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Almost periodic solutions of certain differential equations in Fréchet spaces (**)

1 - An existence theorem

Let $E = E(\tau)$ be a *complete, Hausdorff, locally convex space* (l.c.s.) over the field ϕ ($\phi = \boldsymbol{C}$ or $\boldsymbol{R}$). Its topology τ is generated by a family of continuous seminorms $Q = \{p, q, \dots\}$. A basis of neighbourhoods (of the origin in E) contains sets of the form

$$U = U(\varepsilon;\ p_i,\ 1 \leqslant i \leqslant n) = \{x \in E \mid p_i(x) < \varepsilon,\ 1 \leqslant i \leqslant n\}.$$

We say a continuous function $f\colon \boldsymbol{R} \to E$ is *almost periodic* (a.p.) if for each neighbourhood U, there exists a real number $l = l(U) > 0$ such that every interval $[a, a + l]$ contains at least a point s such that $f(t + s) - f(t) \in U$ for every $t \in \boldsymbol{R}$.

If for each $x^* \in E^*$ (E^* the dual space of E), $x^* f\colon \boldsymbol{R} \to \boldsymbol{R}$ is a.p., then we say f is *weakly almost periodic* (w.a.p.). Now if the topology τ is induced by an invariant and complete metric, we say E is a *Fréchet space*.

We define a *perfect Fréchet space* as a Fréchet space E in which every function $f\colon \boldsymbol{R} \to E$ which satisfies

i) $\{f(t) \mid t \in \boldsymbol{R}\}$ is bounded in E

ii) the derivative $f'(t)$ is a.p.

is necessarily a.p.

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For more information about almost periodicity in a locally convex space, see our paper [1].

Now we are going to state and prove a result inspired by a result of S. Zaidman [3]

Theorem. *Consider in a perfect Fréchet space E, the differential equation*

$$x'(t) = (A+B)x(t) + f(t) \qquad t \in \boldsymbol{R} \tag{1}$$

where the closed linear operator A is the infinitesimal generator of an equi-continuous C_0-group $T(t)$, such that the function $T(t)x: \boldsymbol{R} \to E$ is a.p. for each $x \in E$ and B is a compact linear operator in E. We assume for every $p \in Q$ there exists $q \in Q$ such that $p(T(t)x) \leqslant q(x)$ for every $t \in \boldsymbol{R}$ and every $x \in E$. Finally we suppose $f: \boldsymbol{R} \to E$ is a.p.

Then every w.a.p. solution of (1) *is a.p.*

Before proving this result let us point out two simple facts.

Lemma 1. *If $g: \boldsymbol{R} \to E$ (E l.c.s.) is w.a.p. and $A \in L(E, E)$, then $Ag(t)$ is also w.a.p.*

Lemma 2. *If $T(t)x: \boldsymbol{R} \to E$ is an a.p. function for each $x \in E$ (E l.c.s.), then $\check{T}(t)x = T(-t)x: \boldsymbol{R} \to E$ is also a.p. for each $x \in E$.*

Proof of the Theorem. Let $x(t)$ be a w.a.p. solution of (1). It is

$$x(t) = T(t)x(0) + \int_0^t T(t-\sigma)(Bx(\sigma) + f(\sigma))\,\mathrm{d}\sigma$$

$x(t)$ is weakly bounded (see [2], p. 533). Therefore $x(t)$ is bounded (see Appendix). We can deduce $\{Bx(t) \mid t \in \boldsymbol{R}\}$ is relatively compact in E.

But $Bx(t): \boldsymbol{R} \to E$ is w.a.p. (Lemma 1); therefore $Bx(t)$ is a.p. ([1], Theorem 8). Now, using Lemma 2 above and Lemma 5 of [1], p. 539, we can say $T(-t)(Bx(t) + f(t))$ is a.p.

Put $y(t) = x(t) - T(t)x(0)$. The condition on $T(t)x(0)$ shows that it is a bounded function. Therefore $y(t)$ is bounded as sum of two bounded functions. We can deduce that $T(-t)y(t)$ and therefore

$$F(t) = \int_0^t T(-\sigma)(Bx(\sigma) + f(\sigma))\,\mathrm{d}\sigma$$

are bounded. On another side

$$F'(t) = T(-t)(Bx(t) + f(t))$$

is a.p. As E is a perfect Fréchet space, $F(t)$ is a.p. We then conclude that $y(t) = T(t)F(t)$ is a.p. using Lemma 5 of [1], p. 539.

Appendix. *If f: $\boldsymbol{R} \to E$ (E Fréchet space) is weakly bounded, then f is bounded.*

Proof. f is weakly bounded means $\sup_{t \in \boldsymbol{R}} |x^* f(t)| < \infty$ for every $x^* \in E^*$. Suppose $f(\boldsymbol{R})$ is not bounded, then there exists a semi-norm p such that $p(f(t_n)) \to \infty$, for some sequence $(t_n) \subseteq \boldsymbol{R}$. Let E_p be the completion of the normed space $E/\ker p$ in the norm p. Then E_p is a Banach space and $\tilde{f}(t_n) = f(t_n)/\ker p$ is unbounded in E_p. Consequently there exists $\varphi \in E_p^*$ such that $|\varphi(\tilde{f}(t_n))| \to \infty$ as $n \to \infty$.

The natural map J: $E \to E_p$ is continuous, because J: $E \to E/\ker p \subseteq E_p$ is defined by $J(e) = e/\ker p$. Thus J^*: $E_p^* \to E^*$ is continuous. Finally let us set $\psi = J^*(\varphi) \in E^*$. We have

$$|\psi(f(t_n))| = |J^*(\varphi)(f(t_n))| = |\varphi(J(f(t_n)))| = |\varphi(\tilde{f}(t_n))| \to \infty$$

as $n \to \infty$. This completes the proof.

References

[1] G. M. N'Guerekata, *Almost-periodicity in linear topological spaces and applications to abstract differential equations*, Internat. J. Math. Math. Sci. **7** (1984), 529-540.

[2] G. M. N'Guerekata, *Notes on almost-periodicity in topological vector spaces*, Internat. J. Math. Math. Sci. **9** (1986), 201-204.

[3] S. Zaidman, *Some remarks on almost-periodicity*, Atti Accad. Sci. Torino **106** (1971-1972), 63-67.

Sommario

Si stabilisce un teorema di esistenza di soluzioni quasi periodiche dell'equazione (1).

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