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On a pseudo normal metric manifold (**)

1 - Introduction

In the paper [3], R. S. Mishra defined the *almost contact pseudo normal metric manifolds*. In the present paper some properties of these manifods studied.

2 - Preliminaries

Let M be an n-dimensional C^∞-manifold and let there exist on M a vector valued linear funciton Φ, a vector field T, and a 1-form A such that

$$\overline{\overline{X}} + X = A(X)\,T \tag{2.1}$$

where $\overline{X} = \Phi(X)$, for any vector field X. Then M is called an *almost contact manifod* and the structure (Φ, T, A) is called an almost constact structure.

It follows from (2.1) that on M we have rank$(\Phi) = n-1$, n is odd, i.e. $n = 2m+1$ and

$$\overline{T} = 0 \qquad A(\overline{X}) = 0 \qquad A(T) = 1\,. \tag{2.2}$$

In addition, if on M there exists a metric tensor g satisying

$$g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y) \tag{2.3}$$

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which is equivalent to $g(\overline{X}, \overline{Y}) = -g(\overline{\overline{X}}, Y)$ and $g(X, T) = A(X)$ then M is called almost *contact metric manifold* and (Φ, T, A, g) an *almost contact metric structure* [4].

Putting $F(X, Y) = g(\overline{X}, Y)$ we have

$$F(\overline{X}, \overline{Y}) = F(X, Y) \qquad F(X, Y) = -F(Y, X). \tag{2.4}$$

It has been shown by Mishra [**1**], that if D is the Riemannian connection in an almost contact metric manifold, then we have

$$\begin{aligned} (D_X F)(\overline{Y}, \overline{\overline{Z}}) &= (D_X F)(\overline{\overline{Y}}, \overline{Z}) \\ (D_X F)(\overline{Y}, \overline{Z}) + (D_X F)(\overline{\overline{Y}}, \overline{\overline{Z}}) &= 0 \end{aligned} \tag{2.5}$$

and further that the *Nijenhuis tensor* $\mathcal{N}$ is given by

$$\mathcal{N}(X, Y) = (D_{\overline{X}}\Phi)(Y) - (D_{\overline{Y}}\Phi)(X) - \overline{(D_X\Phi)Y} + \overline{(D_Y\Phi)(X)} \tag{2.6}$$

whence

$$\begin{aligned} N(X, Y, Z) = {} & (D_{\overline{X}}F)(Y, Z) - (D_{\overline{Y}}F)(X, Z) \\ & + (D_X F)(Y, \overline{Z}) - (D_Y F)(X, \overline{Z}) \end{aligned} \tag{2.7}$$

where $N(X, Y, Z) = g(\mathcal{N}(X, Y), Z)$.

If an almost contact metric manifold M satisfies

$$D_T F = 0 \tag{2.8}$$

and

$$(D_{\overline{X}}F)(\overline{Y}, Z) + (D_X F)(Y, Z) - A(Y)(D_X A)(\overline{Z}) = 0 \tag{2.9}$$

then M is called an *almost contact pseudo normal metric manifold* [**3**]. It can be easily seen that for such a manifold relations

$$(D_{\overline{X}}A)(Y) = (D_X A)(\overline{Y}) \qquad (D_X A)(Y) = -(D_{\overline{X}}A)(\overline{Y}) \tag{2.10}$$

are equivalent.

3 - Properties

Theorem 1. *On an almost contact pseudo normal metric manifold, we have*

$$F(Y, D_{\overline{X}} T + \overline{D_X T}) = 0\,. \tag{3.1}$$

Proof. Putting T for Z in (2.9), we get

$$(D_{\overline{X}} F)(\overline{Y}, T) + (D_X F)(Y, T) = 0$$

or

$$\overline{X}(F(\overline{Y}, T)) - F(D_{\overline{X}} \overline{Y}, T) - F(\overline{Y}, D_{\overline{X}} T)$$
$$+X(F(Y, T)) - F(D_X Y, T) - F(Y, D_X T) = 0\,.$$

Since $F(X, T) = 0$, we get

$$g(\overline{\overline{Y}}, D_{\overline{X}} T) + g(\overline{Y}, D_X T) = 0$$

or

$$g(Y, D_{\overline{X}} T + \overline{D_X T}) = A(Y) A(D_{\overline{X}} T)\,.$$

Barring Y, we obtain $g(\overline{Y}, D_{\overline{X}} T + \overline{D_X T}) = 0$, which proves the statement.

Theorem 2. *An almost contact pseudo normal metric manifold is completely integrable if and only if we have*

$$(D_{\overline{X}} F)(\overline{Y}, \overline{Z}) = (D_{\overline{Y}} F)(\overline{X}, \overline{Z})\,. \tag{3.2}$$

Proof. The condition for almost contact metric manifold to be completely integrable [2] is

$$N(\overline{X}, \overline{Y}, \overline{Z}) = 0\,. \tag{3.3}$$

Barring Y in (2.9), we find

$$(D_X F)(\overline{Y}, Z) - (D_{\overline{X}} F)(Y, Z) = A(Y) F(D_{\overline{X}} T, Z)\,. \tag{3.4}$$

In consequence of (3.4), we have

$$\begin{aligned} -(D_X F)(\overline{Y}, Z) + (D_{\overline{X}} F)(Y, Z) + (D_Y F)(\overline{X}, Z) - (D_{\overline{Y}} F)(X, Z) \\ = -A(Y) F(D_{\overline{X}} T, Z) + A(X) F(D_{\overline{Y}} T, Z)\,. \end{aligned} \tag{3.5}$$

Barring Z in above equation, we have

$$(D_{\bar{X}}F)(Y, \bar{Z}) - (D_{\bar{Y}}F)(X, \bar{Z}) - (D_X F)(\bar{Y}, \bar{Z}) + (D_Y F)(\bar{X}, \bar{Z}) = A(X)F(D_{\bar{Y}}T, \bar{Z}) - A(Y)F(D_{\bar{X}}T, \bar{Z}). \tag{3.6}$$

We have [2]

$$(D_X F)(\bar{Y}, \bar{Z}) = -(D_X F)(Y, Z) + A(Y)(D_X A)(\bar{Z}) - A(Z)(D_X A)(\bar{Y}). \tag{3.7}$$

Using (3.7) in (3.6), we obtain

$$\begin{aligned}(D_{\bar{X}}F)(Y, \bar{Z}) &- (D_{\bar{Y}}F)(X, \bar{Z}) + (D_X F)(Y, Z) - (D_Y F)(X, Z)\\ &= A(Y)(D_X A)(\bar{Z}) - A(Z)(D_X A)(\bar{Y}) - A(X)(D_Y A)(\bar{Z}) - A(Z)(D_Y A)(\bar{X})\\ &\quad + A(X)F(D_{\bar{Y}}T, \bar{Z}) - A(Y)F(D_{\bar{X}}T, \bar{Z}).\end{aligned} \tag{3.8}$$

From (2.7) we find

$$N(\bar{X}, \bar{Y}, \bar{Z}) = (D_{\bar{\bar{X}}}F)(\bar{Y}, \bar{Z}) - (D_{\bar{\bar{Y}}}F)(\bar{X}, \bar{Z}) + (D_{\bar{X}}F)(\bar{Y}, \bar{\bar{Z}}) - (D_{\bar{Y}}F)(\bar{X}, \bar{\bar{Z}}).$$

By using (2.8) (3.7) and $(2.10)_2$ the above equation reduces to

$$\begin{aligned}N(\bar{X}, \bar{Y}, \bar{Z}) &= (D_X F)(Y, Z) - (D_Y F)(X, Z) - (D_{\bar{X}}F)(Y, \bar{Z}) + (D_{\bar{Y}}F)(X, \bar{Z})\\ &\quad - 2A(X)(D_{\bar{Y}}A)(\bar{\bar{Z}}) + 2A(Y)(D_{\bar{X}}A)(\bar{\bar{Z}}) - A(Z)[(D_Y A)(\bar{X}) - (D_X A)(\bar{Y})].\end{aligned} \tag{3.9}$$

In consequence of (3.9), (3.8) and (3.3), we get (3.2).

4 - Affine connection

Let D be a riemannian connection and B be an affine connection in an almost contact pseudo normal metric manifold. Let us put

$$\mathcal{H}(X, Y) = B_X Y - D_X Y \quad \text{and} \quad H(X, Y, Z) = g(\mathcal{H}(X, Y), Z). \tag{4.1}$$

So the torsion tensor of B is given by

$$S(X, Y) = \mathcal{H}(X, Y) - \mathcal{H}(Y, X). \tag{4.2}$$

Theorem 3. *For the riemannian metric g we have*

$$(B_X F)(Y, Z) = (B_X g)(\bar{Y}, Z) + g((B_X \Phi)Y, Z). \tag{4.3}$$

Proof. We have

$$
\begin{aligned}
(B_X g)(\overline{Y}, Z) &= X(g(\overline{Y}, Z) - g(B_X \overline{Y}, Z) - g(\overline{Y}, B_X Z) \\
&= X(F(Y, Z) - F(Y, B_X Z) - F(B_X Y, Z) - g(B_X \overline{Y}, Z) + g(\overline{B_X Y}, Z) \\
&= (B_X F)(Y, Z) + g(\overline{B_X Y}, Z) - g(B_X \overline{Y}, Z) = (B_X F)(Y, Z) - g((B_X \Phi) Y, Z),
\end{aligned}
$$

which prove (4.3).

Theorem 4. *On an almost contact pseudo normal metric manifold we have*

$$H(T, \overline{Z}, T) = A(B_T \overline{Z}) \tag{4.4}$$

Proof. From (2.8) we have

$$T(F(X, Y) - F(D_T X, Y) - F(X, D_T Y) = 0$$

or

$$(B_T F)(X, Y) + F(\mathcal{H}(T, X), Y) + F(X, \mathcal{H}(T, Y)) = 0$$

which is equivalent to

$$(B_T F)(X, Y) = H(T, X, \overline{Y}) - H(T, Y, \overline{X}). \tag{4.5}$$

Further, we have

$$(D_X F)(Y, Z) = (B_X F)(Y, Z) - H(X, Y, \overline{Z}) + H(X, Z, \overline{Y}). \tag{4.6}$$

Similarly, we get

$$(D_{\overline{X}} F)(\overline{Y}, Z) = (B_{\overline{X}} F)(\overline{Y}, Z) - H(\overline{X}, \overline{Y}, \overline{Z}) + H(\overline{X}, Z, \overline{\overline{Y}}). \tag{4.7}$$

Also

$$A(Y)(D_X A)(\overline{Z}) = A(Y)[(B_X A)(\overline{Z}) + H(X, \overline{Z}, T)]. \tag{4.8}$$

Thus in consequence of (4.6), (4.7) and (4.8), equation (2.9) takes the form

$$
\begin{aligned}
&(B_X F)(Y, Z) + (B_{\overline{X}} F)(\overline{Y}, Z) - A(Y)(B_X A(\overline{Z}) \\
&= H(X, Y, \overline{Z}) - H(X, Z, \overline{Y}) + H(\overline{X}, \overline{Y}, \overline{Z}) - H(\overline{X}, Z, \overline{\overline{Y}}) + A(Y) H(X, \overline{Z}, T).
\end{aligned} \tag{4.9}
$$

Putting T for X in (4.9), we obtain

$$(B_T F)(Y,Z) - A(Y)(B_T A)(\overline{Z}) = H(T,Y,\overline{Z}) - H(T,Z,\overline{Y}) + A(Y) H(T, \overline{Z}, T). \tag{4.10}$$

Using (4.5) in (4.10), we have

$$-A(Y)(B_T A)(Z) = A(Y) H(T, \bar{Z}, T)$$

which proves the statement.

Theorem 5. *On an almost contact pseudo normal metric manifold we have*

$$H(T, Y, \bar{Z}) + H(T, \bar{Y}, Z) = g((B_T \Phi) Y, Z). \tag{4.11}$$

Proof. Putting T for X in (4.3), we get

$$(B_T F)(Y, Z) = (B_T g)(\bar{Y}, Z) + g((B_T \Phi) Y, Z). \tag{4.12}$$

We have [2]

$$(B_T g)(\bar{Y}, Z) = -H(T, \bar{Y}, Z) - H(T, Z, \bar{Y}) \tag{4.13}$$

Again from (4.5), we get

$$(B_T F)(Y, Z) = H(T, Y, \bar{Z}) - H(T, Z, \bar{Y}). \tag{4.14}$$

Thus in consequence of (4.13) and (4.14), the equation (4.12) reduces to (4.11).

References

[1] R. S. MISHRA, *Affine connections in an almost Grayan manifold*, Tensor **23** (1972), 317-322.

[2] R. S. MISHRA, *Structures on a differentiable manifold and their applications*, Balrampur House, Allahabad 1984.

[3] R. S. MISHRA, *Almost contact metric manifolds*, Monograph **1**, Tensor Soc. of India, Lucknow 1991.

[4] S. SASAKI, *On differentiable manifolds with certain structures which are closely related to almost contact structure*, Tôhoku Mat. J. **12** (1960), 456-476.

Sommario

In questo lavoro si stabiliscono alcune relazioni per le varietà metriche quasi contatto pseudonormali. In particolare si dà una condizione necessaria e sufficiente per la completa integrabilità della struttura.

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