}_8$ is strict.

Let $H(p, 1)$ be the group of real matrices

$$a = \begin{pmatrix} 1 & A & C \\ 0 & I_p & {}^tB \\ 0 & 0 & 1 \end{pmatrix} \tag{3.16}$$

where I_p denotes the $p \times p$ identity matrix, $A = (a_1, \ldots, a_p)$, $B = (b_1, \ldots, b_p) \in \boldsymbol{R}^p$, and $c \in \boldsymbol{R}$. $H(p, 1)$ is a connected, simply connected, nilpotent Lie group of dimension $2p + 1$, called *generalized Heisenberg group*. A global system of coordinates (x_i, x_{p+i}, z), $1 \leqslant i \leqslant p$, on $H(p, 1)$ is given by

$$x_i(a) = a_i \qquad x_{p+i}(a) = b_i \qquad z(a) = c \qquad 1 \leqslant i \leqslant p. \tag{3.17}$$

A basis for the left invariant 1-form on $H(p, 1)$ is

$$\alpha_i = dx_i \qquad \alpha_{p+i} = dx_{p+i} \qquad \gamma = dz - \sum_{j=1}^{p} x_j\, dx_{p+j} \qquad 1 \leqslant i \leqslant p \tag{3.18}$$

and its dual basis of left invariant vector fields is given by

$$X_i = \frac{\partial}{\partial x_i} \qquad X_{p+i} = \frac{\partial}{\partial x_{p+i}} + x_i \frac{\partial}{\partial z} \qquad Z = \frac{\partial}{\partial z} \qquad 1 \leqslant i \leqslant p. \tag{3.19}$$

Define a left invariant metric on $H(p, 1)$ by $g = \sum_{i=1}^{2p} \alpha_i \otimes \alpha_i + \gamma \otimes \gamma$. Put

$$\varphi X_i = X_{p+i} \qquad \varphi X_{p+i} = -X_i \qquad Z = \xi. \tag{3.20}$$

Then (φ, ξ, η, g) is an almost contact metric structure on $H(r, 1)$. By Theorem 2.1 of [5], all contact homogeneous structures T on $(H(p, 1), g)$ are given by

$$\begin{aligned} 2T = & \sum_{i=1}^{p} (\alpha_i \otimes \alpha_{p+i} \wedge \eta + \alpha_{p+i} \otimes \eta \wedge \alpha_i) \\ & + \sum_{i,j}^{p} [a_{ij} \otimes (\alpha_i \wedge \alpha_j + \alpha_{p+i} \wedge \alpha_{p+j}) + 2b_{ij} \otimes \alpha_i \wedge \alpha_{p+j}] \end{aligned} \tag{3.21}$$

where a_{ij} and b_{ij} are 1-forms such that $a_{ij} = -a_{ji}$, $b_{ij} = b_{ji}$ and whose $\tilde{\nabla}$-covariant derivative verifies some special conditions (see (2.4) and (2.5) of [5]).

Furthermore, $T \in \mathcal{T}_3$ if and only if $a_{ij} = 0$, for all i, j, $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = \frac{1}{2}\eta$. By explicit calculation, we can check that $H(p, 1)$ belong to the class

$$\mathcal{C}_7 \oplus \mathcal{C}_8 = \{T \in \mathcal{T} \mid T_{XYZ} = \eta(X)\, T_{\xi\varphi Y \varphi Z} + \eta(Y)\, T_{\varphi X \xi \varphi Z} + \eta(Z)\, T_{\varphi X \varphi Y \xi}\} \tag{3.22}$$

but T does not belong to $\mathcal{C}_7$ nor to $\mathcal{C}_8$.

E. The following is an example of an almost contact homogeneous structure of type $\mathcal{T}_2$ which belongs to $\mathcal{C}_{14}$. Let $G(a, b)$ be the 3-dimensional Lie group of real matrices

$$\begin{pmatrix} e^{-az} & 0 & 0 & e^{-az}x \\ 0 & e^{bz} & 0 & e^{bz}y \\ 0 & 0 & 1 & \frac{z}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

with the left invariant metric

$$g = dz^2 + e^{2az}dx^2 + e^{-2bz}dy^2 \,. \tag{3.24}$$

The case $a + b = 0$ corresponds to the hyperbolic space H^3. The orthonormal basis of left invariant vector fields given by

$$Y_1 = \frac{\partial}{\partial z} \qquad Y_2 = e^{-az}\frac{\partial}{\partial x} \qquad Y_3 = e^{bz}\frac{\partial}{\partial y} \tag{3.25}$$

satisfies $[Y_1, Y_2] = -aY_2$, $[Y_1, Y_3] = bY_3$, the other brackets being zero. Then the conditions

$$\varphi Y_2 = Y_3 \qquad \varphi Y_3 = -Y_2 \qquad Y_1 = \xi \tag{3.26}$$

define an almost contact structure on $G(a, b)$ such that the tensor T of (3.8) satisfies

$$T_{Y_2 Y_2 \xi} = -a \qquad T_{Y_3 Y_3 \xi} = a \,. \tag{3.27}$$

If $a + b \neq 0$, the homogeneous structure T is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ and belongs to $\mathcal{T}_2$ if and only if $a = b$. In this case, an explicit calculation shows that $T \in \mathcal{C}_{14}$.

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Sommario

Il confronto tra due diverse classificazioni delle strutture quasi contatto omogenee conduce ad alcuni risultati geometrici per diverse classi di varietà quasi contatto omogenee. Alcuni esempi vengono sviluppati dettagliatamente.

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