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Bifurcation and non-uniqueness in electrohydrodynamics (**)

1 - Introduction

Bulk motions can be induced in a slightly conducting fluid by application of an electric field. We refer to Turnbull [14] for the description of a related experiment.

There are basically two mechanisms responsible for the fluid's motion. The first one is the existence of a free surface corresponding to sharp discontinuities in electric conductivity and permittivity. For an experiment illustrating this situation we refer to Melcher and Taylor [7]. The second is of intrinsic nature: significant electric body forces are generated in the fluid if the conductivity varies with the temperature (Roberts [9] and Bradley [3]). This case requires the presence of a temperature gradient, in a situation which is reminiscent of the classical Benard's instability. The electric body forces to add in the equations of motion are of the form $q\boldsymbol{E}$, where q is the charge density and $\boldsymbol{E}$ the electric field; these quantities depend on the temperature in a rather involved way. One relevant experimental finding is that the onset of the electric instability does not depend on the polarity of the applied voltage.

Goal of this paper is to prove that the solution of the equations governing the problem is in certain cases non unique. This shall be done by showing that the linearized probem has a positive simple eigenvalue which corresponds physically to a special value of a non-dimensional *electric* parameter. As a consequence it becomes possible to apply a theorem of M. A. Krasnosel'skii [6] on bifurcation from odd eigenvalues, thus proving non-uniqueness.

Our approach is similar to the treatment given by W. Velte [15], [16] to the classical Benard and Taylor's problems.

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2 – Governing equations and boundary conditions

As constitutive relation for the current density $\boldsymbol{J}$ we assume, using the *SI* units system

$$\boldsymbol{J} = \sigma(T)\boldsymbol{E} - k\nabla q \tag{2.1}$$

where $k > 0$ is a diffusion coefficient and $\sigma(T)$ the electric conductivity, a given function of the temperature T, which we assume to be of the form $\sigma(T) = \sigma_0[1 + \alpha\,\tau(T)]$ with α a positive parameter and $\tau(T)$ a regular function such that

$$\tau(T) > 0 \qquad \tau'(T) > 0 \qquad \tau''(T) > 0\,. \tag{2.2}$$

Convection of charge is neglected in (2.1). Moreover k or, better to say, the non-dimensional group β^{-1} defined later, is in practice very small. As usual in electrohydrodynamics [**11**] magnetic effects are neglected and, since we assume steady conditions, we can write

$$-\varepsilon\Delta V = q \qquad \boldsymbol{E} = -\nabla V \tag{2.3}$$

where V is the electric potential and ε the permittivity. By the law of conservation of charge we have

$$\nabla\cdot\boldsymbol{J} = 0\,. \tag{2.4}$$

Substituting (2.1) into (2.4), we obtain, taking into account (2.3)

$$-k\varepsilon\Delta^2 V + \sigma(T)\,\Delta V + \sigma'(T)\,\nabla T\cdot\nabla V = 0\,.$$

The fluid is supposed to be incompressible and gravitational body forces are neglected. Hence we have

$$\nabla\cdot\boldsymbol{v} = 0 \qquad \rho\boldsymbol{v}_{x_k}v_k = -\nabla p + \mu\Delta\boldsymbol{v} + q\boldsymbol{E}\,. \tag{2.5}$$

The last term in (2.5) represents, when ε is a constant, the electric force [**11**], ρ and μ are respectively the mass density and the viscosity, in this scheme both given constants.

Ignoring Joule heating, the energy equation becomes

$$-\kappa\Delta T + \rho c_v\nabla T\cdot\boldsymbol{v} = 0$$

with κ thermal conductivity and c_v specific heat. The motion is assumed to be plane with $U(X, Y)$ and $W(X, Y)$ denoting the non-vanishing components of the velocity in the X and Y directions. We introduce the stream function $\Phi(X, Y)$ which satisfies $\Phi_X = -W$, $\Phi_Y = U$.

We require Φ, V and T to be periodic in X with period L. Therefore we arrive to the following *Problem 1*, which is stated in the set

$$\{(X, Y) \mid -\infty < X < \infty,\ 0 < Y < D\}$$

$$\mu\Delta^2\Phi = \rho(\Delta\Phi_X\Phi_Y - \Delta\Phi_Y\Phi_X) + \varepsilon(\Delta V_X V_Y - \Delta V_Y V_X) \tag{2.6}$$

$$-k\varepsilon\Delta^2 V + \sigma(T)\Delta V + \sigma'(T)\nabla T\cdot\nabla V = 0 \tag{2.7}$$

$$\kappa\Delta T = \rho c_v(T_X\Phi_Y - T_Y\Phi_X) \tag{2.8}$$

$$\Phi(X, 0) = \Phi(X, D) = \Phi_Y(X, 0) = \Phi_Y(X, D) = 0 \tag{2.9}$$

$$V(X, 0) = V_0 \quad V(X, D) = V_1 \quad -\varepsilon V_{YY}(X, 0) = q_0 \quad -\varepsilon V_{YY}(X, D) = q_1 \tag{2.10}$$

$$T(X, 0) = T_0 \quad T(X, D) = T_1\,. \tag{2.11}$$

From the mathematical point of view a second condition on V is needed since (2.7) is of the fourth order by the presence of the diffusion term. With the last two equations of (2.10) the volume charge density is prescribed on the two conducting planes, which bound the region filled with the fluid and to which the difference of potential is applied. Physically, this is what is called an *injection law* and, although with difficulties, a corresponding set-up can be realized in practice.

Remark 1. The solutions of Problem 1 have an elementary, but basic property of symmetry. In fact if $(\Phi(X, Y), V(X, Y), T(X, Y))$ is any solution of Problem 1 and if in (2.10), we change V_i and q_i in $-V_i$ and $-q_i$, then the solution of the corresponding problem is given by $(\Phi(X, Y), -V(X, Y), T(X, Y))$.

There is of course no loss of generality in assuming that $V_0 = 0$ and that the scale of temperature is such that $T_0 = 0$. Moreover, to consider a specific case, we assume that the upper plane is hot i.e. $T_1 > 0$.

Problem 1 has always the motionless solution (Φ^*, T, V^*), where $\Phi^* = 0$, $T^* = T_1 D^{-1} Y$, and $V^*(Y)$ is the unique solution of the problem:

$$-k\varepsilon\frac{d^4V^*}{dY^4} + \frac{d}{dY}\left(\sigma(T^*)\frac{dV^*}{dY}\right) = 0$$

$$V^*(0) = 0 \quad V^*(D) = V_1 \quad \varepsilon\frac{d^2V^*}{dY^2}(0) = -q_0 \quad \varepsilon\frac{d^2V^*}{dY^2}(D) = -q_1\,.$$

We would like to prove that in certain cases (Φ^*, T^*, V^*) is not the unique solution.

The basic approximation used in this paper consists in assuming $\nabla T \approx \nabla T^*$ and $\nabla V \approx \nabla V^*$ in the last term of (2.7). There is no obvious physical justification of this hypothesis. However in this way it becomes relatively easy to prove non-uniqueness. Thus we shall consider *Problem 2*:

$$\mu\Delta^2\Phi = \rho(\Delta\Phi_X\Phi_Y - \Delta\Phi_Y\Phi_X) + \varepsilon(\Delta V_X V_Y - \Delta V_Y V_X) \tag{2.12}$$

$$-k\varepsilon\Delta^2 V + \sigma(T)\Delta V + \sigma'(T)\frac{T_1}{D}\frac{\partial V^*}{\partial Y} = 0 \tag{2.13}$$

$$\kappa\Delta T = \rho c_v(T_X\Phi_Y - T_Y\Phi_X)\,. \tag{2.14}$$

The boundary conditions are still given by (2.9)-(2.11). To rewrite Problem 2 in non-dimensional form, we introduce the following non-dimensional parameters

$$\beta = \frac{\sigma_0 D^2}{k\varepsilon} \qquad \delta_i = -\frac{D^2 q^i}{\varepsilon V_1} \qquad \gamma = \frac{\nu}{c} \qquad \eta = \frac{\varepsilon V_1^2}{\rho\nu^2} \qquad i = 0,\,1$$

where $\nu = \mu\rho^{-1}$ is the kinematic viscosity and $c = \kappa(\rho c_v)^{-1}$ the diffusivity. As basic units for length, stream function, temperature, potential and electric conductivity we take respectively D, ν, T_1, V_1 and σ_0 and define $v(x, y)$, $\phi(x, y)$, $t(x, y)$, $s(t)$, $v^*(y)$ and $t^*(y)$ as follows

$$V_1 v(x, y) = V^*(Dx, Dy) \qquad \nu\phi(x, y) = \Phi(Dx, Dy)$$

$$T_1 t(x, y) = T(Dx, Dy) \qquad s(t) = 1 + \alpha n(t) \quad \text{with } n(t) = \tau(T_1 t)$$

$$V_1 v^*(y) = V(Dy) \qquad t^*(y) = y\,.$$

Let $\psi = v(x, y) - v^*(y)$ and $\theta(x, y) = t(x, y) - t^*(y)$. The non-dimensional form of Problem 2 with homogeneous boundary conditions is given by *Problem 3*:

$$\Delta^2\phi = \Delta\phi_x\phi_y - \Delta\phi_y\phi_x + \eta(\Delta\psi_x\psi_y - \Delta\psi_y\psi_x) + \eta(\Delta\psi_x v_y^* - v_{yyy}^*\psi_x) \tag{2.15}$$

$$\Delta^2\psi - \beta s(\theta + y)\Delta\psi = \beta[s(\theta + y)v_{yy}^* + s'(\theta + y)v_y^*] - vv_{yyyy}^* \tag{2.16}$$

$$\Delta\theta = \gamma(\theta_x\phi_y - \theta_y\phi_x - \phi_x) \tag{2.17}$$

(2.18) $$\phi(x, 0) = \phi(x, 1) = \phi_y(x, 0) = \phi_y(x, 1) = 0$$

(2.19) $$\psi(x, 0) = \psi(x, 1) = \psi_{yy}(x, 0) = \psi_{yy}(x, 1) = 0$$

(2.20) $$\theta(x, 0) = \theta(x, 1) = 0$$

(2.21) $$\phi,\ \psi \text{ and } \theta \text{ periodic with period } \mathcal{L} = \mathrm{L}\mathrm{D}^{-1}\,.$$

As a further restriction to the class of admissible functions we require

(2.22) $$\phi(x, y) = -\phi(-x, y) \qquad \psi(x, y) = \psi(-x, y) \qquad \theta(x, y) = \theta(-x, y)\,.$$

The reason for assuming (2.22) is the simplification introduced in this way in the study of the linearized equation. The problem defining $v^*(y)$ is now

(2.23) $$-\frac{\mathrm{d}^4 v^*}{\mathrm{d}y^4} + \beta \frac{\mathrm{d}}{\mathrm{d}y}\left(s(y)\frac{\mathrm{d}v^*}{\mathrm{d}y}\right) = 0$$

(2.24) $$v^*(0) = 0 \qquad v^*(1) = 1 \qquad \frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2}(0) = \delta_0 \qquad \frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2}(1) = \delta_1\,.$$

Conditions (2.2) translate into

(2.25) $$n(t) > 0 \qquad \frac{\mathrm{d}n}{\mathrm{d}t}(t) > 0 \qquad \frac{\mathrm{d}^2 n}{\mathrm{d}t^2}(t) > 0\,.$$

The bifurcation parameter in Problem 3 shall be η. Let

(2.26) $$r(y) = \frac{\mathrm{d}^2 s}{\mathrm{d}y^2}(y)\frac{\mathrm{d}v^*}{\mathrm{d}y}(y) + \frac{\mathrm{d}s}{\mathrm{d}y}(y)\frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2}(y)$$

where $v^*(y)$ is the solution of problem (2.23), (2.24). To prove non-uniqueness we make the key assumption that the parameters α, β and δ_i are such that the following *Condition* **I** holds

I. $$\frac{\mathrm{d}v^*}{\mathrm{d}y}(y) \geqslant 0 \qquad \frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3}(y) \geqslant 0 \qquad r(y) > 0 \qquad \text{for all } y \in [0, 1]\,.$$

In the next lemma we prove an elementary regular perturbation result, which shall permit to prove the existence of a non-empty region $\mathcal{P}$ of $\boldsymbol{R}^4$ such that Condition **I** holds if $(\alpha, \beta, \delta_1, \delta_2) \in \mathcal{P}$.

Lemma 1. *Let* $v^*(y)$ *be the solution of problem* (2.23), (2.24), *where* $s(y) = 1 + \alpha\, n(y)$ *and* $n(y)$ *satisfies* (2.25). *Put* $\alpha = 0$ *in* (2.23), (2.24) *and as-*

sume $v_0(y)$ *to be the solutions of the corresponding problem*

$$-\frac{d^4 v_0}{dy^4} + \beta \frac{d^2 v_0}{dy^2} = 0 \tag{2.27}$$

$$v_0(0) = 0 \qquad v_0(1) = 1 \qquad \frac{d^2 v_0}{dy^2}(0) = \delta_0 \qquad \frac{d^2 v_0}{dy^2}(1) = \delta_1 . \tag{2.28}$$

Then there exists a constant $C > 0$, *not depending on* α *and* β *such that*

$$\max \left|\frac{dv^*}{dy} - \frac{dv_0}{dy}\right| \leq C\alpha\beta, \quad \max \left|\frac{d^2 v^*}{dy^2} - \frac{d^2 v_0}{dy^2}\right| \leq C\alpha\beta, \quad \max \left|\frac{d^3 v^*}{dy^3} - \frac{d^3 v_0}{dy^3}\right| \leq C\alpha\beta .$$

Proof. Let us multiply (2.27) by $v^* - v_0$. Integrating by parts over $[0, 1]$, one easily obtains

$$\int_0^1 \left|\frac{d^2 v^*}{dy^2}\right|^2 dy \leq C_1 \tag{2.29}$$

where the constant C_1 does not depend on α and β. Define $u = v^* - v_0$. By difference from (2.23) and (2.27) we have

$$-\frac{d^4 u}{dy^4} + \beta \frac{d^2 u}{dy^2} = -\beta\alpha \frac{d}{dy}\left(n(y) \frac{dv^*}{dy}\right) \tag{2.30}$$

$$u(0) = u(1) = \frac{d^2 u}{dy^2}(0) = \frac{d^2 u}{dy^2}(1) = 0 . \tag{2.31}$$

Multiplying (2.30) by u, integrating by parts and recalling (2.29) we have

$$\int_0^1 \left|\frac{du}{dy}\right|^2 dy \leq C_2 \alpha\beta \qquad \int_0^1 \left|\frac{d^2 u}{dy^2}\right|^2 dy \leq C_2 \alpha\beta .$$

In the same way, after multiplication of (2.30) by $\dfrac{d^4 u}{dy^4}$ we get, again by (2.29)

$$\int_0^1 \left|\frac{d^3 u}{dy^3}\right|^2 dy \leq C_3 \alpha\beta \qquad \int_0^1 \left|\frac{d^4 u}{dy^4}\right|^2 dy \leq C_3 \alpha\beta .$$

Hence the conclusion of Lemma 1, follows.

We exhibit in the following example a case, in which the solution $v^*(y)$ of problem (2.23), (2.24) satisfies Condition **I**.

Example. Let $\delta_0 = 0$ and $\delta_1 = \delta > 0$. The solution of problem (2.27), (2.28) can easily be computed in closed form and is given by

$$v_0(y) = \frac{\delta \sinh(y\sqrt{\beta})}{\beta \sinh\sqrt{\beta}} + y\left(1 - \frac{\delta}{\beta}\right).$$

We have in $[0, 1]$ $\min \frac{dv_0}{dy}(y) = \frac{\delta}{\sqrt{\beta}\sinh\sqrt{\beta}} + 1 - \frac{\delta}{\beta}$.

Therefore we obtain

$$\frac{dv_0}{dy}(y) > 0 \quad \text{if } \delta < \mathcal{F}(\beta), \qquad \text{where } \mathcal{F}(\beta) = \left(\frac{1}{\beta} - \frac{1}{\sqrt{\beta}\sinh\sqrt{\beta}}\right)^{-1}.$$

It is easily seen that $\mathcal{F}(\beta) \to 6$ as $\beta \to 0+$ and $\mathcal{F}'(\beta) > 0$ when $\beta > 0$. On the other hand we have $\left(\frac{d^2 v_0}{dy^2}\right)(y) > 0$ in $(0, 1]$ and $\frac{d^3 v_0}{dy^3}(y) > 0$ in $[0, 1]$ for all $\beta > 0$ and $\delta > 0$.

Define $r_0(y) = \frac{d^2 s}{dy^2}\frac{dv_0}{dy} + \frac{ds}{dy}\frac{d^2 v_0}{dy^2}$. We have $r_0(y) > m \geq 0$. Hence there exists a constant C^* such that if $\alpha\beta < C^*$ and $\delta < \mathcal{F}(\beta)$, Condition **I** holds by Lemma 1.

3 - Functional form of Problem 3

Let $C^\infty_{0p}(\Omega)$, $C^\infty_{0d}(\Omega)$ denote the sets of functions of class C^∞, defined in the set

$$\Omega = \{(x, y) \mid -\infty < x < \infty, 0 \leq y \leq 1\}$$

periodic in x with period l, which vanish near $y = 0$ and $y = 1$ and are respectively even and odd in x. Define in $C^\infty_{0p}(\Omega)$ and $C^\infty_{0d}(\Omega)$ the *norm*

$$\|u\| = \left(\int_{\Omega(\mathcal{L})} |\nabla u|^2 \, dx\, dy\right)^{\frac{1}{2}} \tag{3.1}$$

where $\Omega(\mathcal{L}) = \{x, y) \mid 0 < x < \mathcal{L}, 0 < y < 1\}$ and call $H^1_{0p}(\Omega)$, $H^1_{0d}(\Omega)$ the completion of $C^\infty_{0p}(\Omega)$, $C^\infty_{0d}(\Omega)$ with respect to (3.1). The Sobolev spaces $H^k_{0p}(\Omega)$, $H^k_{0d}(\Omega)$ of functions which are periodic in x and respectively even and odd in x can be defined in a similar way. Let $\mathcal{B} = H^2_{0d}(\Omega) \cap H^3(\Omega)$. Fix $\phi \in \mathcal{B}$ and consider the *uncoupled boundary value problem*

$$\Delta^2\omega = \Delta\phi_x\phi_y - \Delta\phi_y\phi_x + \eta(\Delta\psi_x\psi_y - \Delta\psi_y\psi_x) + \eta(\Delta\psi_x v_y^* - v_{yyy}^*\psi_x) \tag{3.2}$$

$$\Delta^2\psi - \beta s(\theta + y)\Delta\psi = \beta[s(\theta + y)v_{yy}^* + s'(\theta + y)v_y^*] - v_{yyyy}^* \tag{3.3}$$

$$\Delta\theta = \gamma(\theta_x\phi_y - \theta_y\phi_y - \phi_x) \tag{3.4}$$

with $\omega \in H^2_{0d}(\Omega)$, $\psi \in H^1_{0p}(\Omega) \cap H^2(\Omega)$, $\psi_{yy} = 0$ on $y = 0$ and $y = 1$, $\theta \in H^1_{0p}(\Omega)$.

By the results of the theory of linear elliptic equations, (3.2)-(3.4) has, for every given $\phi \in \mathcal{B}$, a unique solution. Together with (3.2)-(3.4) we need the linear problem stated below, where ϕ again is a given function in $\mathcal{B}$

$$\Delta^2\omega = \Delta\psi_x v_y^* - v_{yyyy}^*\psi_x \qquad \omega \in H^2_{0d}(\Omega) \tag{3.5}$$

$$\begin{aligned} &\Delta^2\psi - \beta s(y)\Delta\psi = \beta r(y)\theta \\ &\psi \in H^1_{0p}(\Omega) \cap H^2(\Omega) \qquad \psi_{yy} = 0 \qquad \text{on } y = 0 \qquad \text{and } y = 1 \end{aligned} \tag{3.6}$$

$$\Delta\theta = -\gamma\phi_x \qquad \theta \in H^1_{0p}(\Omega) \tag{3.7}$$

where $r(y)$ is given by (2.26). Problems (3.2)-(3.4) and (3.5)-(3.7) are related in the following way

Lemma 2. *Let β and γ be given positive constants. Then we have*

i) *Equations* (3.2)-(3.4) *define an operator* $\omega = \mathrm{A}(\eta, \phi)$ *from* $\boldsymbol{R} \times \mathcal{B}$ *into* $\mathcal{B}$.

ii) *Problem 3 is equivalent to the functional equation* $\phi = \mathrm{A}(\eta, \phi)$.

iii) *The operator* A *is compact.*

iv) *Equations* (3.5)-(3.7) *define a linear operator* $\omega = \mathrm{B}(\phi)$.

v) *The Frechet derivative of* $\mathrm{A}(\eta, \phi)$ *in* $\phi = 0$ *is of the form* $\eta\mathrm{B}$.

Proof.

i) Equation (3.4) is linear in θ. Thus the existence of a weak solution is guaranteed by standard results. The right hand side of equation (3.3) belongs to $L^2(\Omega)$. This permits to solve (3.3) and to obtain in particular $\psi \in H^4$. Finally recalling that $\phi \in \mathcal{B}$, we deduce that the right hand side of equation (3.2) is in $L^2(\Omega)$. We conclude that (3.2) is solvable and define $\mathrm{A}(\eta, \phi)$.

ii) A fixed point of the operator A gives a solution $\{\phi, \psi, \theta\}$ to Problem 3 if ψ and θ are computed using equations (2.16), (2.19) and (2.17), (2.20).

iii) If $\{\phi_k\}$ is a sequence bounded in $\mathcal{B}$, we infer, by results of regularity for elliptic equations, that $\{\omega_k\}$ is bounded in $H^4(\Omega)$. Therefore A is compact by Rellich's theorem.

iv) The operator $\omega = \mathrm{B}(\phi)$ is well-defined again by the linear theory.

v) Equations (3.2), (3.7) are obtained from (3.2) and (3.4) neglecting the quadratic part.

Thus the result follows.

We quote below a theorem of M. A. Krasnosel'skii [6] on which our proof of non-uniqueness rests.

Theorem 1. *Let the continuous and compact operator* $\mathrm{A}(\lambda, \phi)$ *be defined in* $\boldsymbol{R} \times \boldsymbol{X}$, *where* $\boldsymbol{X}$ *is a Banach space and* $\mathrm{A}(\lambda, 0) = 0$. *Assume the Frechet derivative of* A *in* $\phi = 0$ *to be of the form* $\lambda\mathrm{B}$, *where the linear operator* B *does not depend on* λ. *Let* λ_0 *be an eigenvalue of* B *with odd (algebraic) multiplicity. Then* λ_0 *is a bifurcation point for the equation* $\phi = \mathrm{A}(\lambda, \phi)$.

To apply this result we need to study the following linear eigenvalue *Problem 4* corresponding to (3.5)-(3.7). We put in these equations $\beta = 1$ and $\gamma = 1$, since these constants are inessential in our considerations.

$$(3.8) \qquad \Delta^2\phi = \eta(\Delta\psi_x v_y^* - v_{yyy}^*\psi_x) \qquad \Delta^2\psi - s(y)\Delta\psi = r(y)\theta \qquad \Delta\theta = -\phi_x$$

$$\phi(x, 0) = \phi(x, 1) = \phi_y(x, 0) = \phi_y(x, 1) = 0$$

$$\psi(x, 0) = \psi(x, 1) = \psi_{yy}(x, 0) = \psi_{yy}(x, 1) = 0 \qquad \theta(x, 0) = \theta(x, 1) = 0$$

ϕ, ψ and θ are periodic in x with period $\mathcal{L} = \mathrm{LD}^{-1}$, ϕ is odd in x, ψ and θ are even in x.

We claim that Problem 4 has a positive eigenvalue of algebraic multiplicity equal to one. Since all solutions of Problem 4 are of class C^∞ we can write, with $a = 2\pi L^{-1}$

$$(3.9) \qquad \phi(x, y) = \sum_{k=1}^{\infty} \phi_k(y) \sin(kax) \qquad \psi(x, y) = \psi_0(y) + \sum_{k=1}^{\infty} \psi_k(y) \cos(kax)$$

$$\theta(x, y) = \theta_0(y) + \sum_{k=1}^{\infty} \theta_k(y) \cos(kax).$$

Substituting (3.9) into (3.8), the following hierarchy of eigenvalue problems for ϕ_k, ψ_k and θ_k is easily obtained

(3.10) $$\frac{d^4\psi_0}{dy^4} - s(y)\frac{d^2\psi_0}{dy^2} = r(y)\theta_0 \qquad \frac{d^2\theta}{dy^2} = 0$$

(3.11) $$M_k^2\phi_k = \eta ak[(M_k\psi_k)\frac{dv^*}{dy} + \psi_k\frac{d^3v^*}{dy^3}]$$

(3.12) $$M_k^2\psi_k + s(y)M_k\psi_k = r(y)\theta_k \qquad M_k\theta_k = ak\phi_k$$

(3.13) $$\phi_k(0) = \phi_k(1) = \frac{d\phi_k}{dy}(0) = \frac{d\phi_k}{dy}(1) = 0$$

(3.14) $$\psi_k(0) = \psi_k(1) = \frac{d^2\psi_k}{dy^2}(0) = \frac{d^2\psi_k}{dy^2}(1) = 0 \qquad \theta_k(0) = \theta_k(1) = 0$$

where $M_k = -\frac{d^2}{dy^2} + a^2k^2$ and $k = 1, 2, 3, \ldots$.

We find immediately $\psi_0 = 0$, $\theta_0 = 0$. Moreover, as a preliminary step to the study of Problem 4, we prove that each problem (3.11)-(3.14) has a positive and simple eigenvalue. To this end we invoke a theorem of M. G. Krein and M. A. Rutman [**1**], which, to make the paper self-contained, we quote below.

Theorem 2. *Let X be a Banach space and C a convex cone with vertex 0. Suppose C closed,* $\operatorname{int} C \neq \emptyset$ *and* $C \cap (-C) = \{0\}$. *Let* $T \in \mathcal{L}(X, X)$ *be compact and such that* $T(C\backslash\{0\})$ *be contained in* $\operatorname{int} C$. *Then there exists* $\phi \in \operatorname{int} C$ *and* $\eta_1 > 0$ *such that* $\eta_1 T\phi = \phi$. *The algebraic and geometric multiplicity of* η_1 *is* 1 *and we have* $\eta_1 < |\eta|$ *for any other characteristic value of* T.

Let us consider the operator $\omega = T\phi$, defined by the following problem in which the index k is omitted

(3.15) $$M^2\omega = ak(\frac{dv^*}{dy}M\psi + \psi\frac{d^3v^*}{dy^3}) \qquad \omega(0) = \omega(1) = \frac{d\omega}{dy}(0) = \frac{d\omega}{dy}(1) = 0$$

(3.16) $$M^2\psi + s(y)M\psi = r(y)\theta \qquad \psi(0) = \psi(1) = \frac{d^2\psi}{dy^2}(0) = \frac{d^2\psi}{dy^2}(1) = 0$$

(3.17) $$M\theta = ak\phi \qquad \theta(0) = \theta(1) = 0.$$

Hereafter we suppose that Condition **I** holds. Let C be the cone of the functions of $H_0^2([0,1])$ which are non-negative. Assume $\phi \in C \backslash \{0\}$. We have $\theta > 0$ in $(0, 1)$ from (3.17) by the one dimensional maximum principle. Let $\mathrm{M}\psi = w$. By (3.16) we have $\mathrm{M}w + s(y)w = r(y)\theta$, $w(0) = w(1) = 0$. Since $r(y) > 0$ by Condition **I** and $s(y) > 0$ by (2.1), we obtain $\mathrm{M}\psi > 0$ in $(0, 1)$. But $\mathrm{M}\psi = w$, $\psi(0) = \psi(1) = 0$ and therefore also $\psi > 0$. Recalling that $\frac{\mathrm{d}v^*}{\mathrm{d}y} \geqslant 0$ and $\frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3} \geqslant 0$, it is possible to apply to problem (3.15) a theorem of K. Kirchgässner (see [**13**] page 236) which implies $\omega > 0$ in $(0, 1)$. Thus T satisfies the hypotheses of Theorem 2 and we have

Lemma 3. *Problem* (3.11)-(3.14) *has, for every k, a positive and simple eigenvalue η_k^1 such that $\eta_k^1 < |\eta_k|$ for any other eigenvalue η_k.*

Now every simple eigenvalue of (3.11)-(3.14) is also eigenvalue of (3.8), but not necessarily a simple one. It may happen $\eta_n^1 = \eta_m^1$ with $n < m$. We want to prove that it is possible to choose the period $\mathcal{L}$ in such a way to avoid this situation. To this goal, we need the following

Lemma 4. *Let η_k be any positive eigenvalue of* (3.11)-(3.14). *Assume $\alpha \leqslant \min(b_0, 1)$ where $b_0 = \min\{b(y) \mid y \in [0, 1]\}$ and $b(y) = (r(y))^{-1}$. Then one has $\lim_{k\to\infty} \eta_k = \infty$.*

Proof. After the substitution $\phi_k = \sqrt{\eta}\,\xi_k$ and the change of notation $\lambda = ak\sqrt{\eta}$, the eigenvalue problem (3.11)-(3.14) becomes, omitting the index k,

$$(3.18) \quad \mathrm{M}^2\xi = \lambda\Big(\frac{\mathrm{d}v^*}{\mathrm{d}y}\mathrm{M}\psi + \psi\frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3}\Big) \qquad \mathrm{M}^2\psi + s(y)\mathrm{M}\psi = r(y)\theta \qquad \mathrm{M}\theta = \lambda\xi$$

where:

$$\xi(0) = \xi(1) = 0 \qquad \frac{\mathrm{d}\xi}{\mathrm{d}y}(0) = \frac{\mathrm{d}\xi}{\mathrm{d}y}(1) = 0$$

$$\psi(0) = \psi(1) = 0 \qquad \frac{\mathrm{d}^2\psi}{\mathrm{d}y^2}(0) = \frac{\mathrm{d}^2\psi}{\mathrm{d}y^2}(1)$$

$$\theta(0) = \theta(1) = 0$$

with $r(y)$ given by (2.26) and $\mathrm{M} = -\frac{\mathrm{d}^2}{\mathrm{d}y^2} + a^2k^2$. We have $\mathrm{M}\psi(0) = \mathrm{M}\psi(1) = 0$ and by the second equation (3.18) and the condition for θ also $\mathrm{M}^2\psi(0) = \mathrm{M}^2\psi(1) = 0$.

Define

$$(3.19)\qquad \chi = \mathrm{M}\psi + \psi \qquad \zeta = \mathrm{M}\psi + \frac{v^{*'''}}{v^{*'}}\psi \qquad \text{thus } \zeta = \chi + \psi\left(\frac{v^{*'''}}{v^{*'}} - 1\right).$$

We have

$$(3.20)\qquad \chi(0) = \chi(1) = 0 \qquad \mathrm{M}\chi(0) = \mathrm{M}\chi(1) = 0\,.$$

By adding and substracting $\mathrm{M}\psi$ in the left hand side of the second equation of (3.18) and recalling that $s - 1 = \alpha n(y)$, equations (3.18) can be rewritten as follows

$$(3.21)\qquad \mathrm{M}^2\xi = \lambda v^{*'}\zeta \qquad b(\mathrm{M}\chi + \alpha n \mathrm{M}\psi) = 0 \qquad \mathrm{M}\theta = \lambda\xi\,.$$

We would like to eliminate θ in the last two equations. From the first one we have

$$\mathrm{M}\theta = \mathrm{M}(b(y)\,\mathrm{M}\chi) + \alpha \mathrm{M}[b(y)\,n(y)\,\mathrm{M}\psi].$$

Hence we get $\mathrm{M}^2\xi = \lambda v^{*'}\zeta$ and $\mathrm{M}(b\mathrm{M}\chi) = \lambda\xi - \alpha\mathrm{M}(nb\mathrm{M}\psi)$.

Multiplying the first equation by ξ and the second by χ, integrating by parts and summing, we have, recalling (3.20),

$$J = \int_0^1 (\mathrm{M}\xi)^2\,dy + \int_0^1 b(\mathrm{M}\chi)^2\,dy = \lambda\int_0^1 (v^{*'}\zeta\xi + \chi\xi)\,dy - \alpha\int_0^1 (\mathrm{M}\psi + \psi)\,\mathrm{M}(bn\mathrm{M}\psi)\,dy$$

$$= \lambda\int_0^1 (v^{*'}\zeta\xi + \chi\xi)\,dy - \alpha a^2k^2\int_0^1 bn(\mathrm{M}\psi)^2\,dy - \alpha a^2k^2\int_0^1 bn\psi\mathrm{M}\psi\,dy$$

$$+ \alpha\int_0^1 \mathrm{M}\psi''(bn\mathrm{M}\psi)\,dy + \alpha\int_0^1 \psi'' bn\mathrm{M}\psi\,dy\,.$$

Recalling (3.19) and the Cauchy-Schwartz inequality we get

$$J \leqslant C_1\Big[\lambda\int_0^1 (\chi^2 + \xi^2 + \psi^2)\,dy + \alpha\int_0^1 (\mathrm{M}\psi'')^2\,dy + \alpha a^2k^2\int_0^1 (\psi'')^2\,dy + \alpha\int_0^1 (\psi'')^2\,dy\Big]$$

$$+ C_1\Big[\alpha a^2k^2\int_0^1 (\mathrm{M}\psi)^2\,dy + a^2k^2\alpha\int_0^1 (\psi)^2\,dy + a^4k^4\alpha\int_0^1 (\psi)^2\,dy\Big]$$

where the constant C_1 does not depend on α.

On the other hand we have

$$J \geq \int_0^1 (\xi''^2 + 2a^2k^2\xi'^2 + a^4k^4\xi^2)\,dy + b_0 \int_0^1 (\chi''^2 + 2a^2k^2\chi'^2 + a^4k^4\chi^2)\,dy$$

$$\geq a^4k^4 \int_0^1 \xi^2\,dy + \frac{b_0}{2} a^4k^4 \int_0^1 \chi^2\,dy + \frac{b_0}{2} \int_0^1 (\chi''^2 + 2a^2k^2\chi'^2 + a^4k^4\chi^2)\,dy$$

$$= a^4k^4 \int_0^1 \xi^2\,dy + \frac{b_0}{2} a^4k^4 \int_0^1 \chi^2\,dy$$

$$+ \frac{b_0}{2} \Big[\int_0^1 (\mathrm{M}\psi'' + \psi'')^2\,dy + 2a^2k^2 \int_0^1 (\mathrm{M}\psi' + \psi')^2\,dy + a^4k^4 \int_0^1 (\mathrm{M}\psi + \psi)^2\,dy\Big]$$

$$\geq a^4k^4 \Big(\int_0^1 \xi^2\,dy + \frac{b_0}{2} \int_0^1 \chi^2\,dy + \frac{1}{2} \int_0^1 \psi^2\,dy\Big)$$

$$+ \frac{b_0}{2} \Big[\int_0^1 (\mathrm{M}\psi'')^2\,dy + 2\int_0^1 (\psi'''^2 + a^2k^2\psi''^2)\,dy + \int_0^1 \psi''^2\,dy + 2a^2k^2 \int_0^1 (\mathrm{M}\psi')^2\,dy$$

$$+ 2a^2k^2 \int_0^1 (\psi''^2 + a^2k^2\psi'^2)\,dy + 2a^2k^2 \int_0^1 \psi'^2\,dy + a^4k^4 \int_0^1 (\mathrm{M}\psi)^2\,dy$$

$$+ 2a^4k^4 \int_0^1 (\psi'^2 + a^2k^2\psi^2)\,dy + \frac{a^4k^4}{2} \int_0^1 \psi^2\,dy\Big].$$

Using the Poincaré inequality

$$J \geq a^4k^4 \Big(\int_0^1 \xi^2\,dy + \frac{b_0}{2} \int_0^1 \chi^2\,dy + \frac{1}{2} \int_0^1 \psi^2\,dy\Big) + \frac{b_0}{2} \Big[\int_0^1 (\mathrm{M}\psi'')^2\,dy + 2a^2k^2 \int_0^1 (\psi''^2)\,dy\Big]$$

$$+ \frac{b_0}{2} \Big[\int_0^1 \psi''^2\,dy + 4a^2k^2 \int_0^1 (\mathrm{M}\psi)^2\,dy + 4a^2k^2 \int_0^1 (\psi)^2\,dy + \frac{a^4k^4}{2} \int_0^1 \psi^2\,dy\Big].$$

Collecting the above estimates we get

$$a^4k^4 \Big(\int_0^1 \xi^2\,dy + \frac{b_0}{2} \int_0^1 \chi^2\,dy + \frac{1}{2} \int_0^1 \psi^2\,dy\Big) + \frac{b_0}{2} \Big[\int_0^1 (\mathrm{M}\psi'')^2\,dy + 2a^2k^2 \int_0^1 \psi''^2\,dy\Big]$$

$$+ \frac{b_0}{2} \Big[\int_0^1 \psi''^2\,dy + 4a^2k^2 \int_0^1 (\mathrm{M}\psi)^2\,dy + 4a^2k^2 \int_0^1 (\psi)^2\,dy + \frac{a^4k^4}{2} \int_0^1 \psi^2\,dy\Big]$$

$$\leq C_1 \Big[\lambda \int_0^1 (\chi^2 + \xi^2 + \psi^2)\,dy + \alpha \int_0^1 (\mathrm{M}\psi'')^2\,dy + \alpha a^2k^2 \int_0^1 (\psi'')^2\,dy + \alpha \int_0^1 (\psi'')^2\,dy\Big]$$

$$+ C_1 \Big[\alpha a^2k^2 \int_0^1 (\mathrm{M}\psi)^2\,dy + a^2k^2\alpha \int_0^1 (\psi)^2\,dy + a^4k^4\alpha \int_0^1 (\psi)^2\,dy\Big].$$

Thus we have

$$m_1 a^4 k^4 \int_0^1 (\chi^2 + \xi^2 + \psi^2)\, dy + (\frac{b_0}{2} - \alpha) \int_0^1 (M\psi'')^2\, dy + (2 - \alpha) a^2 k^2 \int_0^1 (\psi'')^2\, dy$$

$$+ (1 - 2\alpha) \int_0^1 (\psi'')^2\, dy + (4 - \alpha) a^2 k^2 \int_0^1 (M\psi)^2\, dy + (4 - 2\alpha) a^2 k^2 \int_0^1 (\psi)^2\, dy$$

$$+ (\frac{1}{2} - \alpha) a^4 k^4 \int_0^1 (\psi)^2\, dy \leq C_1 \lambda \int_0^1 (\chi^2 + \xi^2 + \psi^2)\, dy$$

where $m_1 = \min(b_0, 1)$. Hence if $\alpha \leq \min(\frac{1}{2}, b_0)$ we obtain $\eta \geq (\frac{m_1}{C_1})^2 a^6 k^6$ and the result follows.

Lemma 5. *Let* η_k^1 *denote the eigenvalue of problem* (3.11)-(3.14) *given by Lemma 3. The period* $\mathcal{L} = 2\pi\ a^{-1}$ *can be chosen so that* $\eta_1^1 < \eta_k^1$, *for all* $k > 1$.

Proof. By Lemma 4 $\inf\{\eta_k^1, k \geq 1\}$ is positive. Moreover his value is taken only for a finite number of values of k, in general greater than 1, Let $m = \max\{n \in N, \eta_n^1 = \inf\{\eta_k^1, k \geq 1\}\}$. Choose $\mathcal{L}$ arbitrarily and define a new period $\mathcal{L}^* = \mathcal{L} m^{-1}$, with $a^* = ma$. With this choice we have $\eta_1^1 < \eta_k^1$ for all $k > 1$, as required.

Hereafter $\mathcal{L}$ shall be the period found in Lemma 5. The Fourier expansion of the unique eigenvector corresponding to η_1^1 is given by

$$\phi(x, y) = \phi_1(y) \sin(ax) \qquad \psi(x, y) = \psi_1(y) \cos(ax) \qquad \theta(x, y) = \theta_1(\psi) \cos(ax).$$

What remains to prove to apply Krasnosel'kii's theorem is that the algebraic multiplicity of η_1^1 is 1. To this end it suffices to show that the dimension of $\mathrm{Ker}(\mathbf{I} - \eta_1^1 B)^p$ is one for all $p \geq 1$.

Lemma 6. *We have* $\mathrm{Ker}(\mathbf{I} - \eta_1^1 B)^2 = \mathrm{Ker}(\mathbf{I} - \eta_1^1 B)$.

Proof. First of all it is obvious that $\mathrm{Ker}(\mathbf{I} - \eta_1^1 B)^2 \supset \mathrm{Ker}(\mathbf{I} - \eta_1^1 B)$. The other inclusion shall be proved by contradiction. Suppose there exists ϕ^0 such that

$$(3.22) \qquad (\mathbf{I} - \eta_1^1 B)^2 \phi^0 = 0 \quad \text{and} \quad (\mathbf{I} - \eta_1^1 B)\phi^0 = \phi^1$$

with $\phi^1 \neq 0$. It follows that ϕ^1 is an eigenfunction of Problem 4 corresponding to

η_1^1. By Lemma 5 we have $\phi^1(x, y) = \phi_1(y)\sin(ax)$, $\psi^1(x, y) = \psi_1(y)\cos(ax)$, $\theta^1(x, y) = \theta_1(y)\cos(ax)$. On the other hand ϕ^0, ψ^0 and θ^0 admit the Fourier's expansions

$$\phi^0(x, y) = \sum_{k=1}^{\infty} \phi_k^0(y)\sin(akx) \qquad \psi^0(x, y) = \sum_{k=1}^{\infty} \psi_k^0(y)\cos(axk) + \psi_0^0(y)$$
(3.23)
$$\theta^0(x, y) = \sum_{k=1}^{\infty} \theta_k^0(y)\cos(akx) + \theta_0^0(y).$$

Now the second equation (3.22) is equivalent to the problem

$$\Delta^2(\phi^0 - \phi^1) = \eta_1^1(\Delta\psi_x^0 v_y^* - \psi_x^0 v_{yyy}^*)$$
(3.24)
$$\Delta^2\psi^0 - s(y)\Delta\psi^0 = r(y)\theta^0 \qquad \Delta\theta^0 = -\phi_x^0$$

with the boundary conditions of Problem 4. Substituting (3.23) in (3.24) we obtain, if $k = 1$

$$\mathrm{M}_1^2(\phi_1^0 - \phi_1^1) = \eta_1^1 a(\mathrm{M}_1\psi_1^0 \frac{\mathrm{d}v^*}{\mathrm{d}y} + \psi_1^0 \frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3})$$
(3.25)
$$\mathrm{M}_1^2\psi_1^0 + s(y)\mathrm{M}_1\psi_1^0 = r(y)\theta_1^0 \qquad \mathrm{M}_1\theta_1^0 = a\phi_1^0$$

and, when $k \geqslant 2$

$$\mathrm{M}_k^2\phi_k^0 = \eta_1^1 a(\mathrm{M}_k\psi_k^0 \frac{\mathrm{d}v^*}{\mathrm{d}y} + \psi_k^0 \frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3})$$
(3.26)
$$\mathrm{M}_k^2\psi_k^0 + s(y)\mathrm{M}_k\psi_k^0 = r(y)\theta_k^0 \qquad \mathrm{M}_k\theta_k^0 = ak\phi_k^0$$

in both cases with the boundary conditions (3.13), (3.14). By the choice of the period $\mathcal{L}$ we have, if $k > 1$, $\phi_k^0 = \psi_k^0 = \theta_k^0 = 0$ since η_1^1 is not an eigenvalue for equations (3.26).

Problem (3.25) can be transformed into the non-homogeneous integral equation

(3.27)
$$\phi_1^0(y) = \eta_1^1 \int_0^1 \mathrm{H}(y, s)\phi_1^0(s)\,\mathrm{d}s + \phi_1^1(y).$$

Because $\frac{\mathrm{d}v^*}{\mathrm{d}y}$, $\frac{\mathrm{d}^3 v^*}{\mathrm{d}y^3}$ and $r(y)$ are positive in $[0, 1]$, the kernel $\mathrm{H}(y, s)$ is also positive. Let us consider the adjoint equation

(3.28)
$$g(y) = \eta_1^1 \int_0^1 \mathrm{H}(s, y) g(s)\,\mathrm{d}s.$$

By Theorem 2 equation (3.28) has certainly a positive eigenfunction $g^1(y)$. On the other hand $\phi_1^1(y) > 0$ and this contradicts the orthogonality condition $\int_0^1 \phi_1^1(y) g^1(y)\,dy = 0$ needed for the solvability of equation (3. 27). Therefore $(I - \eta_1^1 B)\phi_0^0 = 0$.

By induction on p we conclude that the dimension of $\mathrm{Ker}(I - \eta_1^1 B)^p$ is one. Thus Krasnosels'kii theorem is applicable and we conclude with

Theorem 2. *Let Condition* **I** *holds and* $\alpha \leq \min(b_0, 1)$. *Then, corresponding to a critical value of the parameter* η, *there is loss of uniqueness for Problem* 2.

The experimental fact quoted in the introduction about the possibility of inverting the polarity of the applied voltage without altering the onset of instability is accounted for by the present theory, if together with the voltage we change sign to the injection law. This is immediately seen, recalling Remark 1, if we note that η depends on V_1^2.

4 – The case of constant electrical conductivity

Is bifurcation of electrical origin possible, when the electric conductivity does not depend on the temperature? If we assume $s(t) = s_0 > 0$ and neglect the inertia term, the equations of Problem 3 become

$$\Delta^2\phi = \eta(\Delta\psi_x\psi_y - \Delta\psi_y\psi_x) + \eta(\Delta\psi_x v_y^* - v_{yyy}^*\psi_x) \tag{4.1}$$

$$\Delta^2\psi - \beta\Delta\psi = \beta v_{yyy}^* - v_{yyyy}^* \tag{4.2}$$

$$\Delta\theta = \gamma(\theta_y\phi_y - \theta_y\phi_x - \phi_x) \tag{4.3}$$

$$\theta(x, 0) = \phi(x, 1) = \phi_y(x, 0) = \phi_y(x, 1) = 0 \tag{4.4}$$

$$\psi(x, 0) = \psi(x, 1) = \psi_{yy}(x, 0) = \psi_{yy}(x, 1) = 0 \tag{4.5}$$

$$\theta(x, 0) = \theta(x, 1) = 0 \tag{4.6}$$

$$\phi,\ \psi \text{ and } \theta \text{ periodic with period } \mathcal{L} = \mathrm{LD}^{-1}\,. \tag{4.7}$$

This problem is uncoupled. It is in fact possible to obtain ψ from (4.2), (4.5). Then we can solve the boundary value problems (4.1), (4.4) and (4.3), (4.6) using the result of the linear theory. Since ψ is uniquely determined, we conclude that the solution of (4.1)-(4.7) is also unique and, as a consequence of the constancy of σ, bifurcation is not possible. On the other hand various heuristic approximations have been involved in deriving the scheme studied in the previous sections. In particular the contribution to the current density $\boldsymbol{J}$ due to the convection of charges is neglected in equation (2.1). A more realistic equation is therefore

$$\boldsymbol{J} = \sigma_0 \boldsymbol{E} - k\nabla q + q\boldsymbol{v} \tag{4.8}$$

where σ_0 is a positive constant. We want to investigate in this section if the equations, which follow from (4.8), allow bifurcation from the ground-state solution under reasonable physical conditions. In addition to $\beta = \sigma_0 D^2 (k\varepsilon)^{-1}$ we need the group $\alpha = \nu k^{-1}$ to write the equation for the *non-dimensional* potential. We obtain

$$\Delta^2 v - \beta \Delta v = -\alpha(\phi_x \Delta v_y - \phi_y \Delta v_x)\,. \tag{4.9}$$

The fluid motion is supposed to be slow. Thus we have

$$\Delta^2 \phi = \eta(\Delta v_x v_y - \Delta v_y v_x) \tag{4.10}$$

where $\eta = \varepsilon V_1^2 (\rho \nu^2)^{-1}$. Moreover we define $\delta_i = D^2 q_i (\varepsilon V_1)^{-1}$, $i = 0, 1$ where q_0 and q_1 are the prescribed charge densities on $Y = 0$ and $Y = 1$ respectively. The electric potential of the *ground* solution is now defined by the problem

$$\begin{gathered} \frac{\mathrm{d}^4 v^*}{\mathrm{d}y^4} - \beta \frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2} = 0 \\ v^*(0) = 0 \qquad v^*(1) = 1 \qquad \frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2}(0) = \delta_0 \qquad \frac{\mathrm{d}^2 v^*}{\mathrm{d}y^2}(1) = \delta_1 \end{gathered} \tag{4.11}$$

the solution $v^*(y)$ is of course computed immediately by elementary means. What is needed in the sequel is the value of the constant $C = \dfrac{\mathrm{d}^3 v^*}{\mathrm{d}y^2} - \beta \dfrac{\mathrm{d}v^*}{\mathrm{d}y}$ given by

$$C = \delta_1 - \delta_0 - \beta\,. \tag{4.12}$$

Problem 5 formed by (4.9), (4.10) and the boundary conditions

$$v(x,0)=0 \quad v(x,1)=1 \quad v_{yy}(x,0)=\delta_0 \quad v_{yy}(x,1)=\delta_1 \tag{4.13}$$
$$\phi(x,0)=\phi(x,1)=\phi_y(x,0)=\phi_y(x,1)=0$$

with the usual periodicity conditions on x, has always the solution $v(x,y)=v^*(y)$, $\phi(x,y)=0$. Let $\psi=v-v^*$. We can rewrite and collect the equations in *Problem 6*:

$$\Delta^2\phi=\eta(\Delta\psi_x\psi_y-\Delta\psi_y\psi_x)+\eta(\Delta\psi_x v_y^*-\psi_x v_{yyy}^*) \tag{4.14}$$
$$\Delta^2\psi-\beta\Delta\psi=-\alpha(\phi_x\Delta\psi_y-\phi_y\Delta\psi_x)-\alpha\phi_x v_{yyy}^*$$

$$\psi(x,0)=\psi(x,1)=\psi_{yy}(x,0)=\psi_{yy}(x,1)=0 \tag{4.15}$$
$$\phi(x,0)=\phi(x,1)=\phi_y(x,0)=\phi_y(x,1)=0\,.$$

Let us consider the operator A: $H_0^2(\Omega)\to H_0^2(\Omega)$, defined implicitly by the following problem

$$\Delta^2\omega=(\Delta\psi_x\psi_y-\Delta\psi_y\psi_x)+(\Delta\psi_x v_y^*-\psi_x v_{yyy}^*) \tag{4.16}$$
$$\Delta^2\psi-\beta\Delta\psi=-\alpha(\phi_x\Delta\psi_y-\phi_y\Delta\psi_x)-\alpha\phi_x v_{yyy}^*$$

$$\omega(x,0)=\omega(x,1)=\omega_y(x,0)=\omega_y(x,1)=0 \tag{4.17}$$
$$\psi(x,0)=\psi(x,1)=\psi_{yy}(x,0)=\psi_{yy}(x,1)=0\,.$$

Problem 6 can clearly be written as a functional equation of the from

$$\phi=\eta A(\phi)\,. \tag{4.18}$$

The Frechet's derivative $A'(0)$ of A is implicitly defined by

$$\Delta^2\omega=(\Delta\psi_x v_y^*-\psi_x v_{yyy}^*) \qquad \Delta^2\psi-\beta\Delta\psi=-\alpha\phi_x v_{yyy}^*$$

with the boundary conditions (4.17). We quote below a well-know elementary result [**6**].

Theorem 3. *Let $\mathcal{B}$ be a Banach space and* A: $\mathcal{B}\to\mathcal{B}$ *an operator* (*not necessarily compact*) *such that* $A(0)=0$. *Assume* $A'(0)$ *exists, then the set of the bifurcation points of* $\lambda\phi=A\,\phi$ *is contained in the spectrum of* $A'(0)$.

A crucial information on the spectrum of A′(0) can easily be obtained as follows. Consider the equations

$$\Delta^2\phi = \eta(\Delta\psi_x v_y^* - \psi_x v_{yyy}^*) \qquad \Delta^2\psi - \beta\Delta\psi = -\alpha\phi_x v_{yyy}^* . \tag{4.19}$$

Now, let us multiply the first equation by ϕ and the second by ψ and then by $\Delta\psi$. Integrating by parts over Ω we have

$$\frac{\beta}{\eta}\int_\Omega |\Delta\phi|^2 \,dx\,dy = -\beta\int_\Omega \phi_x \Delta\psi\beta_y^* \,dx\,dy + \beta\int_\Omega \phi_x v_{yyy}^* \psi \,dx\,dy \tag{4.20}$$

$$\frac{\beta}{\alpha}\int_\Omega |\Delta\psi|^2 \,dx\,dy + \frac{\beta^2}{\alpha}\int_\Omega |\nabla\psi|^2 \,dx\,dy = -\beta\int_\Omega \phi_x v_{yyy}^* \psi \,dx\,dy \tag{4.21}$$

$$\frac{1}{\alpha}\int_\Omega |\nabla(\Delta\psi)|^2 \,dx\,dy + \frac{\beta}{\alpha}\int_\Omega |\Delta\psi|^2 \,dx\,dy = \int_\Omega \phi_x v_{yyy}^* \Delta\psi \,dx\,dy . \tag{4.22}$$

Adding (4.20), (4.21) and (4.22), we obtain, recalling (4.12) and the Poincaré inequality

$$\frac{1}{\alpha}\int_\Omega |\nabla(\Delta\psi)|^2 \,dx\,dy + \frac{2\beta}{\alpha}\int_\Omega |\Delta\psi|^2 \,dx\,dy + \frac{\beta^2}{\alpha}\int_\Omega |\nabla\psi|^2 \,dx\,dy + \frac{\beta}{\eta}\int_\Omega |\Delta\phi|^2 \,dx\,dy$$

$$= C\int_\Omega \phi_x \Delta\psi \,dx\,dy \leqslant \frac{C}{4}\int_\Omega \phi_{xy}^* \,dx\,dy + \frac{C}{2}\int_\Omega |\Delta\psi|^2 \,dx\,dy .$$

Since $2\int_\Omega |\Delta\psi|^2 \,dx\,dy \leqslant \int_\Omega |\nabla(\Delta\psi)|^2 \,dx\,dy$ we finally obtain

$$\left(\frac{2}{\alpha} + \frac{2\beta}{\alpha} - \frac{C}{2}\right)\int_\Omega |\Delta\psi|^2 \,dx\,dy + \frac{\beta}{\eta}\int_\Omega (\phi_{xx}^2 + \phi_{yy}^2)\,dx\,dy + \left(\frac{2\beta}{\eta} - \frac{C}{4}\right)\int_\Omega \phi_{xy}^2 \,dx\,dy \leqslant 0 .$$

Therefore if

$$\frac{2}{\alpha} + \frac{2\beta}{\alpha} \geqslant \frac{C}{2} \tag{4.23}$$

$$\frac{2\beta}{\eta} \geqslant \frac{C}{4} \tag{4.24}$$

problem (4.14), (4.15) can have only the trivial solution.

Remark 2. Attent and Moreau [2] study the electroconvective vortices, taking as starting equation

$$J = \mu qE + qv . \tag{4.26}$$

This makes the equation for the electric potential similar to the corresponding equation in the so-called *space charge problem* (see [4] and references therein). Adopting (4.26), they are able to prove instability of purely electric origin even neglecting the dependence on the temperature.

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Sommario

Il teorema di biforcazione dell'autovalore semplice è applicato per provare la non unicità della soluzione per il problema di uno strato di liquido dielettrico soggetto a un campo elettrico, a un gradiente di temperatura e ad iniezione di cariche elettriche.

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