Riv. Mat. Univ. Parma (5) 3 (1994), 193-202

H. K. PATHAK (*)

Application of fixed point theorems to abstract Volterra integrodifferential equations (**)

1 - Introduction

In this note we intend to find the common mild solution of abstract Volterra integrodifferential equations of the type (see Theorem 1):

$$(1.1)\quad \begin{aligned} &u'(t) + Au(t) = f(t, u(t)) + \int_{t_0}^{t} g(t, s, u(s), \int_{t_0}^{s} K_1(s, \tau, u(\tau))\, d\tau)\, ds \\ &+ \int_{t_0}^{t} h(t, s, u(s), \int_{t_0}^{\infty} K_2(s, \tau, u(\tau))\, d\tau)\, ds \qquad t > t_0 \geq 0\,, \qquad u(t_0) = u_0 \end{aligned}$$

$$(1.2)\quad \begin{aligned} &u'(t) + Au(t) = \bar{f}(t, u(t)) + \int_{t_0}^{t} \bar{g}(t, s, u(s), \int_{t_0}^{s} \overline{K}_1(s, \tau, u(\tau))\, d\tau)\, ds \\ &+ \int_{t_0}^{t} \bar{h}(t, s, u(s), \int_{t_0}^{\infty} \overline{K}_2(s, \tau, u(\tau))\, d\tau)\, ds \qquad t > t_0 \geq 0\,, \qquad u(t_0) = u_0 \end{aligned}$$

where $-A$ is the infinitesimal generator of C_0-semigroup $\{T(t)\,|\,t \geq 0\}$ of bounded linear operators on a Banach space B with norm $\|\cdot\|$, $f, \bar{f} \in C(\boldsymbol{R}_+ \times B, B)$, $g, \bar{g}, h, \bar{h} \in C(\boldsymbol{R}_+ \times \boldsymbol{R}_+ \times B \times B, B)$, $K_1, \overline{K}_1, K_2, \overline{K}_2 \in C(\boldsymbol{R}_+ \times \boldsymbol{R}_+ \times B, B)$ and $\boldsymbol{R}_+ = (0, \infty)$.

In the sequel, we extend Theorem 1 to the study of the common mild solution of (1.1) and of the infinite family of integrodifferential equations (see Theo-

(*) Dept. of Math., Kalyan Mahavidyalaya, Bhilai Nagar (M.P.), 490006 India.
(**) Received March 23, 1993. AMS classification 47 G 20. Research partially supported by UGC, New Dehi, India.

rem 2):

$$
(1.3)\quad \begin{aligned} u'(t) + Au(t) &= f_j(t, u(t)) + \int_{t_0}^{t} g_j(t, s, u(s), \int_{t_0}^{s} K_{1j}(s, \tau, u(\tau))\, d\tau)\, ds \\ &+ \int_{t_0}^{t} h_j(t, s, u(s), \int_{t_0}^{\infty} K_{2j}(s, \tau, u(\tau))\, d\tau)\, ds \qquad t > t_0 \geq 0 \qquad u(t_0) = u_0 \end{aligned}
$$

where f_j, g_j K_{1j} and K_{2j} $(j = 1, 2, \ldots)$ play the roles of $\bar{f}$, $\bar{g}$, $\bar{K}_1$ and $\bar{K}_2$ of (1.2).

It may be mentioned that equations of the type (1.1) arise naturally in study of initial value problems on the infinite half line. Moreover, many problems arising in various branches of physics and in other areas of mathematical sciences find themselves incorporated in the abstract formulation (1.1) (see, for instance, [2], [6], [7]). Existence, uniqueness and other properties of the solution of several forms of (1.1) using various assumptions and different methods have been studied, among others, by Barbu [1], Fitzgibbon [3], Hussain [5], Martin Jr. [6], Miller [7], Miller, Nohel and Wong [8], Sinestrari [11], Singh [12], Shuart [13], Travis and Webb [14], Vainberg [15] and Webb [16].

In this note, we intend to study existence and uniqueness of the common solution of the equations (1.1) and (1.2). To prove the existence of a common solution of equations (1.1) and (1.2), we utilize a fixed point theorem of Yen [17] for two contractive type operators (see Lemma 1). On the other hand, the uniqueness of the solution is established using a result of Pachpatte [9] (see Lemma 2). In the sequel, some results of Singh [12] and Hussain [5] are obtained as a particular case of our results (see Corollaries 1, 2 and 3). The proof of Theorem 2 is prefaced by a special case of fixed point theorem of Husain and Sehgal [4] (see Lemma 3) and Lemma 2.

2 - Prelininaries

Throughout in this note B stands for a *Banach space* with norm $\|\cdot\|$ and $-A$ for the infinitesimal generator of C_0-semigroup of operators $\{T(t), t \geq 0\}$ on B. A family $\{T(t): t \in R_+\}$ of bounded operators from B into B is a *C_0-semigroup* if:

i. $T(0) =$ the identity operator and $T(t+s) = T(t)T(s)$ for all $t, s \leqslant 0$

ii. $T(\cdot)$ is strongly continuous in $t \in \boldsymbol{R}_+$

iii. $\|T(t)\| \leqslant Me^{\omega t}$ for some $M \geqslant 0$, real ω and $t \in \boldsymbol{R}_+$ ([8]).

For the sake of brevity, we assume that

$$H(T, t_0, f, g, h, K_1, K_2) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s) f(s, u(s))\, ds$$

$$+ \int_{t_0}^{t} T(t-s) \Big(\int_{t_0}^{s} g(s, \tau, u(\tau), \int_{t_0}^{\tau} K_1(\tau, \xi, u(\xi))\, d\xi)\, d\tau\Big)\, ds$$

$$+ \int_{t_0}^{t} T(t-s) \Big(\int_{t_0}^{s} h(s, \tau, u(\tau), \int_{t_0}^{\infty} K_2(\tau, \xi, u(\xi))\, d\xi)\, d\tau\Big)\, ds .$$

A continuous $u(t)$ is a *mild solution* of (1.1) if

$$u(t) = H(T, t_0, f, g, h, K_1, K_2).$$

Moreover, a continuous $u(t)$ is a *common mild solution of* (1.1) *and of* (1.2) if

$$H(T, t_0, f, g, h, K_1, K_2) = u(t) = H(T, t_0, \bar{f}, \bar{g}, \bar{h}, \overline{K}_1, \overline{K}_2).$$

We use the following assumptions in our first theorem. For all t, s in (t_0, ∞) and x_i, y_i in B, $i = 1, 2$, let non-negative numbers L_i, $i = 1, 2, 3, 4, 5, 6, 7$ exist, such that

A$_1$. $$\|K_1(t, s, y_1) - \overline{K}_1(t, s, y_2)\| \leqslant L_1 \|y_1 - y_2\|$$

A$_2$. $$\sup_{t \geqslant t_0} \int_{t_0}^{\infty} \|K_2(t, s, y_1) - \overline{K}_2(t, s, y_2)\|\, ds \leqslant L_2 \|y_1 - y_2\|$$

A$_3$. $$\|g(t, s, x_1, y_1) - \bar{g}(t, s, x_2, y_2)\| \leqslant L_3 \|x_1 - x_2\| + L_4 \|y_1 - y_2\|$$

A$_4$. $$\|h(t, s, x_1, y_1) - \bar{h}(t, s, x_2, y_2)\| \leqslant L_5 \|x_1 - x_2\| + L_6 \|y_1 - y_2\|$$

A$_5$. $$\|f(t, x_1) - \bar{f}(t, x_2)\| \leqslant L_7 \|x_1 - x_2\| .$$

In the proof of our main theorem, we require the lemmas stated below. The following fixed point theorem is essentially given by Yen [**17**] (see also Husain and Sehgal [4] Cor. 2 and Rhoades [**10**] Th. 14).

Lemma 1. *Let T_1 and T_2 be maps on a complete metric space X. If there exist a positive integer m and a positive number $k < 1$ such that, for all x, y in X, $d(T_1^m x, T_2^m y) \leqslant k d(x, y)$, then T_1 and T_2 have a unique common fixed point.*

The following lemma, due to Pachpatte [9], plays the key role in proving the uniqueness of the common mild solution of (1.1) and (1.2) and the Lipschitz continuity of certain maps.

Lemma 2. *Let $x(t)$, $a(t)$, $b(t)$ and $c(t)$ be real-valued non-negative continuous functions defined on R_+, for which the inequality*

$$x(t) \leqslant x_0 + \int_0^t a(s)\,x(s)\,\mathrm{d}s + \int_0^t a(s)\Big(\int_0^s b(r)\,x(r)\,\mathrm{d}r\Big)\,\mathrm{d}s + \int_0^s a(s)\Big(\int_0^s b(r)\Big(\int_0^r c(z)\,x(z)\,\mathrm{d}z\Big)\,\mathrm{d}r\Big)\,\mathrm{d}s$$

holds for all t in R_+, where x_0 is a non-negative constant. Then

$$x(t) \leqslant x_0\Big(1 + \int_0^t a(s)\,\exp\Big(\int_0^s a(r)\,\mathrm{d}r\Big)\Big\{1 + \int_0^s b(r)\,\exp\Big(\int_0^r (b(z) + c(z))\,\mathrm{d}z\Big)\,\mathrm{d}r\Big\}\,\mathrm{d}s\Big).$$

3 - Main results

We establish

Theorem 1. *Suppose that A_1-A_5 are satisfied. Then, for u_0 in B, the initial value problems (1.1) and (1.2) have a unique common mild solution $u(t)$ in $C([t_0, t_1], B)$ for $t \geqslant t_0$, where t_1 is arbitrarily fixed, with $t_1 > t_0$. Moreover, the map $u_0 \to u$ from B into $C([t_0, t_1], B)$ is Lipschitz continuous.*

Proof. Let $C = C([t_0, t_1), B)$. Define the norm $\|\cdot\|_C$ in C as

$$\|u\|_C = \max_{t \in [t_0, t_1]} \|u(t)\|.$$

Then $\|\cdot\|_C$ is a Banach space. Let $F, \bar{F}: C \to C$ be such that

$$(3.1) \qquad (Fu)(t) = H(T, t_0, f, g, h, K_1, K_2) \qquad t_0 \leqslant t < \infty$$

$$(3.2) \qquad (\bar{F}u)(t) = H(T, t_0, \bar{f}, \bar{g}, \bar{h}, \bar{K}_1, \bar{K}_2) \qquad t_0 \leqslant t < \infty .$$

Evidently, a common solution of equations (1.1) and (1.2) is also a common fixed point of the operators F and $\bar{F}$.

Let $\bar{M}$ be an upper bound of $\|T(t-s)\|$ on $[t_0, t_1]$. Evidently, $\bar{M} = M$ if $\omega \leqslant 0$ and $\bar{M} = M \exp(\omega t_1)$ if $\omega > 0$ (see iii).

Then from (3.1), (3.2) and A_1-A_5, we obtain

$$\|(Fu)(t) - (\bar{F}v)(t)\| \leqslant \int_{t_0}^t \bar{M} L_7 \|u(s) - v(s)\|\,\mathrm{d}s + \int_{t_0}^t \bar{M} \int_{t_0}^s (L_5 + L_6 L_2)\,\|u(\tau) - v(\tau)\|\,\mathrm{d}\tau\,\mathrm{d}s$$

$$+\int_{t_0}^{t}\overline{M}\int_{t_0}^{s}[L_3\|u(\tau)-v(\tau)\|+L_4\int_{t_0}^{\tau}L_1\|u(\xi)-v(\xi)\|\,d\xi]\,d\tau\,ds \leqslant \overline{M}L_7\|u-v\|_C(t-t_0)$$

$$+\overline{M}(L_3+L_5+L_6L_2)\|u-v\|_C\frac{(t-t_0)^2}{2}+\overline{M}L_4L_1\|u-v\|_C\frac{(t-t_0)^3}{6}$$

$$=\overline{M}(t-t_0)[\{L_7+L_3\frac{(t-t_0)}{2}+L_4L_1\frac{(t-t_0)^2}{6}\}+(L_5+L_6L_2)\frac{(t-t_0)}{2}]\|u-v\|_C$$

$$=\overline{M}\alpha[\{L_7+L_3\frac{\alpha}{2}+L_4L_1\frac{\alpha^2}{6}\}+(L_5+L_6L_2)\frac{\alpha}{2}]\|u-v\|_C$$

where $\alpha=t-t_0$.

Repeating this process again, we have

$$\|(F^2u)(t)-\overline{F}^2u)(t)\|\leqslant\int_{t_0}^{t}\overline{M}L_7\|(Fu)(s)-(\overline{F}u)(s)\|\,ds$$

$$+\int_{t_0}^{t}\overline{M}\int_{t_0}^{s}[L_3\|(Fu)(\tau)-(\overline{F}u)(\tau)\|+L_4\int_{t_0}^{\tau}L_1\|(Fu)(\xi)-(\overline{F}v)(\xi)\|\,d\xi]\,d\tau\,ds$$

$$+\int_{t_0}^{t}\overline{M}\int_{t_0}^{s}(L_5+L_6L_2)\|(Fu)(\tau)-(\overline{F}v)(\tau)\|\,d\tau\,ds$$

$$\leqslant\frac{\overline{M}^2}{2}[L_7\int_{t_0}^{t}\{2L_7(s-t_0)+(L_3+L_5+L_6L_2)(s-t_0)^2+L_4L_1\frac{(s-t_0)^3}{3}\}\,ds$$

$$+(L_3+L_5+L_6L_2)\int_{t_0}^{t}\int_{t_0}^{s}\{2L_7(\tau-t_0)+(L_3+L_5+L_6L_2)(\tau-t_0)^2+L_4L_1\frac{(\tau-t_0)^3}{3}\}\,d\tau\,ds$$

$$+L_4L_1\int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{\tau}\{2L_7(\xi-t_0)+(L_3+L_5+L_6L_2)(\xi-t_0)^2+L_4L_1\frac{(\xi-t_0)^3}{3}\}\,d\xi\,d\tau\,ds]\|u-v\|_C$$

$$=\frac{\overline{M}^2}{2}[L_7\{L_7(t-t_0)^2+(L_3+L_5+L_6L_2)\frac{(t-t_0)^3}{3}+L_4L_1\frac{(t-t_0)^4}{12}\}$$

$$+(L_3+L_5+L_6L_2)\{L_7\frac{(t-t_0)^3}{3}+(L_3+L_5+L_6L_2)\frac{(t-t_0)^4}{12}+L_4L_1\frac{(t-t_0)^5}{60}\}$$

$$+L_4L_1\{L_7\frac{(t-t_0)^4}{12}+(L_3+L_5+L_6L_2)\frac{(t-t_0)^5}{60}+L_4L_1\frac{(t-t_0)^6}{360}\}]\|u-v\|_C$$

$$= \frac{\alpha^2}{2} \overline{M}^2 [L_7 \{L_7 + (L_3 + L_5 + L_6 L_2) \frac{\alpha}{3} + L_4 L_1 \frac{\alpha^2}{12} + L_4 L_1 \frac{\alpha^3}{60}\}$$

$$+ L_4 L_1 \{L_7 \frac{\alpha^2}{12} + (L_3 + L_5 + L_6 L_2)\} \frac{\alpha^3}{60} + L_4 L_1 \frac{\alpha^4}{360}\}] \|u - v\|_C$$

$$= \frac{\alpha^2}{2} \overline{M}^2 [\{L_7^2 + L_3^2 \frac{\alpha^2}{12} + L_4^2 L_1^2 \frac{\alpha^4}{360} + L_7 L_3 \frac{\alpha}{3} + L_7 L_4 L_1 \frac{\alpha^2}{6} + L_3 L_4 L_1 \frac{\alpha^3}{60}\}$$

$$+ (L_7 \frac{1}{3} + L_3 \frac{\alpha}{12} + L_4 L_1 \frac{\alpha^2}{60})(L_5 + L_6 L_2) \alpha + (L_5 + L_6 L_1)^2 \frac{\alpha^2}{12}] \|u - v\|_C$$

$$\leqslant \frac{\alpha^2}{2} \overline{M}^2 [\{L_7^2 + L_3^2 \frac{\alpha^2}{4} + L_4^2 L_1^2 \frac{\alpha^4}{360} + L_7 L_3 \alpha + L_7 L_4 L_1 \frac{\alpha^2}{3} + L_3 L_4 L_1 \frac{\alpha^3}{6}\}$$

$$+ 2(L_7 + L_3 \frac{\alpha}{2} + L_4 L_1 \frac{\alpha^2}{6})(L_5 + L_6 L_2) \frac{\alpha}{2} + (L_5 + L_6 L_2)^2 \frac{\alpha^2}{4}] \|u - v\|_C$$

$$= \frac{\alpha^2}{2!} \overline{M}^2 [\{L_7 + L_3 \frac{\alpha}{2} + L_4 L_1 \frac{\alpha^2}{6}\} + (L_5 + L_6 L_2) \frac{\alpha}{2}]^2 \|u - v\|_C .$$

Further, continuing this process $(n-2)$ times, it can be seen that

$$\|(F^n u)(t) - (\overline{F}^n v)(t)\| \leqslant (\frac{\alpha^n}{n!}) \overline{M}^n [\{L_7 + L_3 \frac{\alpha}{2} + L_4 L_1 \frac{\alpha^2}{6}\} + (L_5 + L_6 L_2) \frac{\alpha}{2}]^n \|u - v\|_C.$$

Therefore $$\|F^n u - \overline{F}^n v\|_C \leqslant k \|u - v\|_C$$

where $$k = (\frac{1}{n!})(\overline{\alpha}\overline{M})^n [\{L_7 + L_3 \frac{\overline{\alpha}}{2} + L_4 L_1 \frac{\overline{\alpha}}{6}\} + (L_5 + L_6 L_2) \frac{\overline{\alpha}}{2}]^n$$

since when we take the maximum with respect to t, $\alpha = t - t_0$ becomes $\overline{\alpha} = t_1 - t_0$.

For sufficiently large n, we can make $k < 1$, and so all the hypotheses of Lemma 1 are satisfied. Consequently, there exists a unique u in C such that $(Fu)(t) = u(t) = (\overline{F}u)(t)$.

Moreover, this unique common fixed point is a common mild solution of (1.1) and (1.2).

In order to show that (1.1) and (1.2) have exactly one common mild solution, we assume that v is another common mild solution of (1.1) and (1.2) with $v(t_0) = v_0$.

Suppose $L = \max\{L_i \mid i = 1, 2, 3, 4, 5, 6, 7\}$. Then

$$\|u(t) - v(t)\| = \|(Fu)(t) - (Fv)(t)\| \leqslant \overline{M}\|u_0 - v_0\| + \int_{t_0}^{t} \overline{M}L\,\|u(s) - v(s))\|\,ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} \overline{M}L\,[\|u(\tau) - v(\tau)\| + \int_{t_0}^{\tau} L\,\|u(\xi) - v(\xi)\|\,d\xi]\,d\tau\,ds$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} \overline{M}L\,[\|u(\tau) - v(\tau)\| + \int_{t_0}^{\infty} L\,\|u(\xi) - v(\xi)\|\,d\xi]\,d\tau\,ds\,.$$

Now Lemma 2 yields $\|u(t) - v(t)\| \leqslant \overline{M}\,\|u_0 - v_0\|\,R(t)$ where:

$$2R(t) = [1 + \int_{t_0}^{t} \overline{M}L\,\exp(\int_{t_0}^{s} \overline{M}L\,d\tau)\{1 + \int_{t_0}^{s}\exp(\int_{t_0}^{\tau} 2[1 + L]\,d\xi)\,d\tau\}\,ds]$$

$$+[1 + \int_{t_0}^{t} \overline{M}L\,\exp(\int_{t_0}^{s} \overline{M}L\,d\tau)\{1 + \int_{t_0}^{s}\exp[2(1 + L)\,d\tau\}\,ds]\,.$$

Hence $\|u - v\|_C \leqslant \overline{M}\,\|u_0 - v_0\|\,\overline{R}$ $\qquad \overline{R} = \max\{R(t),\ \ t \in [t_0, t_1]\}$.

By induction we can prove that

$$\|u - v\|_C \leqslant \frac{(\overline{M}\overline{R})^n}{n!}\,\|u_0 - v_0\|$$

which tends to zero as $n \to \infty$. This yields the unicity of the common mild solution, and the Lipschitz continuity of the map $u_0 \to u$. This completes the proof.

The following results easily follow from Theorem 1.

Corollary 1 [12]. *Suppose that* A_1, A_3, A_5 *and* $h = \bar{h} = 0$ *with* $K_2 = \overline{K}_2$ *are satisfied. Then, for* u_0 *in* B, *the initial value problems* (1.1) *and* (1.2) *have a unique common mild solution* u *in* $C([t_0, t_1], B)$ *for* $t \geqslant t_0$, *such that* $t_0 \leqslant t \leqslant t_1$. *Moreover, the map* $u_0 \to u$ *from* B *into* $C([t_0, t_1], B)$ *is Lipschitz continuous.*

Corollary 2 [2]. *Suppose that* A_1 *with* $K_1 = \overline{K}_1$, A_3 *with* $g = \bar{g}$ *and* $L_3 = L_4$, A_5 *with* $f = \bar{f}$ *and* $h = \bar{h} = 0$ *with* $K_2 = \overline{K}_2$ *are satisfied. Then, for* $u_0 \in B$, *the initial value problem* (1.1) *has a unique mild solution* $u \in C([t_0, t_1], B)$ *for* $t \geqslant t_0$ *such that* $t_0 \leqslant t \leqslant t_1$. *Moreover, the map* $u_0 \to u$ *from* B *into* $C([t_0, t_1], B)$ *is Lipschitz continuous.*

Remark 1. Fitzgibbon [3] and Webb [16] have studied certain special forms of (1.1) by using different assumptions and methods.

Our next theorem is prefaced by the following result, which is a special case of the fixed point theorem of Husain and Sehgal [4] (Cor. 2) and Rhoades [10] (Th. 20).

Lemma 3. *Let T and T_i, $i = 1, 2, \dots$ be maps on a complete metric space X. If there exist positive integers m_i and positive numbers $k_i < 1$, $i = 1, 2, \dots$, such that $d(T^{m_i}x, T_i^{m_i}y) \leqslant k_i d(x, y)$ for all x, y in X, then there exists a unique element u in X such that $Tu = u = T_i u$.*

We use the following assumptions in Theorem 2 (see also A_1-A_5).

For all t, s in $[t_0, \infty)$ and x_q, y_q in B, $q = 1, 2$, let non-negative numbers L_{ij}, $i = 1, 2, 3, 4, 5, 6, 7$, $j = 1, 2, 3, \dots$, exist, such that

B_1. $$\|K_1(t, s, y_1) - K_{1j}(t, s, y_2)\| \leqslant L_{1j}\|y_1 - y_2\|$$

B_2. $$\sup_{t \geqslant t_0} \int_{t_0}^{\infty} \|K_2(t, s, y_1) - K_{2j}(t, s, y_2)\| \, ds \leqslant L_{2j}\|y_1 - y_2\|$$

B_3. $$\|g(t, s, x_1, y_1) - g_j(t, s, x_1, y_2)\| \leqslant L_{3j}\|x_1 - x_2\| + L_{4j}\|y_1 - y_2\|$$

B_4. $$\|h(t, s, x_1, y_1) - h_j(t, s, x_2, y_2)\| \leqslant L_{5j}\|x_1 - x_2\| + L_{6j}\|y_1 - y_2\|$$

B_5. $$\|f(t, x_1) - f_j(t, x_2)\| \leqslant L_{7j}\|x_1 - x_2\|.$$

Theorem 2. *Suppose that B_1-B_5 are satisfied for each $j = 1, 2, \dots$. Then, for u_0 in B, the initial value problems (1.1) and (1.3) have a unique common mild solution u in $C([t_0, t_1], B)$. Moreover, the map $u_0 \to u$ from B into $C([t_0, t_1], B)$ is Lipschitz continuous.*

Proof. Let C and $F: C \to C$ be defined as in the proof of Theorem 1 (see (3.1)). Further, let $F_j: C \to C$, $j = 1, 2, \dots$, be such that

$$(F_j u)(t) = H(T, t_0, f_j, g_j, h_j, K_{1j}, K_{2j}) \qquad t_0 \leqslant t \leqslant \infty. \tag{3.3}$$

Then a common solution of (3.1) and (3.3) is a common fixed point of the operators F and F_j, $j = 1, 2, \dots$. It can be seen that for each $j \in \{1, 2, \dots\}$, we get

$$\|F^n u - F_j^n v\|_C \leqslant \|u - v\|_C$$

where $k_j = \frac{1}{n!}(\bar{\alpha}\overline{M})^n [\{L_{7j} + L_{3j}\frac{\bar{\alpha}}{2} + L_{4j}L_{1j}\frac{\bar{\alpha}^2}{6}\} + (L_{5j} + L_{6j}L_{2j})\frac{\bar{\alpha}}{2}]^n$.

For sufficiently large n, we can make $k_j < 1$. Consequently, Lemma 3 guarantees the existence of a unique u in C such that $Fu = u = F_j u$ for any $j = 1, 2 \dots$.

The rest of the proof is similar to that given in the proof of Theorem 1.

Remark 2. As Lemma 3 is true for T and an uncountable family of maps. Theorem 2 is true for (1.1) and an uncountable family of integrodifferential equations of type (1.3).

It may be mentioned that the following result easily follows from Theorem 2.

Corollary 3 [12]. *Suppose that* B_1, B_3, B_5 *and* $h = h_j = 0$ *for* $K_2 = K_{2j}$ *are satisfied for each* $j = 1, 2, \dots$. *Then, for* u_0 *in* B, *the initial value problems* (1.1) *and* (1.3) *have a unique common mild solution* u *in* $C([t_0, t_1], B)$ *for* $t \geq t_0$ *such that* $t_0 \leq t \leq t_1$. *Moreover, the map* $u_0 \to u$ *from* B *into* $C([t_0, t_1], B)$ *is Lipschitz continuous.*

References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leiden 1976.

[2] A. Belleni-Morante, *An integro-differential equation arising from the theory of heat conduction in rigid materials with memory*, Boll. Un. Mat. Ital. B **15** (1978), 470-482.

[3] W. E. Fitzgibbon, *Semilinear integrodifferential equations in Banach space*, Nonlinear Anal. 4 (1980), 745-760.

[4] S. A. Husain and V. M. Sehgal, *On common fixed points for a family of mappings*, Bull. Austral. Math. Soc. **13** (1975), 261-267.

[5] M. A. Hussain, *On a nonlinear integrodifferential equation in Banach space*, Indian J. Pure Appl. Math. **19** (1980), 516-529.

[6] R. H. Martin Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York 1976.

[7] R. K. Miller, *Volterra integral equation in a Banach spaces*, Funkcial Ekvac. **18** (1975), 163-193.

[8] R. K. Miller, T. A. Nohel and J. S. W. Wong, *A stability theorem for nonlinear mixed integral equations*, J. Math. Anal. Appl. **25** (1965), 446-449.

[9] B. G. Pachpatte, *On some integral inequalities and their applications to integrodifferential equations*, Indian J. Pure Appl. Math. 8 (1977), 1157-1175.

[10] B. E. RHOADES, *A comparison of various difinitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257-290.

[11] F. SINESTRARI, *Time-dependent integrodifferential equations in Banach space*, Nonlinear Phenomena in Mathematical Sciences, ed. V. Lakshimikantham, Academic Press, New York 1982.

[12] S. L. SINGH, *Application of fixed point theorems to nonlinear integrodifferential equations*, Riv. Mat. Univ. Parma **16** (1990), 205-212.

[13] C. A. STUART, *Existence theorems for a class of nonlinear equations*, Math. Z. **137** (1974), 49-66.

[14] C. C. TRAVIS and G. F. WEBB, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395-418.

[15] M. M. VAINBERG, *Existence theorems for system of nonlinear integral equations*, Uchen. Zap. Mosk. Obl. Pedints 18, Trud. Kafedr. Fizamata. 2 (1951), 56-73.

[16] G. F. WEBB, *An abstract semilinear Volterra integro-differential equation*, Amer. Math. Soc. **69** (1978), 155-260.

[17] C. L. YEN, *Remark on common fixed points*, Tamkang J. Math. 3 (1972), 95-96.

Sommario

Proposito di questa nota è di ottenere una comune «mild solution» per una coppia o per una famiglia di equazioni integro-differenziali di Volterra astratte con data condizione iniziale. L'esistenza di una soluzione comune è assicurata applicando alcuni teoremi di punto fisso.

* * *