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The twistor transform of a Verlinde formula (**)

1 - Introduction

Let Σ be a compact Riemann surface of genus g. The moduli space $\mathcal{M}_g = \mathcal{M}_g(2, 1)$ of stable rank 2 holomorphic bundles over Σ with fixed determinant bundle of degree 1 is a smooth complex $(3g-3)$-dimensional manifold [25]. The anticanonical bundle of M is the square of a holomorphic line bundle L, some power of which embeds $\mathcal{M}_g$ into a projective space. The dimensions of the vector spaces $H^0(\mathcal{M}_g, \mathcal{O}(L^{m-1}))$ of holomorphic sections of powers of L are known to be independent of the choice of complex structure on Σ, and are given by the formula

$$h^0(\mathcal{M}_g, \mathcal{O}(L^{m-1})) = -m^{g-1} \sum_{i=1}^{2m-1} (-1)^i \operatorname{cosec}^{2g-2}\left(\frac{i\pi}{2m}\right) \tag{1.1}$$

predicted by Verlinde [29]. This is closely related to the structure of the cohomology ring of $\mathcal{M}_g$, and a number of independent proofs and generalizations of (1.1) are now known. Below we shall follow closely the approach of Szenes [27].

In the case in which Σ is a hyperelliptic surface, and is therefore a 2-fold branched covering of $\mathbf{CP}^1$, Desale and Ramanan [9] exhibit $\mathcal{M}_g$ as a complex submanifold of the flag manifold $\mathcal{F}_g = SO(2g+2)/(U(g-1) \times SO(4))$. As explained in [27] this reduces verification of (1.1) to certain $SO(2g+2)$-equivariant calculations. Our contribution is to observe that $\mathcal{F}_g$ is the twistor space of the real oriented Grassmannian $\mathcal{G}_g = SO(2g+2)/(SO(2g-2) \times SO(4))$ in the sense of [7], [8] for all $g \geq 3$. This enables us to relate the cohomology of the symmetric space $\mathcal{G}_g$ directly to the coholomogy of $\mathcal{M}_g$, and we obtain a set of

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generators for the latter which may be compared to the universal ones described in [21], [28], [10]. As a feasibility study, we illustrate the theory in the present paper for the case $g = 3$ which is worthy of special attention since the fibration $\mathcal{F}_3 \to \mathcal{G}_3$ encapsulates the quaternionic structure of the base space in a manner first identified by Wolf [31].

In Section 2 we investigate the cohomology of $\mathcal{G}_3 = SO(8)/(SO(4) \times SO(4))$. Using its quaternionic spin structure, we prove that the odd Pontrjagin classes of $\mathcal{G}_3$ vanish, and that its $\hat{A}$ class simplifies remarkably. In the third section we recover Ramanan's description [23] of the Chern ring of $\mathcal{M}_3$ in the context of the natural mapping $\mathcal{M}_3 \to \mathcal{G}_3$, enabling $h^0(\mathcal{M}_3, L^{m-1})$ to be computed rapidly. Whilst this provides only a particularly simple instance of (1.1), results of the fourth section identify $H^0(\mathcal{M}_3, L^k)$ with a virtual representation of $SO(8)$ that also arises from the kernels of coupled Dirac operators on $\mathcal{G}_3$. Similar techniques can in theory be applied to higher genus cases, and formulae such as $p_1^g = 0$ on $\mathcal{M}_g$ [28], [16] may be expected to interact with properties of $\mathcal{G}_g$ such as the constancy of the elliptic genera considered in [30], [15].

2 - Grassmannian cohomology

From now on we denote by $\mathcal{G}$ the Grassmannian

$$\mathcal{G}_3 = \frac{SO(8)}{SO(4) \times SO(4)} \tag{2.1}$$

that parametrizes real oriented 4-dimensional subspaces of $\boldsymbol{R}^8$. Let W denote the tautological real rank 4 vector bundle over $\mathcal{G}$, and $W^\perp$ its orthogonal complement in the trivial bundle over $\mathcal{G}$ with fibre $\boldsymbol{R}^8$. The bundles W and $W^\perp$ arise from the standard representations of the two $SO(4)$ factors constituting the isotropy subgroup in (2.1), and it follows that

$$T\mathcal{G} \cong W \otimes W^\perp . \tag{2.2}$$

The $SO(8)$-invariant Riemannian metric on $\mathcal{G}$ determines an isomorphism $W \cong W^*$ of vector bundles.

The decomposition (2.2) may be refined by lifting the $SO(4)$ structure of W to Spin$(4) \cong SU(2) \times SU(2)$ on a suitable open dense subset $\mathcal{G}'$ of $\mathcal{G}$. This procedure is one that is familiar from the study of Riemannian 4-manifolds, and

$$W_C \cong U \otimes_C V$$

where U and V are each complex rank 2 vector bundles over $\mathcal{G}'$. The resulting isomorphism

$$(T\mathcal{G})_C \simeq U \otimes (V \otimes W_C^{\perp}) \tag{2.3}$$

reflects the fact that $\mathcal{G}$ is a quaternion-Kähler manifold [31], [24]. In (2.3), U may be thought of as a quaternionic line bundle (usually called H), and its cofactor $V \otimes W_C^{\perp}$ (usually called E) has structure group $SU(2) \times SO(4)$ extending to $Sp(4)$.

The Betti numbers of a quaternion-Kähler $4n$-manifold of positive scalar curvature satisfy $b_{2k+1} = 0$ for all k and $b_{2k-4} \leqslant b_{2k}$ for $k \leqslant n+1$. They are also subject to the linear constraint of [17] which for $n = 4$ takes the form

$$3(b_2 + b_4) = 1 + b_6 + 2b_8 .$$

This is well illustrated by $\mathcal{G}$, which has Poincaré polynomial

$$P_t(\mathcal{G}) = 1 + 3t^4 + 4t^8 + 3t^{12} + t^{16},$$

and is the only real Grassmannian to have $b_4 > 2$. (These facts may be deduced from [12], chapter XI). We shall in fact only be concerned with the subring generated by the Euler class $e = e(W)$ and the first Pontrjagin class $f = p_1(W)$.

Although the classes e and f are very natural, it will ultimately be more convenient to consider

$$u = -c_2(U) \qquad v = -c_2(V).$$

Because of the $\boldsymbol{Z}_2$-ambiguity in the definition U, V, the classes u, v are not integral, but the symmetric products $\odot^2 U$, $\odot^2 V$ are globally defined so $4u$, $4v$ belong to $H^4(\mathcal{G}, \boldsymbol{Z})$. If we write formally $4u = l^2$ then

$$\mathrm{ch}(U) = e^{\frac{l}{2}} + e^{-\frac{l}{2}} = 2 + u + \frac{1}{12}u^2 + \frac{1}{360}u^3 + \frac{1}{20160}u^4 . \tag{2.4}$$

The class l is given geometrical significance by the splitting (3.2). An analogous expression to (2.4) holds for $\mathrm{ch}(V)$, and from $\mathrm{ch}(W_C) = \mathrm{ch}(U)\,\mathrm{ch}(V)$, we obtain

$$e = u - v \qquad f = 2(u + v). \tag{2.5}$$

We may add that $p_2(W) = c_4(W_C) = (u - v)^2$ confirming the well-known relation

$$p_2(W) = e^2 . \tag{2.6}$$

Moreover, the space $H^4(\mathcal{G}, \boldsymbol{Z})$ is generated by e, f together with $e(W^{\perp})$ [**19**].

Proposition 1. *Evaluation on the fundamental cycle* $[\mathcal{G}]$ *yields*

$$e^4 = 2 = e^2 f^2 \qquad e^3 f = 0 = ef^3 \qquad f^4 = 4$$

$$u^4 = \frac{21}{64} = v^4 \qquad u^3 v = -\frac{7}{64} = uv^3 \qquad u^2 v^2 = \frac{5}{64} .$$

We shall deduce these Schubert-type relations from a description of the total Pontrjagin class and the $\hat{A}$ class

$$P(T\mathcal{G}) = 1 + P_1 + P_2 + P_3 + P_4$$

$$\hat{A}(T\mathcal{G}) = 1 + \hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_4$$

of the tangent bundle (2. 2) of $\mathcal{G}$. (Upper case P_i's are used to prevent a future clash of notation.) The classes $\hat{A}_i$, $1 \leqslant i \leqslant 4$ are determined in terms of the P_i in the usual way [**14**], and

Proposition 2. $P_1 = 0 = P_3$ *and* $\hat{A}(\mathcal{G}) = 1 - \dfrac{1}{240} f^2$.

Proof of both Propositions. It is easy to check that, in the presence of (2.5), the two sets of equations of Proposition 1 are equivalent. The equalities $u^4 = v^4$ and $u^3 v = uv^3$ are immediate from the symmetry between U and V, and these are equivalent to $e^3 f = 0 = ef^3$. Using (2.6), we have

$$\begin{aligned} \operatorname{ch}(W_C) &= 4 + f + \frac{1}{12}(-2e^2 + f^2) \\ &+ \frac{1}{360}(-3e^2 f + f^3) + \frac{1}{20160}(2e^4 - 4e^2 f^2 + f^4). \end{aligned} \tag{2.7}$$

From (2.2) and (2.7)

$$\begin{aligned} \operatorname{ch}(T\mathcal{G})_C &= (\operatorname{ch} W_C)(8 - \operatorname{ch} W_C) \\ &= 16 - f^2 + \frac{1}{6}(2e^2 f - f^3) + \frac{1}{720}(-20e^4 + 32e^2 f^2 - 9f^4). \end{aligned} \tag{2.8}$$

In particular $P_1 = 0$, and so we also have

$$\text{(2.9)} \qquad \operatorname{ch}(T\mathcal{G})_C = 16 - \frac{1}{6} P_2 + \frac{1}{120} P_3 + \frac{1}{10080} (P_2^2 - 2P_4).$$

Comparing (2.8) and (2.9) gives

$$\text{(2.10)} \qquad P_2 = 6f^2 \qquad P_3 = 20(2e^2 f - f^3) \qquad P_4 = 140e^4 - 224e^2 f^2 + 81f^4 .$$

The remainder of the proof is based on the following less obvious facts.

i. $\mathcal{G}$ is a spin manifold (see forward to (4.1)) carrying a metric of positive scalar curvature. Therefore its $\hat{A}$ genus

$$\text{(2.11)} \qquad \hat{A}_4 = \frac{1}{2^{16} 3^4 5^2 7} (762\, P_1^4 - 1808\, P_1^2 P_2 + 416\, P_2^2 + 1024\, P_1 P_3 - 384\, P_4)$$

vanishes. Thus

$$\text{(2.12)} \qquad 0 = 416\,(6f)^2 - 384(140e^4 - 224e^2 f^2 + 81f^4) = 5376(-10e^4 + 16e^2 f^2 - 3f^4).$$

ii. The dimension d of the isometry group of any quaternion-Kähler 16-manifold with positive scalar curvature is given by $d = 7 - \frac{8}{3} P_1 u^3 + 64u^4$, [24], p. 170. In the present case, $d = \dim SO(8) = 28$ and we obtain

$$\text{(2.13)} \qquad 21 = 64u^4 = \frac{1}{4}(16e^4 + 24e^2 f^2 + f^4).$$

iii. On any compact quaternion-Kähler $4n$-manifold M with positive scalar curvature and $n > 2$, the index $\hat{A}(M, \odot^2 U) = \langle \operatorname{ch}(\odot^2 U)\hat{A}, [M] \rangle$ vanishes; this is a consequence of [24], Corollary 6.7 which is explained in [17]. Given that

$$\operatorname{ch}(\odot^2 U) = 3 + 4u + \frac{4}{3} u^2 + \frac{8}{45} u^3 + \frac{4}{315} u^4$$

$$\hat{A} = 1 - \frac{1}{24} P_1 - \frac{1}{2^5 3^2 5} P_2 - \frac{1}{2^6 3^3 5^1 7} P_3 = 1 - \frac{1}{240} f^2 + \frac{1}{1008} (2e^2 f - f^3)$$

and $u = \frac{1}{4}(2e + f)$, it follows that

$$\text{(2.14)} \qquad 24e^4 - 26e^2 f^2 + f^4 = 0 .$$

Proposition 1 now follows from (2.12), (2.13), (2.14), and it only remains to prove that $P_3 = 0$. Because of the symmetry between W and $W^{\perp}$, it suffices to prove that $P_3 e = 0 = P_3 f$, but this follows from (2.10) and Proposition 1.

Remark. The vanishing of $\hat{A}_4$ and (2.12) above is in fact equivalent to the vanishing of the index $\hat{A}(M, T)$ of the Dirac operator coupled to the tangent bundle (see (4.2)), essentially the so-called Rarita-Schwinger operator. This index is known to be equivariantly constant on any spin manifold with S^1 action [30], and always vanishes in the homogeneous setting [15].

3 - The flag manifold and moduli space

We denote by $\mathcal{F}$ the complex 9-dimensional homogeneous space

$$\mathcal{F}_3 = \frac{SO(8)}{U(2) \times SO(4)} \tag{3.1}$$

that parametrizes complex 2-dimensional subspaces Π of $\boldsymbol{C}^8$ that are isotropic with respect to a standard $SO(8)$-invariant bilinear from. It has a complex contact structure that was studied in [31] and exhibits it as the twistor space of $\mathcal{G}$ in the sense of [24]. Projecting Π to a real 4-dimensional subspace of $\boldsymbol{R}^8$ determines an $SO(8)$-equivariant mapping $\pi: \mathcal{F} \to \mathcal{G}$ and each fibre of π is isomorphic to $SO(4)/U(2)$ and defines a rational curve in the complex manifold $\mathcal{F}$.

From standard facts about twistor spaces [6], [24], [22], one knows that $\mathrm{Pic}(\mathcal{F})$ is generated by a holomorphic line bundle L on $\mathcal{F}$ such that

i. the restriction of L to each fibre $\pi^{-1}(x) \simeq \boldsymbol{CP}^1$ equals $\mathcal{O}(2)$

ii. L^5 is isomorphic to the anticanonical bundle κ^{-1} of $\mathcal{F}$.

The line bundle L admits a square root over an open set $\mathcal{G}'$ of $\mathcal{G}$ on which U and V are defined, there is a C^∞ isomorphism

$$\pi^* U \simeq L^{1/2} \oplus L^{-1/2}. \tag{3.2}$$

Let l denote the fundamental class $c_1(L)$ in $H^2(\mathcal{F}, \boldsymbol{Z})$. From the Leray-Hirsch theorem, there is an identity $(\frac{l}{2})^2 + \pi^* c_2(U) = 0$ of real cohomology classes. In terms of integral classes and omitting π^*

$$l^2 = 4u. \tag{3.3}$$

In the notation of the Introduction, let $\mathcal{M} = \mathcal{M}_3$. Szenes exhibits the latter as the zero set of a non-degenerate holomorphic section $s \in H^0(\mathcal{F}, \mathcal{O}(\sigma^*))$, where $\sigma = \odot^2\tau$ and τ denotes the tautological rank 2 complex vector bundle acquired from the embedding $\mathcal{F} \subset Gr_2(\boldsymbol{C}^8)$. (Such a Section s corresponds to a quadratic form on $\boldsymbol{C}^8$, but we shall not mention this again until the end of Section 4.) From the coset description (3.1), it follows that

$$\tau \simeq L^{-1/2} \otimes \pi^* V \tag{3.4}$$

the right-hand side is well defined on $\mathcal{F}$, even though the individual factors only make sense locally (for example on $\pi^{-1}(\mathcal{G}'))$. Since $V \simeq V^*$, we have $\sigma^* \simeq L \otimes \pi^* \odot^2 V$. The resulting holomorphic structure on $\pi^* \odot^2 V$ coincides with that induced in a standard way from the fact that $\odot^2 V$ has a self-dual connection on the quaternion-Kähler manifold $\mathcal{G}$, in the sense of [18]. In particular, $\pi^* \odot^2 V$ is trivial over each fibre $\pi^{-1}(x) \simeq \boldsymbol{CP}^1$. From now on we shall write $\odot^2 V$ in place of $\pi^* \odot^2 V$, and often omit tensor product signs.

The cohomology classes l, u, v may be pulled back from both $\mathcal{G}$ and $\mathcal{F}$ to $\mathcal{M}$, and we shall denote the resulting elements of $H^i(\mathcal{M}, \boldsymbol{R})$ by the same symbols.

Proposition 3. *On $\mathcal{M}$, $3u^2 + 10uv + 3v^2 = 0$, and evaluation on $[\mathcal{M}]$ yields*

$$u^3 = \frac{7}{2} = -v^3, \qquad uv^2 = \frac{3}{2} = -u^2 v.$$

Proof. The submanifold $\mathcal{M}$ of $\mathcal{F}$ is Poincaré dual to the Euler class $c_3(\sigma^*)$, which is readily computed from the formula $\mathrm{ch}(\sigma^*) = e^l\,\mathrm{ch}(\odot^2 V)$ (see (3.4)) and equals $4l(u - v)$. Then, for example,

$$\langle u^3, [\mathcal{M}]\rangle = \langle u^3 c_3(\sigma^*), [\mathcal{F}]\rangle = \langle 4l(u^4 - u^3 v), [\mathcal{F}]\rangle = 8\langle u^4 - u^3 v, [\mathcal{G}]\rangle = \frac{7}{2},$$

the last equality from Proposition 1. The evaluation of $u^2 v$, uv^2 and v^3 follows in exactly the same way.

Since $H^8(\mathcal{M}, \boldsymbol{R}) \simeq H^4(\mathcal{M}, \boldsymbol{R})$ is 2-dimensional [20], there must be a non-trivial linear relation $au^2 + buv + cv^2 = 0$. The solution $\dfrac{a}{b} = \dfrac{c}{b} = \dfrac{3}{10}$ can be found by multiplying the left-hand side by u and v in turn.

The next result gives an independent derivation of the characteristic ring in the context of the twistor fibration $\mathcal{F} \to \mathcal{G}$.

Proposition 4. *The Chern and Pontrjagin classes of* $\mathcal{M}$ *are given by*

$$c_1 = 2l \qquad c_2 = 4(3u+v) \qquad c_3 = 8lu \qquad c_4 = -\frac{112}{3}uv \qquad c_5 = c_6 = 0$$

$$p_1 = -8(u+v) \qquad p_2 = \frac{3}{8}p_1^2 \qquad p_3 = 0\,.$$

Proof. It is known [24] that the fibration π gives a C^∞ splitting of the holomorphic tangent bundle of $\mathcal{F}$: $T^{1,0}\mathcal{F} \simeq L \oplus L^{1/2}(V \otimes W_C^\perp)$.

Combining this with the isomorphism

$$T^{1,0}\mathcal{F}|_{\mathcal{M}} \simeq T^{1,0}\mathcal{M} \oplus (L \odot^2 V)|_{\mathcal{M}},$$

we obtain $\quad \mathrm{ch}(T^{1,0}\mathcal{M}) = e^l + e^{\frac{l}{2}}\,\mathrm{ch}\,(VW_C^\perp) - e^l\mathrm{ch}(\odot^2 V)$

$$= e^l(1 + e^{-\frac{l}{2}}\,\mathrm{ch}\,V(8 - \mathrm{ch}\,W_C) - \mathrm{ch}\,(\odot^2 V))\,.$$

This yields the required expressions for c_1, c_2, c_3. We also get $c_4 = 28(u+v)^2$ which reduces to $-\frac{112}{3}uv$ from Proposition 3. We next obtain $c_5 = -32\,lv(u+v)$, so that $c_5 l = 0$ and the vanishing of c_5 follows from the fact that $H^2(\mathcal{M}, \mathbf{R})$ is 1-dimensional [20]. Finally, all these equalities combine to yield

$$c_6 = \frac{1}{3}(504u^3 + 2824u^2v + 1928uv^2 + 120v^3),$$

and Proposition 3 implies that $c_6 = 0$. The Pontrjagin classes p_i of $\mathcal{M}$ are now determined from the Chern classes by the usual relations.

Remark. The cohomology ring and Chern classes of $\mathcal{M}$ were computed in [23], Theorem 4, and comparison with that shows that

$$h = l \qquad \nu = \frac{1}{2}(3u+v)\,.$$

In general, it is known that the total Pontrjagin class of $\mathcal{M}_g$ equals $(1 + \frac{1}{2g-2}p_1)^{2g-2}$ [21]. Moreover, $p_1^g = 0$ [16], [28] and $c_i = 0$ if $i > 2g-2$ [11].

The above enable the dimension d_k of $H^0(\mathcal{M}, \mathcal{O}(L^k))$ to be computed quickly. For this purpose it is convenient to set $k = m - 1 \geqslant 0$.

Theorem 1. $d_{m-1} = \frac{1}{45} m^2(11 + 20m^2 + 14m^4)$.

Proof. Given that $c_1(T^{1,0}\mathcal{F}) = 2l$, the Todd class $\mathrm{td}(T^{1,0}\mathcal{M})$ equals

$$e^l \hat{A}(T\mathcal{M}) = e^l[1 - \frac{1}{24} p_1 + \frac{1}{2^7 3^2 5}(7p_1^2 - 4p_2)].$$

Using Propositions 3, 4 and the Riemann-Roch theorem, we obtain

$$\begin{aligned} d_{m-1} &= \langle e^{ml}(1 + \frac{1}{3}(u+v) - \frac{11}{135} uv), [\mathcal{M}]\rangle \\ &= -\frac{22}{135} m^2 u^2 v + \frac{2}{9} m^4 (u^3 + u^2 v) + \frac{4}{45} m^6 u^3 \end{aligned}$$

and the result follows.

4 - Equivariant indexes

In this section, we begin by considering the Dirac operator over the Grassmannian $\mathcal{G}$. Recall from (2.3) that the quaternionic structure of $\mathcal{G}$ is characterized by the vector bundles $H = U$ and $E \cong VW_C$ (juxtaposition denotes tensor product). For $p \leqslant 4$, the exterior power $\wedge^p E$ contains a proper subbundle $\wedge_0^p E$ with the property that $\wedge^p E \cong \wedge_0^p E \oplus \wedge^{p-2} E$ and, as described in [4], the total spin bundle Δ of $\mathcal{G}$ decomposes as $\Delta_+ \oplus \Delta_-$ where

(4.1) $$\Delta_+ \cong \odot^4 U \oplus \odot^2 U \wedge_0^2 E \oplus \wedge_0^4 E \qquad \Delta_- \cong \odot^3 UE \oplus U \wedge_0^3 E.$$

The fact that all the summands on the right-hand side are globally defined confirms that $\mathcal{G}$ is spin, though we shall not in fact need the decompositions (4.1).

Now let X be any other complex vector bundle over $\mathcal{G}$. The choice of a connection on X allows one to extend the Dirac operator on $\mathcal{G}$ to an elliptic operator

$$D_X : \Gamma(\Delta_+ X) \to \Gamma(\Delta_- X).$$

The index of this coupled Dirac operator is by definition $\dim(\ker D_X) - \dim(\mathrm{coker}\, D_X)$. This extends to a homomorphism $K(\mathcal{G}) \to \boldsymbol{Z}$, so that the index of D_X is also defined when X is a virtual vector bundle. The Atiyah-Singer index theorem [3] asserts that the index of D_X equals

(4.2) $$\hat{A}(\mathcal{G}, X) = \langle \mathrm{ch}(X) \hat{A}(T\mathcal{G}), [\mathcal{G}]\rangle.$$

In our situation, this fact is closely related to the Riemann-Roch theorem on $\mathcal{F}$ which provides the following interpretation of d_k.

Theorem 2. *Let* $X_k = \odot^{2k+4}U - \odot^{2k+2}U \odot^2 V + \odot^{2k}U \odot^2 V - \odot^{2k-2}U$, $k \geq 1$. *Then* $d_k = \hat{A}(\mathcal{G}, X_k)$.

Proof. Let σ denote the rank 3 vector bundle $\odot^2\tau$ as above, and let (k) denote the operation of tensoring with L^k. The description of $\mathcal{M}$ as the zero set of a section of $\sigma^* \simeq \odot^2 V(1)$ provides a Koszul complex

$$0 \to \mathcal{O}_{\mathcal{F}}(\Lambda^3 \sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\Lambda^2 \sigma(k)) \to \mathcal{O}_{\mathcal{F}}(\sigma(k)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0$$

or equivalently,

$$0 \to \mathcal{O}_{\mathcal{F}}(k-3) \to \mathcal{O}_{\mathcal{F}}(\odot^2 V(k-2)) \to \mathcal{O}_{\mathcal{F}}(\odot^2 V(k-1)) \to \mathcal{O}_{\mathcal{F}}(k) \to \mathcal{O}_{\mathcal{M}}(k) \to 0 .$$

It follows that

$$\chi(\mathcal{M}, \mathcal{O}(k)) = a_k - b_{k-1} + b_{k-2} - a_{k-3} \tag{4.3}$$

where

$$a_k = \chi(\mathcal{F}, \mathcal{O}(k)) \qquad b_k = \chi(\mathcal{F}, \mathcal{O}(\odot^2 V(k))) . \tag{4.4}$$

These holomorphic Euler characteristics may be computed using the Riemann-Roch theorem and the cohomological version [24], (7.2) of the twistor transform; the result is

$$a_k = \hat{A}(\mathcal{G}, \odot^{2k+4}U) \qquad b_k = \hat{A}(\mathcal{G}, \odot^{2k+4}U \odot^2 V) . \tag{4.5}$$

Finally, Proposition 4 implies that the canonical bundle $\kappa(\mathcal{M})$ is isomorphic to L^{-2}, so by Serre duality and Kodaira vanishing, $H^i(\mathcal{M}, \mathcal{O}(k)) = 0$ for all $i \geq 1$ and $k \geq -1$. In particular, $\chi(\mathcal{M}, \mathcal{O}(k)) = \dim H^0(\mathcal{M}, \mathcal{O}(k))$ for all $k \geq -1$, and the theorem now follows from (4.3).

The isometry group $SO(8)$ of $\mathcal{G}$ acts naturally on the cohomology groups over $\mathcal{F}$ of the sheaves $\mathcal{O}(k)$, $\mathcal{O}(\odot^2 V(k))$ considered above. The integers a_k, b_k and

$$d_k = a_k - b_{k-1} + b_{k-2} - a_{k-3}$$

are therefore the dimensions of certain virtual $SO(8)$-modules, and we identify these shortly.

Let $V(\gamma)$ denote the complex irreducible representation of $SO(8)$ with dominant weight γ, where $\gamma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$. We adopt standard coordinates so that $V(1, 0, 0, 0) = \boldsymbol{C}^8$ is the fundamental representation, and $V(1, 1, 0, 0) = \mathfrak{so}(8, \boldsymbol{C})$ is the complexified adjoint representation.

Proposition 5. *Let* $A_k = V(k, k, 0, 0)$ *and* $B_k = V(k+1, k-1, 0, 0)$. *Then* $a_k = \dim A_k$ *and* $b_k = \dim B_k$.

Proof. The Weyl dimension formula states that

$$\dim (V(\gamma)) = \prod_{\alpha \in R_+} \frac{\langle \alpha, d + \gamma \rangle}{\langle \alpha, d \rangle}\,.$$

where R_+ denotes the set of positive roots and d is half of their sum. With the above coordinates,

$$R_+ = \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1),$$

$$(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1), (0, 1, -1, 0), (0, 1, 0, -1), (0, 0, 1, -1)\}$$

$d = (3, 2, 1, 0)$ and we obtain

$$\dim A_k = \frac{1}{4320}(k+1)(k+2)^3(2k+5)(k+3)^3(k+4)$$

$$\dim B_k = \frac{1}{1440} k(k+1)^2(k+2)(2k+5)(k+3)(k+4)^2(k+5)\,.$$

We claim that the right-hand sides are equal to a_k and b_k respectively. It follows from (4.4) that a_k and b_k are polynomials in k of degree 9, and by Serre duality,

$$a_{-k} = -a_{k-5} \qquad b_{-k} = -b_{k-5} \qquad k \in \boldsymbol{Z}\,. \tag{4.6}$$

By (4.6) and suitable vanishing theorems [5], $a_k = 0 = b_k$ for $k = -4, -3, -\frac{5}{2}, -2, -1$. In addition, $\mathcal{F}$ has Todd genus $a_0 = 1 = -a_{-5}$, and $b_0 = 0 = b_{-5}$. Accordingly,

$$a_k = \frac{1}{4320}(k+1)(k+2)(2k+5)(k+3)(k+4)\,\widetilde{a}_k$$

$$b_k = \frac{1}{1440} k(k+1)(k+2)(2k+5)(k+3)(k+4)(k+5)\,\widetilde{b}_k$$

where $\widetilde{a}_k$ is a quartic polynomial in k with $\widetilde{a}_0 = 36$ and $\widetilde{b}_k$ is quadratic in k.

Let $n = 2k + 4$. The formulae (4.5) involve $\mathrm{ch}(\odot^n U) = f(n)$, where

$$f(x) = \frac{e^{(x+1)\frac{l}{2}} - e^{-(x+1)\frac{l}{2}}}{e^{\frac{l}{2}} - e^{-\frac{l}{2}}} \tag{4.7}$$

(see (3.2)). To evade an explicit calculation of $\mathrm{ch}(\odot^n U)$, we exploit the following formulae which are easily deduced from (4.7).

Lemma. $f'(0) = \dfrac{\frac{l}{2}}{\tanh\frac{l}{2}}$, $f''(0) = u$.

The right-hand side of the first equation is the series used in the definition of Hirzebruch's L-genus, and using (3.3) can be rewritten as

$$\begin{aligned}\frac{d}{dn}\bigg|_{n=0} \mathrm{ch}(\odot^n U) &= 1 - \sum_{j \geqslant 1} (-1)^j \frac{2^{2j} B_j}{(2j)!} u^{2j} \\ &= \frac{1}{2}\left(1 - \frac{1}{3} u - \frac{1}{45} u^2 + \frac{2}{945} u^3 - \frac{1}{4725} u^4\right)\end{aligned}$$

where B_j are the Bernoulli numbers [14]. From above, we obtain

$$\frac{d}{dk}\bigg|_{k=-2} a_k = \frac{1}{270}(u^4 + 2u^3 v + u^2 v^2) - \frac{1}{4725} u^4 = 0 = \frac{d^2}{dk^2}\bigg|_{k=-2} a_k .$$

It follows that $\tilde{a}_k$ is divisible by $(k+2)^2$, and by Serre duality by $(k+3)^2$. We obtain $\tilde{a}_k = (k+2)^2(k+3)^2$. The identification $\tilde{b}_k = (k+1)(k+4)$ is similar, and proceeds using a less-enlightening version of the previous Lemma; we omit the details.

The following table displays some of the above dimension functions in terms of k.

k	0	1	2	3	4	5	6	7	8
a_k	1	28	300	1925	8918	32928	102816	282150	698775
b_k	0	35	567	4312	21840	85050	274890	772464	1945944
d_k	1	28	265	1392	5145	15100	37681	83392	168273

Applying Serre duality and Kodaira vanishing over $\mathcal{F}$, recalling that $\kappa(\mathcal{F}) \simeq L^{-5}$, one shows that there is in fact an $SO(8)$-equivariant isomorphism $A_k \simeq H^0(\mathcal{F}, \mathcal{O}(k))$. In particular, A_1 may be identified with both the space of holomorphic sections of L and the Lie algebra $\mathfrak{so}(8, \boldsymbol{C})$ of infinitesimal automorphisms of the contact structure of $\mathcal{F}$. There is an associated moment mapping $\mathcal{F} \to \boldsymbol{P}(\mathfrak{so}(8, \boldsymbol{C})^*) \simeq \boldsymbol{C}P^{27}$ that identifies $\mathcal{F}$ with the projectivization of the nilpotent orbit of minimal dimension [26]. Accordingly, the $SO(8)$-equivariant linear mapping

$$\phi_k : \odot^k (H^0(\mathcal{F}, \mathcal{O}(1))) \to H^0(\mathcal{F}, \mathcal{O}(k)) \tag{4.8}$$

is onto for all $k \geq 1$. Indeed, A_k is the irreducible summand of $\odot^k A_1$ of highest weight, and it suffices to show that the restriction of ϕ_k to A_k is an isomorphism. Observe that A_k contains a decomposable tensor product $\xi^{\otimes k}$ for some non-zero $\xi \in A_1$ and $\phi_k(\xi^{\otimes k})$, being the k-th power of ξ regarded as a section of L, is also non-zero. The irreducibility of A_k and Schur's lemma establishes the claim.

A similar argument can be given to establish an $SO(8)$-equivariant isomorphism $B_k \simeq H^0(\mathcal{F}, \mathcal{O}(\odot^2 V(k)))$, given that $H^i(\mathcal{F}, \mathcal{O}(\odot^2 V(k)))$ vanishes for all $i > 0$ and $k \geq 0$. One considers the mapping

$$\psi_k : H^0(\mathcal{F}, \mathcal{O}(\odot^2 V(1))) \otimes H^0(\mathcal{F}, \mathcal{O}(k-1)) \to H^0(\mathcal{F}, \mathcal{O}(\odot^2 V(k)))$$

in which $H^0(\mathcal{F}, \mathcal{O}(\odot^2 V(1)))$ is isomorphic to the irreducible 35-dimensional $SO(8)$-module $\odot_0^2 \boldsymbol{C}^8$ with highest weight $(2, 0, 0, 0)$. The irreducible summand of highest weight in the tensor product is isomorphic to B_k and the restriction of ψ_k to this is an isomorphism.

The above arguments can be streamlined by applying more sophisticated twistor transform machinery contained, for example, in [5]. In particular, A_k and B_k are known to be isomorphic to the respective kernels of natural twistor operators

$$\alpha_k : \odot^{2k} U \to E \quad \odot^{2k+1} U \qquad\qquad \beta_k : \odot^{2k} U \quad \odot^2 V \to E \quad \odot^{2k+1} U \quad \odot^2 V.$$

Recall that $\mathcal{M}$ is the zero set of an element s of the space $B_1 \simeq \odot_0^2 \boldsymbol{C}^8$. For suitable hyperelliptic surfaces Σ, the section s will be a real element; at each point of $\mathcal{G}$ it then defines a section of $\odot^2(W \oplus W^\perp)$, which is a trivial bundle with fibre $\boldsymbol{R}^8$ (see (2.1)). In these terms the element $\tilde{s} \in \ker \beta_1$ determined by s

is essentially the image of s by the homomorphism

$$\odot^2(W \oplus W^{\perp})_C \to \odot^2 W_C \to \odot^2 U \ \odot^2 V \simeq \mathrm{Hom}(\odot^2 V, \odot^2 U).$$

This may be used to describe $\mathcal{M}$ as a branched cover of a real subvariety of $\mathcal{G}$.

The Horrocks instanton bundle over $\boldsymbol{CP}^5$ discussed at the end of [18] provides an analogous situation in which a geometric object is defined by a non-degenerate solution of a twistor equation over a homogeneous space. Such situations are worthy of more systematic investigation.

References

[1] M. F. Atiyah and R. Bott, *Yang-Mills equations over Riemann surfaces*, Philos. Trans. R. Soc. London **308** (1982), 523-615.

[2] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. Lond. **362** (1978), 425-461.

[3] M. F. Atiyah and I. M. Singer, *The index theory of elliptic operators* III, Ann. Math. 87 (1968), 546-604.

[4] R. Barker and S. M. Salamon, *Analysis on a generalized Heisenberg group*, J. London Math. Soc. 28 (1983), 184-192.

[5] R. J. Baston and M. G. Eastwood, *The Penrose transform: its interaction with representation theory*, Oxford Univ. Press 1989.

[6] A. Besse, *Einstein manifolds*, Springer, Berlin 1987.

[7] R. L. Bryant, *Lie groups and twistor spaces*, Duke Math. J. **52** (1985), 223-261.

[8] F. E. Burstall and J. H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lect. Notes Math. **1424**, Springer, Berlin 1990.

[9] U. V. Desale and S. Ramanan, *Classification of vector bundles of rank 2 over hyperelliptic curves*, Invent. Math. 38 (1976), 161-185.

[10] S. K. Donaldson, *Gluing techniques in the cohomology of moduli spaces*, Topological Methods in Modern Mathematics, Publish or Perish, Boston 1993.

[11] D. Gieseker, *A degeneration of the moduli space of stable bundles*, J. Diff. Geometry **19** (1984), 173-206.

[12] W. Greub, S. Halperin and R. Vanstone, *Curvature, connections and characteristic classes*, Vol. 3, Academic Press, Boston 1976.

[13] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212** (1978), 215-248.

[14] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, Berlin 1966.

[15] F. HIRZEBRUCH and P. SLADOWY, *Ellipic genera, involutions, and homogeneous spin manifolds*, Geom. Dedicata **35** (1990), 309-343.

[16] F. C. KIRWAN, *The cohomology rings of moduli spaces of bundles over Riemann surfaces*, J. Amer. Math. Soc.. **5** (1992), 853-906.

[17] C. R. LEBRUN and S. M. SALAMON, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), 109-132.

[18] M. MAMONE CAPRIA and S. SALAMON, *Yang-Mills fields on quaternionic spaces*, Nonlinearity **1** (1988), 517-530.

[19] J. W. MILNOR and J. D. STASHEFF, *Characteristic classes*, Annals of Math. Studies 76, Princeton Univ. Press 1974.

[20] P. E. NEWSTEAD, *Topological properties of some spaces of stable bundles*, Topology **6** (1967), 241-262.

[21] P. E. NEWSTEAD, *Characteristic classes of stable bundles over an algebraic curve*, Trans. Am. Math. Soc. **169** (1972), 337-345.

[22] Y. S. POON and S. M. SALAMON, *Eight-dimensional quaternion-Kähler manifolds with positive scalar curvature*, J. Diff. Geometry **33** (1991), 363-378.

[23] S. RAMANAN, *The moduli space of vector bundles over an algebraic curve*, Math. Ann. **200** (1973), 69-84.

[24] S. SALAMON, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143-171.

[25] C. S. SESHADRI, *Space of unitary vector bundles on a compact Riemann surface*, Ann. of Math. **85** (1967), 303-336.

[26] A. F. SWANN, *Hyperkähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), 421-450.

[27] A. SZENES, *Hilbert polynomials of moduli spaces of rank 2 vector bundles* I, Topology **32** (1993), 587-597.

[28] M. THADDEUS, *Conformal field theory and the moduli space of stable bundles*, J. Diff. Geometry **35** (1992), 131-149.

[29] E. VERLINDE, *Fusion rules and modular transformations in 2d conformal field theory*, Nucl. Phys. B. **300** (1988), 360-376.

[30] E. WITTEN, *The index of the Dirac operator in loop space*, Elliptic curves and modular forms in algebraic topology, Lect. Notes Math. **1326**, Springer, Berlin 1988.

[31] J. A. WOLF, *Complex homogeneous contact structures and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033-1047.

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