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Curvature properties of almost Hermitian manifolds (**)

Introduction and preliminaries

Let (M, g, J) be a *2n-dimensional almost Hermitian manifold* ($n \geq 2$), with fundamental 2-form ω such that $\omega(X, Y) = g(JX, Y)$ and Riemannian connection ∇. The covariant derivative $\nabla\omega$ is a section of the vector bundle $\mathcal{W}(M)$ on M, whose fibre, at any $x \in M$, is the linear space

$$\mathcal{W}_x = \{\alpha \in (T_x^* M)^3 \mid \alpha(X, Y, Z) = -\alpha(X, Z, Y) = -\alpha(X, JY, JZ)\} = \bigoplus_{i=1}^{4} (\mathcal{W}_i)_x$$

where, for any i, $(\mathcal{W}_i)_x$ is the linear space considered in [9].

The conditions

$$\alpha(X, Y, Z) = -\alpha(JX, JY, Z) \qquad \alpha(X, Y, Z) = \alpha(JX, JY, Z)$$

characterize the sections of $\mathcal{W}_1(M) \oplus \mathcal{W}_2(M)$, $\mathcal{W}_3(M) \oplus W_4(M)$, respectively.

The *metric connection* $\overline{\nabla}$, defined by

$$(1.1) \qquad \overline{\nabla}_X Y = \nabla_X Y + \tau(X, Y) \qquad \text{where}$$

$$(1.2) \qquad \tau(X, Y) = -\frac{1}{2} J((\nabla_X J) Y)$$

preserves J but, in general, is not symmetric. Let Σ be the *torsion form* of $\overline{\nabla}$

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and N the *Nijenhuis tensor* of J. Then, one can easily prove the formulas:

$$\Sigma(X, Y) = \tau(X, Y) - \tau(Y, X) \tag{1.3}$$

$$2g(\tau(X, Y), Z) = \nabla_X \omega(Y, JZ) = \sum_{i=1}^{4} \tau_i(X, Y, JZ) \tag{1.4}$$

$$\begin{aligned} g(N(X, Y), Z) &= 4g(\Sigma(X, Y) - \Sigma(JX, JY), Z) \\ &= 4(2\tau_1(JZ, X, Y) - \tau_2(JZ, X, Y)). \end{aligned} \tag{1.5}$$

$$\Sigma(X, Y) + \Sigma(JX, JY) = \sum_{j=1}^{2n} \{(\tau_3 + \tau_4)(X, Y, Je_j) - (\tau_3 + \tau_4)(Y, X, Je_j)\} e_j . \tag{1.6}$$

where τ_i, $i \in \{1, 2, 3, 4\}$ is the projection of $\nabla\omega$ on $\mathcal{W}_i(M)$ and $\{e_j\}_{1 \leq j \leq 2n}$ is a local orthonormal frame.

In the following we also use the relation

$$\begin{aligned} \tau_4(X, Y, Z) = & -g(X, Y)\beta(JZ) - g(X, JY)\beta(Z) \\ & + g(X, Z)\beta(JY) + g(X, JZ)\beta(Y) \end{aligned} \tag{1.7}$$

where $\beta = -\dfrac{1}{2(n-1)} \delta\omega \circ J$ is the *Lee form*.

As a consequence of (1.1), (1.3), (1.4), since $\overline{\nabla}$ is metric and unitary, the *curvature* $\overline{R}$ of $\overline{\nabla}$ and the *Riemannian curvature* R are related by

$$\begin{aligned} (\overline{R} - R)(X, Y, Z, W) = \frac{1}{2} \sum_{i=1}^{4} \{ & \overline{\nabla}_X \tau_i(Y, Z, JW) - \overline{\nabla}_Y \tau_i(X, Z, JW) \\ & + \tau_i(\Sigma(X, Y), Z, JW) - \tau_i(X, \tau(Y, Z), JW) + \tau_i(Y, \tau(X, Z), JW)\}. \end{aligned} \tag{1.8}$$

So $\overline{R} - R$ is regarded as a section of the bundle $\wedge^1(M) \otimes \mathcal{W}(M)$, since one can prove that, for any vector field X, $\overline{\nabla}_X \tau_i$ is a section of $\mathcal{W}_i(M)$.

Moreover, since $\overline{R}(X, Y, Z, W) = \overline{R}(X, Y, JZ, JW)$, (1.8) implies

$$\begin{aligned} \lambda(X, Y, Z, W) &= R(X, Y, Z, W) - R(X, Y, JZ, JW) \\ &= -\sum_{i=1}^{4} \{\overline{\nabla}_X \tau_i(Y, Z, JW) - \overline{\nabla}_Y \tau_i(X, Z, JW) + \tau_i(\Sigma(X, Y), Z, JW)\}. \end{aligned} \tag{1.9}$$

The vanishing of λ is equivalent to the condition that R satisfies the Kähler identity, that is (M, g, J) is a $\mathcal{R}_1$-*manifold*.

Now, let $\mathcal{R}(M)$ be the bundle of the algebraic curvature tensor fields on M, equipped with the metric considered in [**19**].

If S is a section of $\mathcal{R}(M)$, putting $S(X, Y) = S(X, Y, X, Y)$, we get

$$
\begin{aligned}
6S(X,Y,Z,W) &= S(X,Y+Z) - S(X,Y+W) + S(Y,X+W) - S(Y,X+Z) \\
&\quad + S(Z, X+W) - S(Z, Y+W) + S(W, Y+Z) - S(W, X+Z) \\
&\quad + S(X+Z,Y+W) - S(X+W,Y+Z) + S(X,W) - S(X,Z) + S(Y,Z) - S(Y,W).
\end{aligned} \tag{1.10}
$$

Thus, S is uniquely determined by the values $S(X, Y)$, for any pair (X, Y) of vector fields on M.

Moreover, one can consider the splitting $\mathcal{R}(M) = \mathcal{K} \oplus \mathcal{K}^{\perp}$, as an orthogonal direct sum, where the sections of $\mathcal{K}$ are the algebraic curvature tensor fields which satisfy the Kähler identity.

The Riemannian curvature R is a sections of $K^{\perp}$, iff the holomorphic sectional curvature of (M, g, J) vanishes ([5]).

This equivalence gives a motivation for a detailed investigation of the projections of R on each of the subbundles of $\mathcal{K}^{\perp}$ considered in [**19**].

On the other hand, until now, the problem of the classification of the $\mathcal{R}_1$-manifolds (or, according to [**17**], para-Kähler manifods) is still open. In fact, as far as the authors know, the 6-dimensional quasi-Kähler manifolds considered in [**21**] are the only example of $\mathcal{R}_1$-manifolds other than the flat or the Kähler ones. The authors hope that a detailed study of the projections of R in $\mathcal{K}^{\perp}$ can be fruitful to produce other example of $\mathcal{R}_1$-manifolds.

The general properties of the $\mathcal{R}_1$-manifolds are stated in [**17**]. According to [5], the projections of R in $\mathcal{K}^{\perp}$ are denoted by $\mathcal{K}_{-1}, \mathcal{K}_{-2}, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_8, \mathcal{C}_6, \mathcal{C}_7$. They coincide with the projections $p_4(R)$, $p_5(R)$, $p_6(R)$, $p_7(R)$, $p_8(R)$, $p_9(R)$, $p_{10}(R)$, of [**19**], respectively.

When $n = 2$, $\mathcal{K}_{-2}$, $\mathcal{C}_4$ and $\mathcal{C}_7$ vanish. $\mathcal{C}_4$ vanishes also if $n = 3$.

In Section **2** the projections of R in $\mathcal{K}^{\perp}$ are explicitly expressed by means of the covariant derivatives $\overline{\nabla}\tau_i$ and suitable contractions of the symmetric products $\tau_i \odot \tau_j$, $i, j \in \{1, 2, 3, 4\}$. These results agree with the Tables 1, 2, 3 in [5], although in [5] there are few explicit formulas.

In the last part of this paper the results of Section **2** assist in the study of the curvature of some Lie groups.

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2 - On the $\mathcal{K}^{\perp}$-projections of $\mathcal{R}$.

Lemma 1. *For any pair (X, Y) of vector fields, we have*

$$(2.1)\quad \begin{aligned}(\mathcal{K}_{-1}+\mathcal{K}_{-2}+\mathcal{C}_4)(X,Y)=\frac{3}{8}\{&\overline{\nabla}_X(\tau_3+\tau_4)(Y,Y,JX)+\overline{\nabla}_Y(\tau_3+\tau_4)(X,X,JY)\\ -\overline{\nabla}_{JX}(\tau_3+\tau_4)(Y,Y,X)-\overline{\nabla}_{JY}(\tau_3+\tau_4)(X,X,Y)&-\frac{1}{4}(\tau_1+\tau_2)(N(X,Y),X,JY)\}.\end{aligned}$$

Applying Theorem 8.1 and Definition 3.3 in [19], we have

$$8(K_{-1}+K_{-2}+C_4)(X,Y,X,Y)=(I-L_1)(I+L_2)(I+L_3)R(X,Y,X,Y)$$

$$=\frac{3}{2}\{\lambda(X,Y,X,Y)-\lambda(JX,JY,X,Y)+\lambda(JX,Y,X,JY)+\lambda(X,JY,X,JY)\}.$$

Then, using (1.9), we have

$$(2.2)\quad \begin{aligned}&\lambda(X,Y,X,Y)-\lambda(JX,JY,X,Y)+\lambda(JX,Y,X,JY)+\lambda(X,JY,X,JY)\\ &=-\overline{\nabla}_X(\tau_1+\tau_2)(Y,X,JY)+\overline{\nabla}_{JX}(\tau_1+\tau_2)(JY,X,JY)+\overline{\nabla}_{JX}(\tau_1+\tau_2)(Y,X,Y)\\ &+\overline{\nabla}_X(\tau_1+\tau_2)(JY,X,Y)+\overline{\nabla}_Y(\tau_1+\tau_2)(X,X,JY)-\overline{\nabla}_{JY}(\tau_1+\tau_2)(JX,X,JY)\\ &-\overline{\nabla}_Y(\tau_1+\tau_2)(JX,X,Y)-\overline{\nabla}_{JY}(\tau_1+\tau_2)(X,X,Y)\\ &-(\tau_1+\tau_2)(\Sigma(X,Y)-\Sigma(JX,JY),X,JY)+(\tau_1+\tau_2)(\Sigma(JX,Y)+\Sigma(X,JY),X,Y)\\ &-\overline{\nabla}_X(\tau_3+\tau_4)(Y,X,JY)+\overline{\nabla}_{JX}(\tau_3+\tau_4)(JY,X,JY)+\overline{\nabla}_{JX}(\tau_3+\tau_4)(Y,X,Y)\\ &+\overline{\nabla}_X(\tau_3+\tau_4)(JY,X,Y)+\overline{\nabla}_Y(\tau_3+\tau_4)(X,X,JY)-\overline{\nabla}_{JY}(\tau_3+\tau_4)(JX,X,JY)\\ &-\overline{\nabla}_{JY}(\tau_3+\tau_4)(X,X,Y)-\overline{\nabla}_Y(\tau_3+\tau_4)(JX,X,Y)\\ &-(\tau_3+\tau_4)(\Sigma(X,Y)-\Sigma(JX,JY),X,JY)+(\tau_3+\tau_4)(\Sigma(JX,Y)+\Sigma(X,JY),X,Y\}.\end{aligned}$$

Since $\overline{\nabla}(\tau_1+\tau_2)$ is a section of $\wedge^1(M)\otimes(\mathcal{W}_1\oplus\mathcal{W}_2)(M)$, the term in (2.2) depending on $\overline{\nabla}(\tau_1+\tau_2)$ vanishes. The summands in $\overline{\nabla}(\tau_3+\tau_4)$ sum up pair by pair, since $\overline{\nabla}(\tau_3+\tau_4)$ is a section of $\wedge^1\otimes(\mathcal{W}_3\oplus\mathcal{W}_4)(M)$. For the remaining terms in (2.2), using relation $N(X,JY)=-J(N(X,Y))$, and (1.5), we obtain

$$(\tau_1+\tau_2)(\Sigma(X,Y)-\Sigma(JX,JY),X,JY)=-(\tau_1+\tau_2)(\Sigma(JX,Y)+\Sigma(X,JY),X,Y)$$

$$(\tau_3+\tau_4)(\Sigma(X,Y)-\Sigma(JX,JY),X,JY)=(\tau_3+\tau_4)(\Sigma(JX,Y)+\Sigma(X,JY),X,Y).$$

Finally, these relations imply (2.1).

Lemma 1 represents the first step for the formulation of an expression of the components $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_4$. To this aim, it is useful to recall that $\mathcal{K}_{-2}$ depends on the symmetric and J-invariant tensor field $(\rho-\rho^*)(R+L_3R)-\dfrac{(\tau-\tau^*)(R)}{n}g$, where $\rho(R)$, $\rho^*(R)$, $\tau(R)$, $\tau^*(R)$ stand for the Ricci-curvature, the $*$-Ricci curvature, the scalar and the $*$-scalar curvatures. Moreover, $(\tau - \tau^*)(R)$ determines the component $\mathcal{K}_{-1}$ ([19]).

Lemma 2. *Let $\{e_i, Je_i\}_{1 \leq i \leq n}$ be a local orthonormal frame on M. Then we have*

$$\begin{aligned}(\rho - \rho^*)(R + L_3R)(X, Y) = \frac{1}{2} \sum_{i=1}^{2n} \{ & \overline{\nabla}_Y \tau_4(e_i, e_i, JX) + \overline{\nabla}_X \tau_4(e_i, e_i JY) \\ & - \overline{\nabla}_{JY} \tau_4(e_i, e_i, X) - \overline{\nabla}_{JX} \tau_4(e_i, e_i, Y) \\ & + 2\overline{\nabla}_{e_i}(\tau_3 + \tau_4)(Y, X, Je_i) + 2\overline{\nabla}_{e_i}(\tau_3 + \tau_4)(X, Y, Je_i) \\ & - \frac{1}{4}(\tau_1 + \tau_2)(N(X, e_i), Y, Je_i) - \frac{1}{4}(\tau_1 + \tau_2)(N(Y, e_i), X, Je_i)\}\end{aligned} \tag{2.3}$$

$$(\tau - \tau^*)(R) = 2 \sum_{i,j=1}^{2n} \overline{\nabla}_{e_i} \tau_4(e_j, e_j, Je_i) - \|\tau_1\|^2 + \frac{1}{2}\|\tau_2\|^2 . \tag{2.4}$$

In fact, a direct computation, together with Lemma 1, gives

$$\begin{aligned}(\rho - \rho^*)(R + L_3R)(X, X) &= \frac{1}{2} \sum_{i=1}^{2n} (I - L_1)(I + L_2)(I + L_3)R(X, e_i, X, e_i) \\ &= \sum_{i=1}^{2n} \{\overline{\nabla}_X \tau_4(e_i, e_i, JX) - \overline{\nabla}_{JX} \tau_4(e_i, e_i X) + 2\overline{\nabla}_{e_i}(\tau_3 + \tau_4)(X, X, Je_i) \\ &\quad - \frac{1}{4}(\tau_1 + \tau_2)(N(X, e_i), X, Je_i)\} .\end{aligned} \tag{2.5}$$

Then (2.3) is a consequence of (2.5) and of the symmetry of $(\rho - \rho^*) \cdot (R + L_3R)$.

Moreover, contracting and using (2.5), we get

$$(\tau - \tau^*)(R) = \sum_{i,j=1}^{2n} \{2\overline{\nabla}_{e_i} \tau_4(e_j, e_j, Je_i) - \frac{1}{8}(\tau_1 + \tau_2)(N(e_i, e_j)), e_i, Je_j)\} .$$

The formula (1.5) implies

$$\sum_{i,j=1}^{2n} (\tau_1 + \tau_2)(N(e_i, e_j), e_i, Je_j)$$

$$= \sum_{i,j,h=1}^{2n} \{8\tau_1^2(Je_h, e_i, e_j) - 4\tau_2^2(Je_h, e_i e_j) + 4\tau_1(Je_h, e_i, e_j)\tau_2(Je_h, e_i, e_j)\} .$$

A direct computation, using the skew-symmetry of τ_1 and the condition $\underset{X,Y,Z}{\sigma} \tau_2(X, Y, Z) = 0$, yields $\sum_{i,j,h=1}^{2n} \tau_1(Je_h, e_i, e_j)\tau_2(Je_h, e_i e_j) = 0$.

Thus, one obtains (2.4) and the proof of Lemma 2 is complete.

Remark 1. The formula (2.4) reduces to:

$(\tau - \tau^*)(R) = -\|\nabla J\|^2$ in the nearly-Kähler case,

$(\tau - \tau^*)(R) = \frac{1}{2}\|\nabla J\|^2$ in the almost-Kähler case,

$(\tau - \tau^*)(R) = 4(n-1)(\mathrm{div}\, B - (n-1)\|\beta\|^2)$, when $\nabla\omega$ is a section of $\mathcal{W}_4(M)$,

where B denotes the vector field associated to the Lee form β with respect to g.

These relations are well-known ([13], [8], [4]). More generally, $\overline{\nabla}\tau_4$, $\|\tau_1\|^2$ and $\|\tau_2\|^2$ determine $\mathcal{K}_{-1}$.

Proposition 1. *If $n \geq 4$, the projection $\mathcal{K}_{-2}$ depends on $\overline{\nabla}\tau_3, \overline{\nabla}\tau_4, \tau_1 \odot \tau_1$, $\tau_1 \odot \tau_2$, $\tau_2 \odot \tau_2$. If $n = 3$, then $\mathcal{K}_{-2}$ depends only on $\overline{\nabla}\tau_3$, $\overline{\nabla}\tau_4$, $\tau_1 \odot \tau_2$, $\tau_2 \odot \tau_2$.*

The statement is a consequence of the Lemma 2 after one has proved that, if $n = 3$, $\mathcal{K}_{-2}$ does not contain the $\tau_1 \odot \tau_1$-summand.

In fact, (1.5), (2.3), (2.4) imply that this summand is determined by the symmetric and J-invariant tensor field T defined by

$$T(X, Y) = -2 \sum_{i,k=1}^{2n} \tau_1(e_k, X, e_i)\tau_1(e_k, Y, e_i) + \frac{1}{n}\|\tau_1\|^2 g(X, Y).$$

Suppose that $\{e_i, Je_i\}_{1 \leq i \leq 3}$ is defined in the open set U. A direct computation yields:

$$\|\tau_1\|^2_{|U} = 24\{\tau_1^2(e_1, e_2, e_3) + \tau_1^2(e_1, e_2, Je_3)\}$$

$$T_{|U}(e_1, e_1) = -2\sum_{i,k=1}^{6} \tau_1^2(e_k, e_1, e_i) + \frac{1}{3}\|\tau_1\|^2$$

$$= -8\{\tau_1^2(e_1, e_2, e_3) + \tau_1^2(e_1, e_2, Je_3)\} + \frac{1}{3}\|\tau_1\|^2 = 0\,.$$

Analogously, we can prove that $T_{|U}(e_i, e_j) = 0$ $i, j \in \{1, 2, 3\}$. Therefore T vanishes.

Proposition 2. *If $n \geq 5$, the projection $\mathcal{C}_4$ depends on $\overline{\nabla}\tau_3$, $\tau_1 \odot \tau_1$, $\tau_1 \odot \tau_2$, $\tau_2 \odot \tau_2$. If $n = 4$, then $\mathcal{C}_4$ depends only on $\overline{\nabla}\tau_3$, $\tau_1 \odot \tau_2$, $\tau_2 \odot \tau_2$.*

As a consequence of Lemmas 1 and 2, $\mathcal{C}_4$ turns out to depend on $\overline{\nabla}\tau_3$, $\overline{\nabla}\tau_4$, $\tau_1 \odot \tau_1$, $\tau_1 \odot \tau_2$, $\tau_2 \odot \tau_2$. Using the definitions of $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$ in [**19**] and Lemmas 1 and 2, one obtains the explicit expression for the values $S(X, Y)$, where S represents the $\overline{\nabla}\tau_4$-summand in $\mathcal{C}_4$. A direct computation, together with the formula

$$\begin{aligned}\overline{\nabla}_X \tau_4(Y, Z, W) = &-g(Y, Z)\overline{\nabla}_X\beta(JW) + g(JY, Z)\overline{\nabla}_X\beta(W)\\ &+g(Y, W)\overline{\nabla}_X\beta(JZ) - g(JY, W)\overline{\nabla}_X\beta(Z)\end{aligned} \tag{2.6}$$

implies $S(X, Y) = 0$, that is $S = 0$.

Finally, if $n = 4$, a long computation yields the vanishing of the $\tau_1 \odot \tau_1$-summand.

Proposition 3. *The projection $\mathcal{C}_5$ depends on $\overline{\nabla}\tau_2$, $\tau_1 \odot \tau_3$, $\tau_2 \odot \tau_3$, $\tau_2 \odot \tau_4$.*

The proof is carried on using the same method as in Lemma 1. In fact, taking account of the definition of $\mathcal{C}_5$ (see [**19**]), we have

$$\mathcal{C}_5(X,Y) = \frac{1}{8}\{\lambda(X,Y,X,Y) - \lambda(JX,JY,X,Y) - \lambda(JX,Y,JX,Y) - \lambda(X,JY,JX,Y)\}.$$

Then, we use (1.9). In the resulting formula, the contribution coming from $\overline{\nabla}\tau_1$ vanishes, since $\overline{\nabla}_X\tau_1$ is skew-symmetric for any vector field X.

Also the terms in $\overline{\nabla}(\tau_3 + \tau_4)$ vanish, since $\overline{\nabla}(\tau_3 + \tau_4)$ is a section of $\Lambda^1(M) \otimes (\mathcal{W}_3 \oplus \mathcal{W}_4)(M)$. Therefore we obtain

$$\begin{aligned}\mathcal{C}_5(X,Y) = &-\frac{1}{4}\{\overline{\nabla}_X\tau_2(Y,X,JY) - \overline{\nabla}_Y\tau_2(X,X,JY) + \overline{\nabla}_{JX}\tau_2(Y,X,Y) - \overline{\nabla}_{JY}\tau_2(X,X,Y)\}\\ &-\frac{1}{32}\{(\tau_1 + \tau_2)(N(X, Y), X, JY) - (\tau_1 + \tau_2)(N(JX, Y), JX, JY)\\ &+(\tau_3 + \tau_4)(N(X, Y), X, JY) - (\tau_3 + \tau_4)(N(JX, Y), JX, JY\}.\end{aligned}$$

Since $N(JX, Y) = -J(N(X, Y))$, applying also (1.5) and (1.7), we get

$$(\tau_1 + \tau_2)(N(JX, Y), JX, JY) = (\tau_1 + \tau_2)(N(X, Y), X, JY)$$

$$\begin{aligned}&(\tau_3 + \tau_4)(N(X,Y),X,JY) - (\tau_3 + \tau_4)(N(JX,Y),JX,JY) = 2(\tau_3 + \tau_4)(N(X,Y),X,JY)\\ &= -8\{\tau_2(JX,X,Y)\beta(Y) + \tau_2(X,X,Y)\beta(JY) - \tau_2(Y,X,Y)\beta(JX) - \tau_2(JY,X,Y)\beta(X)\}\\ &\quad +2\tau_3(N(X, Y), X, JY).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{C}_5(X, Y) = -\frac{1}{4}\{&\overline{\nabla}_X\tau_2(Y, X, JY) - \overline{\nabla}_Y\tau_2(X, X, JY) + \overline{\nabla}_{JX}\tau_2(Y, X, Y)\\ &-\overline{\nabla}_{JY}\tau_2(X, X, Y) + \frac{1}{4}\tau_3(N(X, Y), X, JY)\\ -\tau_2(JX,X,Y)\beta(Y)+&\tau_2(JY,X,Y)\beta(X)-\tau_2(X,X,Y)\beta(JY)+\tau_2(Y,X,Y)\beta(JX)\}.\end{aligned} \tag{2.7}$$

Remark 2. The vanishing of $\mathcal{C}_5$ in the hermitian or in the nearly-Kähler case can now be obtained using Proposition 3. A characterization of the condition $\mathcal{C}_5 = 0$ in terms of properties of $\nabla\omega$ is not known up to now. In Section 3 an example of almost Kähler manifold such that $\mathcal{C}_5 = 0$ is given.

Lemma 3. *For any pair (X, Y) of vector fields on M, we have*

$$\begin{aligned}&(\mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8)(X, Y)\\ &= \frac{1}{2}\{\overline{\nabla}_X(\tau_2+\tau_3+\tau_4)(Y, Y, JX) + \overline{\nabla}_Y(\tau_2+\tau_3+\tau_4)(X, X, JY)\\ &+\overline{\nabla}_{JX}(\tau_2+\tau_3+\tau_4)(JY, JY, X) + \overline{\nabla}_{JY}(\tau_2+\tau_3+\tau_4)(JX, JX, Y)\\ &-(\tau_1+\tau_2)(\Sigma(X,Y)+\Sigma(JX,JY),X,JY)-(\tau_3+\tau_4)(\Sigma(X,Y)+\Sigma(JX,JY),X,JY)\}.\end{aligned} \tag{2.8}$$

In fact, using Theorem 8.1 and Definition 3.3 of [19], we have

$$\begin{aligned}&2(C_6 + C_7 + C_8)(X, Y, X, Y)\\ &= (R - L_3R)(X, Y, X, Y) = \lambda(X, Y, X, Y) - \lambda(JX, JY, JX, JY).\end{aligned}$$

Thus (1.9) and the skew-symmetry of $\overline{\nabla}_X\tau_1$ imply the statement.

Lemma 4. *For any pair (X, Y) of vector fields on M, we have*

$$\begin{aligned}&\rho^*(R - L_3R)(X, Y)\\ &= \frac{1}{6}\sum_{i=1}^{2n}\{6\overline{\nabla}_{e_i}\tau_2(Je_i, X, Y) + 4\overline{\nabla}_{e_i}\tau_3(Je_i, X, Y) + 4\overline{\nabla}_{e_i}\tau_4(Je_i, X, Y)\\ &-\overline{\nabla}_X\tau_4(e_i,e_i,JY)-\overline{\nabla}_{JX}\tau_4(e_i,e_i,Y)+\overline{\nabla}_Y\tau_4(e_i,e_i,JX)\}+\overline{\nabla}_{JY}\tau_4(e_i,e_i,X)\\ &-\frac{1}{6}\{4(n+1)\tau_1(JB,X,Y)+2(2n-1)\tau_2(JB,X,Y)+4(n-1)\tau_3(JB,X,Y\}\\ &-\frac{1}{6}\sum_{i,j=1}^{2n}\{(\tau_1+\tau_2)(e_j, Y, Je_i)(\tau_3(e_i, X, Je_j) - \tau_3(X, e_i, Je_j))\\ &\qquad-(\tau_1+\tau_2)(e_j, X, Je_i)(\tau_3(e_i, Y, Je_j) - \tau_3(Y, e_i, Je_j))\}.\end{aligned} \tag{2.9}$$

In fact, by means of (2.8) and (1.10), one obtains the values of

$(R - L_3R)(X, Y, Z, W)$. Then, since $\overline{\nabla}\tau_i \in \Lambda^1(M) \otimes \mathcal{W}_i(M)$, $i \in \{1, 2, 3, 4\}$

$$\rho^*(R - L_3R)(X, Y) = \frac{1}{6}\sum_{i=1}^{2n}\{6\overline{\nabla}_{e_i}\tau_2(Je_i, X, Y) + 4\overline{\nabla}_{e_i}\tau_3(Je_i, X, Y)$$
$$-\overline{\nabla}_X\tau_4(e_i,e_i,JY) - \overline{\nabla}_{JX}\tau_4(e_i,e_i,Y) + \overline{\nabla}_Y\tau_4(e_i,e_i,JX) + \overline{\nabla}_{JY}\tau_4(e_i,e_i,X) + 4\overline{\nabla}_{e_i}\tau_4(Je_i,X,Y)\}$$
$$+\frac{1}{6}\sum_{i=1}^{2n}\{t(\Sigma(X, e_i) + \Sigma(JX, Je_i), Y, Je_i) - t(\Sigma(Y, e_i) + \Sigma(JY, Je_i), X, Je_i)$$
$$+t(\Sigma(e_i, Je_i) - \Sigma(Je_i, e_i), X, Y)\}$$

where t stands for $\tau_1 + \tau_2 + \tau_3 + \tau_4$.

Moreover, (1.7) and (1.6) imply

$$\sum_{i=1}^{2n}\{(\tau_1+\tau_2)(\Sigma(X,e_i)+\Sigma(JX,Je_i),Y,Je_i) - (\tau_1 + \tau_2)(\Sigma(Y, e_i) + \Sigma(JY, Je_i), X, Je_i)\}$$
$$= \sum_{i,j=1}^{2n}\{-(\tau_1 + \tau_2)(e_j, Y, Je_i)(\tau_3(e_i, X, Je_j) - \tau_3(X, e_i, Je_j))\} - 8\tau_1(JB, X, Y)$$
$$+ \sum_{i,j=1}^{2n}\{(\tau_1 + \tau_2)(e_j, X, Je_i)(\tau_3(e_i, Y, Je_j) - \tau_3(Y, e_i, Je_j))\} - 2\tau_2(JB, X, Y).$$

$$\sum_{i=1}^{2n}\{(\tau_3+\tau_4)(\Sigma(X,e_i)+\Sigma(JX,Je_i),Y,Je_i) - (\tau_3+\tau_4)(\Sigma(Y,e_i)+\Sigma(JY,Je_i),X,Je_i)\} = 0$$

where $\{e_i, Je_i\}$ is a local orthonormal frame. Finally, using the formulas

$$\sum_{i=1}^{2n}\Sigma(e_i, Je_i) - \Sigma(Je_i, e_i) = -4(n-1)JB \qquad \tau_4(JB, X, Y) = 0 \qquad \text{we get}$$

$$\sum_{i=1}^{2n}(\tau_1+\tau_2+\tau_3+\tau_4)(\Sigma(e_i,Je_i) - \Sigma(Je_i,e_i),X,Y) = -4(n-1)(\tau_1+\tau_2+\tau_3)(JB,X,Y).$$

These relations yield (2.9).

Proposition 4. *The projection* $\mathcal{C}_6$ *depends on* $\overline{\nabla}\tau_2$, $\overline{\nabla}\tau_3$, $\overline{\nabla}\tau_4$, $\tau_1 \odot \tau_3$, $\tau_2 \odot \tau_3$, $\tau_1 \odot \tau_4$, $\tau_2 \odot \tau_4$, $\tau_3 \odot \tau_4$. *In particular, if* $n = 3$, *the* $\tau_1 \odot \tau_3$*-component vanishes.*

Since the tensor field $\rho^*(R - L_3R)$ determines $\mathcal{C}_6$, ([19]), the first part of the statement is a consequence of Lemma 4.

Moreover, the $\tau_1 \odot \tau_3$-component of $\mathcal{C}_6$ is determined by the J-antiinvariant 2-form

$$T(X, Y) = \sum_{i,j=1}^{2n}\{\tau_1(e_j, Y, Je_i)(\tau_3(e_i, X, Je_j) - \tau_3(X, e_i, Je_j))$$
$$-\tau_1(e_j, X, Je_i)(\tau_3(e_i, Y, Je_j) - \tau_3(Y, e_i, Je_j))\}.$$

When $n = 3$, since τ_1 is skew-symmetric, for a given $x \in M$, there exists an orthonormal frame $\{e_i, Je_i\}_{1 \leq i \leq 3}$ defined in a neighbourhood U of x such that $\tau_1(e_1, e_2, Je_3) = 0$.

With respect to this frame, using the condition

$$2 \sum_{i=1}^{3} \tau_3(e_i, e_i, X) = \sum_{i=1}^{3} (\tau_3(e_i, e_i, X) + \tau_3(Je_i, Je_i, X) = 0$$

we obtain $\quad T_{|U}(e_i, e_j) = T_{|U}(e_i, Je_j) = 0 \qquad i < j \quad i, j \in \{1, 2, 3\}$.

Therefore, $T_{|U} = 0$, and the proof is complete.

Proposition 5. *The projection $\mathcal{C}_8$ depends on $\overline{\nabla}\tau_2$, $\overline{\nabla}\tau_4$, $\tau_1 \odot \tau_3$, $\tau_2 \odot \tau_3$, $\tau_2 \odot \tau_4$, $\tau_3 \odot \tau_3$, $\tau_4 \odot \tau_4$.*

In fact, the symmetric tensor field $\rho(R - L_3 R)$ determines $\mathcal{C}_8$ ([**19**]). By means of Theorem 8.1 in [**19**] and of (2.8), (1.7) one obtains

$$\begin{aligned} \rho(R-L_3R)(X,X) = &\sum_{i=1}^{2n} \{2\overline{\nabla}_{e_i}\tau_2(X,X,Je_i) + \overline{\nabla}_X\tau_4(e_i, e_i, JX) + \overline{\nabla}_{JX}\tau_4(e_i, e_i, X)\} \\ &- \sum_{i,j=1}^{2n} \{(\tau_1 + \tau_2)(e_j, X, Je_i)\tau_3(X, e_i, Je_j) - \sum_{i,j=1}^{2n} \tau_3(e_j,X,Je_i)\tau_3(e_i,X,Je_j)\} \\ &+ 2\tau_2(X, X, JB) - 2(n-1)(\beta(JX)^2 - \beta(X)^2) \end{aligned} \tag{2.10}$$

where $\{e_i, Je_i\}$ is a local orthonormal frame.

Proposition 6. *The projection $\mathcal{C}_7$ depends on $\overline{\nabla}\tau_2$, $\overline{\nabla}\tau_3$, $\tau_1 \odot \tau_3$, $\tau_2 \odot \tau_3$, $\tau_2 \odot \tau_4$, $\tau_3 \odot \tau_3$, $\tau_3 \odot \tau_4$. If $n = 3$, the $\tau_1 \odot \tau_3$-component vanishes.*

In fact, Theorem 8.1 in [**19**] yields to

$$\begin{aligned} \mathcal{C}_7(X, Y) = &\frac{1}{2}(R - L_3R)(X, Y) - \frac{3}{2(n+1)} g(X, JY)\rho^*(R - L_3R)(X, JY) \\ &- \frac{1}{4(n-1)} \{g(X, X)\rho(R - L_3R)(Y, Y) + g(Y, Y)\rho(R - L_3R)(X, X) \\ &- 2g(X, Y)\rho(R - L_3R)(X, Y)\}. \end{aligned}$$

Then, applying (2.8), (2.9), (2.10) we express C_7 as a function of $\overline{\nabla}\tau_2$, $\overline{\nabla}\tau_3$, $\overline{\nabla}\tau_4$ and of suitable contractions of $\tau_1 \odot \tau_3$, $\tau_2 \odot \tau_3$, $\tau_1 \odot \tau_4$, $\tau_2 \odot \tau_4$, $\tau_3 \odot \tau_3$, $\tau_3 \odot \tau_4$, $\tau_4 \odot \tau_4$.

Now, the $\tau_1 \odot \tau_4$-component in $\mathcal{C}_7$ is determined by the algebraic curvature tensor field T defined by

$$T(X,Y) = -\frac{1}{2}\sum_{i=1}^{2n}\{\tau_4(X,Y,Je_i)-\tau_4(Y,X,Je_i)\}\,\tau_1(e_i,X,JY)-g(X,JY)\,\tau_1(X,Y,B).$$

Then, applying (1.7), we get $T(X, Y) = 0$, that is $T = 0$.

The vanishing of the $\tau_4 \odot \tau_4$-component is proved in the same way. Relation (2.6) is applied to prove that the term in $\mathcal{C}_7(X, Y)$, expressed by means of $\overline{\nabla}\tau_4$, vanishes.

Finally, let T denote the $\tau_1 \odot \tau_3$-component of $\mathcal{C}_7$. Then T is a section of $\mathcal{R}(M)$ and the values $T(X, Y)$ can be easily evaluated using (2.8), (2.9) and (2.10). By means of (1.10), we obtain $T(X, Y, Z, W)$, for any quadruplet of vector fields. When $n = 3$, there exists a local orthonormal frame $\{e_i, Je_i\}$ such that $\tau_1(e_1, e_2, Je_3) = 0$. It is easy to prove now that T vanishes for any quadruplets of vector fields choosen in this frame. Thus the proof of Proposition 6 is complete.

3 - Some applications to the Lie groups

Let (G, g, J) be the almost Hermitian manifold consisting of a connected Lie group G equipped with a metric and an almost complex structure which are both left-invariant. Then, the Riemannian connection ∇ is detemined by

$$(3.1) \qquad 2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X)$$

for any X, Y, Z in the Lie algebra $\mathfrak{g}$ of G.

The aim of this section is to compute the $\mathcal{K}^{\perp}$-projections of R, for suitable manifolds (G, g, J). These projections are determined by the values on the quadruplets (X, Y, X, Y) with $X, Y \in \mathfrak{g}$.

Table 1, 2, 3 present an outline of the results which will be stated. They give the real dimension, the Gray-Hervella class and the non-vanishing $\mathcal{K}^{\perp}$-projection of the curvature of the examined manifolds.

Table 1

4	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\mathcal{K}_{-1} + \mathcal{C}_5 + \mathcal{C}_8$
4	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$\mathcal{C}_5 + \mathcal{C}_8$
4	$\mathcal{W}_2$	$\mathcal{K}_{-1} + \mathcal{C}_8$
4	$\mathcal{W}_4$	$\mathcal{K}_{-1} + \mathcal{C}_8$

Table 2

6	$\mathcal{W}_3$	$\mathcal{K}_{-2}$
6	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\mathcal{K}_{-1} + \mathcal{K}_{-2}$

Table 3

6	$\mathcal{W}_1$	$\mathcal{K}_{-1}$
8	$\mathcal{W}_1$	$\mathcal{K}_{-1} + \mathcal{K}_{-2}$
$2n$, $n \geq 5$	$\mathcal{W}_1$	$\mathcal{K}_{-1} + \mathcal{C}_4$
$2n$, $n \geq 5$	$\mathcal{W}_1$	$\mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4$

For any $(\lambda, \mu) \in (\boldsymbol{R}^2)^*$, let $G_{(\lambda,\mu)}$ be the 4-dimensional Lie group consisting of matrices considered in [1], Sec. 6, Example c, equipped with the left invariant metric $g_{(\lambda,\mu)}$ such that

$$\{e_1 = \frac{\partial}{\partial t}\,, \quad e_2 = e^{\lambda t}\,\frac{\partial}{\partial x}\,, \quad e_3 = e^{\mu t}\,\frac{\partial}{\partial y}\,, \quad e_4 = e^{-(\lambda+\mu)t}\,\frac{\partial}{\partial z}\}$$

is an orthonormal frame on $\mathfrak{g}$ ([1], [14]).

Let J be the left-invariant almost complex structure such that

$$J(e_1) = ce_4 \qquad J(e_2) = de_3 \qquad c, d \in \boldsymbol{R}\,, \qquad c^2 = d^2 = 1\,.$$

Then, using (3.1), we obtain the possibly non-vanishing components of ∇, i.e.

$$\begin{aligned} &\nabla_{e_2} e_1 = -\lambda e_2 \qquad \nabla_{e_2} e_2 = \lambda e_1 \qquad \nabla_{e_3} e_1 = -\mu e_3 \qquad \nabla_{e_3} e_3 = \mu e_1 \\ &\nabla_{e_4} e_1 = (\lambda+\mu)\, e_4 \qquad \nabla_{e_4} e_4 = -(\lambda+\mu)\, e_1\,. \end{aligned} \tag{3.2}$$

Proposition 7. *For any pair $(\lambda, \mu) \in (\boldsymbol{R}^2)^*$, $(G_{(\lambda,\mu)}, g_{(\lambda,\mu)}, J)$ is not a Kähler manifold. Indeed, it is a global conformal Kähler manifold iff $\lambda = \mu$, an almost Kähler manifold iff $\lambda = -\mu$. For the $\mathcal{K}^{\perp}$-projections of R we have $\mathcal{C}_6 = 0$ and $\mathcal{C}_8 \neq 0$. Moreover, $\mathcal{C}_5$ vanishes iff $\lambda = \pm\mu$ and $\mathcal{K}_{-1}$ vanishes iff $\lambda = 0$ or $\mu = 0$.*

In this case, τ_2 and τ_4 determine $\nabla\omega$. It is easy to prove that the possibly non-vanishing components of τ_2 are obtained by means of

$$\begin{aligned} \tau_2(e_2, e_1, e_3) = \tau_2(e_3, e_1, e_2) \quad &= \frac{1}{2}\, d(\lambda-\mu) \\ \tau_2(e_2, e_2, e_4) = -\tau_2(e_3, e_3, e_4) &= -\frac{1}{2}\, c(\lambda-\mu)\,. \end{aligned}$$

The Lee form is exact, since one has $\beta = -\dfrac{1}{2}(\lambda+\mu)\, dt$. Thus, the first part of the statement is proved. By means of (2.4) and (2.6) we obtain $(\tau - \tau^*)(R) = -4\lambda\mu$. Therefore $\mathcal{H}_{-1}$ vanishes iff $\lambda = 0$ or $\mu = 0$.

Moreover, using (2.7) and (1.10), one can prove that the possibly non-vanishing values of C_5 are determined by

$$\mathcal{C}_5(e_1, e_2, e_1, e_2) = -\mathcal{C}_5(e_1, e_3, e_1, e_3) = \frac{1}{4}(\lambda^2 - \mu^2)$$
$$\mathcal{C}_5(e_1, e_2, e_3, e_4) = -\mathcal{C}_5(e_1, e_3, e_2, e_4) = \frac{1}{4}(\lambda^2 - \mu^2)\, cd\,.$$

So, $\mathcal{C}_5$ vanishes iff $\lambda = \mu$ or $\lambda = -\mu$.

Moreover, $\mathcal{C}_6$ vanishes, since one has

$$\rho^*(R - L_3 R)(e_1, e_2) = \rho^*(R - L_3 R)(e_1, e_3) = 0\,.$$

Note also that the values $\rho^*(R - L_3R)(e_1, e_2)$, $\rho^*(R - L_3R)(e_1, e_3)$ determine $\rho^*(R - L_3R)$. Finally, $\mathcal{C}_8$ never vanishes, since using (2.10) we have $\rho(R - L_3R)(e_1, e_1) = 2(\lambda^2 + \mu^2 + \lambda\mu)$.

Remark 3. Table 1 represents a scheme of the statement of Proposition 7. In fact, the first row correponds to the choice of (λ, μ) such that $\lambda, \mu, \lambda + \mu, \lambda - \mu \neq 0$ If $\lambda = 0$ or $\mu = 0$, then the situation corresponding to the second row occurs. In the case $\lambda = -\mu$ the manifold $(G_{(\lambda,-\lambda)}, g_{(\lambda,-\lambda)}, J)$ turns out to be an almost-Kähler manifold such that $\mathcal{C}_5 = 0$. Finally, the last row of Table 1 corresponds to the case $\lambda = \mu$,

Now, let G be the group consisting of the real matrices

$$\begin{pmatrix} 1 & x^1 & x^2 & 0 & -y^1 & -y^2 \\ 0 & 1 & x^3 & 0 & 0 & -y^3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & y^1 & y^2 & 1 & x^1 & x^2 \\ 0 & 0 & y^3 & 0 & 1 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

equipped with the left-invariant metric g such that:

$$\{e_1 = \frac{\partial}{\partial x^1}, e_2 = \frac{\partial}{\partial x^2}, e_3 = x^1\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} + y^1\frac{\partial}{\partial y^2},$$
$$e_4 = \frac{\partial}{\partial y^1}, e_5 = \frac{\partial}{\partial y^2}, e_6 = -y^1\frac{\partial}{\partial x^2} + x^1\frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^3}\}$$

is an orthonormal frame of Lie algebra $\mathfrak{g}$ of G. The only non-vanishing brackets are $[e_1, e_3] = -[e_4, e_6] = e_2$, $[e_1, e_6] = -[e_3, e_4] = e_5$.

According to [2], let J be the almost complex structure on G, defined by

$$J(e_1) = e_4\,, \quad J(e_2) = e_5\,, \quad J(e_3) = e_6\,.$$

Then (G, J) is a complex Lie group and we have

$$(3.3) \qquad [X, JY] = [JX, Y] = J([X, Y]) \qquad X, Y \in \mathfrak{g}\,.$$

In the present case, the covariant derivative $\nabla\omega$ is a section of $\mathcal{W}_3(G)$. Moreover, by means of (3.1), (3.3) and (1.1) we get

$$(3.4) \qquad \tau_3(X,Y,Z) = -g(X,[Y,JZ]) \qquad g(\overline{\nabla}_X Y,Z) = \frac{1}{2}\,(g([X,Y],Z) + g([Z, X], Y))\,.$$

for any X, Y, Z, of $\mathfrak{g}$.

Proposition 8. *The projection $\mathcal{K}_{-2}$ is the only non-vanishing $\mathcal{K}^{\perp}$-component of the curvature of (G, g, J).*

As an immediate consequence of (2.4) and (2.7) we obtain the vanishing of $\mathcal{K}_{-1}$ and $\mathcal{C}_5$. In the present case (2.8) reduces to

$$(\mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8)(X, Y) = \frac{1}{2}\,\{\overline{\nabla}_X \tau_3(Y, Y, JX) + \overline{\nabla}_Y \tau_3(X, X, JY)$$
$$+\overline{\nabla}_{JX} \tau_3(Y, Y, X) + \overline{\nabla}_{JY} \tau_3(X, X, Y) - 2\tau_3(\Sigma(X, Y), X, JY)\} \qquad X, Y \in \mathfrak{g}\,.$$

Moreover, (3.3) and (3.4) imply also $\overline{\nabla}_X(JY) + \overline{\nabla}_{JX} Y = [X, JY]$ and this relations gives

$$(3.5) \qquad \overline{\nabla}_X \tau_3(Y, Y, JX) + \overline{\nabla}_{JX} \tau_3(Y, Y, X) = 0\,.$$

Using (1.3) and (3.4), we get

$$(3.6) \qquad \tau_3(\Sigma(X, Y), X, JY) = \frac{1}{2}\,\{g(X,[[Y, X], Y]) + g(Y,[[X, Y], X]\} = 0$$

since, for any $X, Y \in \mathfrak{g}$, $[X, Y]$ is a linear combination of e_2, e_5 and, for any Z, $[e_2, Z] = [e_5, Z] = 0$. Thus, (3.5) and (3.6) imply the vanishing of $\mathcal{C}_6$, $\mathcal{C}_7$, $\mathcal{C}_8$.

Finally, since $\tau(R) = \tau^*(R)$, the projection $\mathcal{K}_{-2}$ is determined by the tensor field $(\rho - \rho^*)(R + L_3 R)$ and using (2.5) and (3.4) we have

$$(\rho - \rho^*)(R + L_3R)(e_2, e_2) = 2 \sum_{i=1}^{6} \overline{\nabla}_{e_i} \tau_3(e_2, e_2, Je_i) = 4 \sum_{i=1}^{3} \overline{\nabla}_{e_i} \tau_3(e_2, e_2, Je_i)$$
$$= -2 \sum_{i=1}^{3} \sum_{h=1}^{6} g(e_2, [e_h, e_i])^2 = -4 .$$

Now, let J' be the almost complex structure such that

$$J'(e_1) = e_4, \qquad J'(e_2) = -e_5, \qquad J'(e_3) = e_6 .$$

Then, (G, g, J') is a quasi-Kähler manifold, i.e. $\nabla\omega'$ is a section of $\mathcal{W}_1(G) \oplus \mathcal{W}_2(G)$.

Moreover, since

$$\nabla_X(J'Y) + J'(\nabla_X Y) = 0 \tag{3.7}$$

one has $\overline{\nabla}_X Y = 0$ $X, Y \in \mathfrak{g}$. This means that $\overline{\nabla}$ coincides with the Cartan-Schouten connection on (G, g). Therefore, any left-invariant tensor field is $\overline{\nabla}$-parallel.

Proposition 9. *The only non-vanishing $\mathcal{K}^{\perp}$-projections of the curvature of (G, g, J') are $\mathcal{K}_{-1}$ and $\mathcal{K}_{-2}$.*

Since $\nabla\omega'$ is a section of $\mathcal{W}_1(G) \oplus \mathcal{W}_2(G)$, (2.7) and (2.8) imply that $\mathcal{C}_5$ and $\mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8$ depend on the covariant derivative $\overline{\nabla}\tau_2$. But $\overline{\nabla}\tau_2 = 0$, since τ_2 is left-invariant. So, one has: $\mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = \mathcal{C}_8 = 0$.

Moreover, (3.7) implies $\Sigma(X, Y) = -\nabla_X Y + \nabla_Y X = -[X, Y]$ $X, Y \in \mathfrak{g}$. Therefore, applying (2.4) and (1.5), we have

$$(\tau - \tau^*)(R) = -\|\tau_1\|^2 + \frac{1}{2}\|\tau_2\|^2 = 8$$

that is $\mathcal{K}_{-1} \neq 0$.

Analogously, using (2.5) we obtain

$$(\rho - \rho^*)(R + L_3R)(X, Y) = 2 \sum_{i=1}^{6} (\tau_1 + \tau_2)([X, e_i], X, J'e_i) .$$

In particular, $(\rho - \rho^*)(R + L_3R)(e_2, e_2)$ vanishes and so $\mathcal{K}_{-2} \neq 0$.

The results stated in Propositions 8 and 9 are summarized in Table 2.

Table 3 refers to the following class of examples.

Let G be a connected n-dimensional Lie group, equipped with a biinvariant metric g. For sake of simplicity, G is assumed to be simply connected.

We recall that, if $n = 3$, g is an Einstein metric with constant sectional curvature and the Lie algebra $\mathfrak{g}$, up to isomorphisms, is $\boldsymbol{R}^3$ or $su(2)$ ([**16**]).

As in [2], we consider the nearly-Kähler manifold $(G \times G, g', J)$, where g' and J are both left-invariant and are defined by:

$$g'(X^v, Y^v) = g'(X^h, Y^h) = g(X, Y) \qquad g'(X^v, Y^h) = g'(X^h, Y^v) = 0$$

$$J(X^v) = X^h, \qquad J(X^h) = -X^v$$

where, for any X of $\mathfrak{g}$, $X^v = (0, X)$ and $X^h = \dfrac{1}{\sqrt{3}}(2X, X)$.

As a consequence of the relations

$$\begin{aligned} \tau(X^v, Y^v) &= -\tau(X^h, Y^h) = -\frac{1}{6}[X, Y]^v \\ \tau(X^v, Y^h) &= \quad \tau(X^h, Y^v) = \frac{1}{6}[X, Y]^h \end{aligned} \tag{3.8}$$

$(G \times G, g', J)$ is Kähler manifold, iff $\mathfrak{g}$ is abelian.

In the non-abelian case, which can occur if $n \geqslant 3$, for the $\mathcal{K}^{\perp}$-projections of the curvature R' of $G \times G$, one has: $\mathcal{K}_{-1} \neq 0$, and, possibly, only $\mathcal{K}_{-2}$ and $\mathcal{C}_4$ don't vanish. The following lemmas are useful to obtain conditions for the vanishing of $\mathcal{K}_{-2}$.

Lemma 5. *For any* $X, Y \in \mathfrak{g}$, *we have*

$$\begin{aligned} \rho(R')(X^v, Y^v) &= \rho(R')(X^h, Y^h) = \frac{10}{9}\rho(R)(X, Y) \\ \rho(R')(X^v, Y^h) &= \rho(R')(X^h, Y^v) = 0\,. \end{aligned} \tag{3.9}$$

A direct computation gives

$$\begin{aligned} &R'(X^v, Y^v, Z^v) = R(X, Y, Z)^v \qquad R'(X^h, Y^h, Z^h) = R(X, Y, Z)^h \\ &R'(X^h, Y^v, Z^v) = \frac{1}{3}\{R(X, Y, Z) + \frac{2}{3}R(Z, X, Y)\}^h \qquad\qquad X, Y, Z \in \mathfrak{g}\,. \end{aligned} \tag{3.10}$$

With respect to an orthonormal frame $\{e_i^v, e_i^h\}_{1 \leqslant i \leqslant n}$ of $\mathfrak{g} \oplus \mathfrak{g}$, where $\{e_i\}_{1 \leqslant i \leqslant n}$ is an orthonormal frame of $\mathfrak{g}$, using (3.10), we obtain

$$\rho(R')(X^v, X^v) = \frac{10}{9}\rho(R)(X, X) \qquad \rho(R')(X^v, Y^h) = 0.$$

These relations and the J-invariance of $g(R')$ imply (3.9).

Lemma 6. *The $*$-Ricci tensor of $(G\times G, g', J)$ is given by*

$$\rho^*(R') = \frac{1}{5}\rho(R'). \tag{3.11}$$

Indeed, choosing the same orthonormal frame as in Lemma 5 and applying formulas (2.3), (1.5), (1.3), (1.4), (3.8), (3.9), we have

$$2(\rho-\rho^*)(R')(X^v, Y^v) = -8\sum_{i=1}^{n}\tau_1(\tau(Y^v, e_i^v), X^v, e_i^h)$$
$$= -\frac{4}{9}\sum_{i=1}^{n} g([[Y, e_i], X], e_i) = \frac{16}{9}\rho(R)(X, Y) = \frac{8}{5}\rho(R')(X^v, Y^v)$$

$$2(\rho-\rho^*)(R')(X^h, Y^v) = -8\sum_{i=1}^{n}\tau_1(\tau(Y^v, e_i^v), X^h, e_i^h)$$
$$= \frac{8}{3}\sum_{i=1}^{n} g'(\tau([Y, e_i]^v, X^h), e_i^v) = 0\,.$$

These relations, together with the J-invariance of $\rho(R')$ and $\rho^*(R')$ imply (3.11).

An immediate consequence of Lemmas 5, 6 is

Proposition 10. *The following statements are equivalent*

i. *(G, g) is an Einstein manifold*

ii. *$(G\times G, g')$ is an Einstein manifold*

iii. *$(G\times G, g', J)$ is a $*$-Einstein manifold*

iv. *The projection $\mathcal{K}_{-2}$ of R' vanishes.*

As far as regards the projection $\mathcal{C}_4$ of R', it vanishes if $n=3$. Moreover, when $n=3$, $SU(2)$ is, up to isomorphisms, the only simply connected, connected and not abelian Lie group which admits biinvariant metrics. So, $SU(2)\times SU(2)$ is the model corresponding to the first row of Table 3. The projection $\mathcal{C}_4$ vanishes also when $n=4$ (see Proposition 2).

The group $\boldsymbol{R}\times SU(2)$, equipped with the product metric of biinvariant metrics on each of the factors, is an example of a non-Einstein 4-dimensional manifold. Therefore, the product manifold $(G\times G, g', J)$, with $G=\boldsymbol{R}\times SU(2)$, is a model of the nearly-Kähler manifolds considered in the second row of Table 3. As a consequence of the following proposition, we realize that this manifold is the only model.

First of all, using Lemmas 1, 2, 5 and 6 we get

$$\begin{aligned}\mathcal{C}_4(X^v, Y^v) = &-\frac{1}{12}\|[X,Y]\|^2 - \frac{1}{3(n-2)}(\|X\|^2\rho(R)(Y,Y) + \|Y\|^2\rho(R)(X,X) \\ &-2g(X,Y)\rho(R)(X,Y)) + \frac{\tau(R)}{3(n-1)(n-2)}(\|X\|^2\|Y\|^2 - g(X,Y)^2).\end{aligned} \tag{3.12}$$

Proposition 11. *Let G be a connected, simply connected non-abelian Lie group, equipped with a biinvariant metric g. If the projection $\mathcal{C}_4$ of R' vanishes, then G is 3 or 4-dimensional and the manifold (G, g) is isometric to $(SU(2), g_2)$ or to $(\mathbf{R} \times SU(2), g_1 \times g_2)$, where g_1, g_2 are biinvariant metrics on $\mathbf{R}$, $SU(2)$, respectively.*

First of all, G is assumed to be a simple Lie group. Then, (G, g) is an Einstein manifold, since we have $g(X, Y) = \lambda B(X, Y)$ $X, Y \in \mathfrak{g}$, where $\lambda \in \mathbf{R}^*$ and B is the Killing form of $\mathfrak{g}$.

Applying (3.12), the vanishing of $\mathcal{C}_4$ implies that (G, g) has constant sectional curvature given by $K = \dfrac{\tau(R)}{n(n-1)}$, where $n = \dim G$. Therefore, we get

$$\begin{aligned}&-\frac{1}{4}g([X,Y],[Z,W]) = R(X,Y,Z,W) \\ &= \frac{\tau(R)}{n(n-1)}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}.\end{aligned} \tag{3.13}$$

Since G is not abelian, the scalar curvature $\tau(R)$ does not vanish.

Let $\{e_i\}_{1 \leq i \leq n}$ be an orthonormal frame on $\mathfrak{g}$. Then (3.13) implies that the $\dfrac{n(n-1)}{2}$ vectors $\{[e_i, e_p]\}_{1 \leq i < p \leq n}$ are linearly independent. Therefore, one has $n = 3$ and (G, g) is isometric to $(SU(2), g_2)$.

If G is not simple, we consider the orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, with $m = \dim \mathfrak{z} \geq 1$.

Then $G, g)$ is isometric to the manifold $(\mathbf{R}^n \times H, g_1 \times g_2)$, where H is a compact Lie group equipped with a biinvariant metric g_2.

Now, we prove that $m = 1$. In fact, if $m \geq 2$, for a choice of an orthonormal pair (X, Y) in $\mathfrak{z}$, we have $\mathcal{C}_4(X^v, Y^v) = \dfrac{\tau(R)}{3(n-1)(n-2)}$. So, the hypothesis $\mathcal{C}_4 = 0$ implies $\tau(R) = 0$. Moreover, the hypothesis $\mathcal{C}_4 = 0$ and (3.12) yield $\|X\|^2\rho(R)(Y, Y) = 0$, $X \in \mathfrak{z}$, $Y \in [\mathfrak{g}, \mathfrak{g}]$.

Thus, $\rho(R)$ vanishes, and, using again (3.12), one has: $[X, Y]=0$, $X, Y \in [\mathfrak{g}, \mathfrak{g}]$. This proves that $\mathfrak{g}$ is an abelian Lie algebra, contradicting the hypothesis.

Moreover, (3.12) yields $\|X\|^2\{\frac{\tau(R)}{n-1}\|Y\|^2-\rho(R)(Y,Y)\}=0$, $X\in\mathfrak{z}$, $Y\in[\mathfrak{g},\mathfrak{g}]$. This proves that g_2 is an Einstein metric.

Applying the condition $\mathcal{C}_4(X^v, Y^v)=0$ to the pairs (X, Y) of vectors in $[\mathfrak{g}, \mathfrak{g}]$, we find that (H, g_2) has constant sectional curvature. Then, using the same argument as in the previous case, we have $n-1=\dim[\mathfrak{g}, \mathfrak{g}] \leqslant 3$. If $n=3$, G should be an abelian or a simple Lie group. So, $n=4$ and H is isomorphic to $SU(2)$.

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Sommario

Si studiano le proiezioni della curvatura riemanniana su opportuni sottofibrati del fibrato dei campi tensoriali di curvatura algebrici di una varietà quasi hermitiana. Si ottengono quindi proprietà di curvatura per alcune classi di varietà quasi hermitiane e per opportuni gruppi di Lie.

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