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On some semi-invariant submanifolds of a trans-Sasakian manifold (**)

0 - Introduction

Bejancu and Papaghiuc introduced and studied semi-invariant submanifolds of a Sasakian manifold [2], [3]. Roughly speaking, a semi-invariant submanifold of a Sasakian manifold is a notion corresponding to that of *CR*-submanifolds in a Kaehler manifold [1]. On the other hand semi-invariant submanifolds of a Kenmotsu manifold have been studied by Kobayashi [12].

More general, are the notions of α-Sasakian structure and β-Kenmotsu structure [9]. In [13] J. A. Oubina introduced a new class of almost contact Riemannian manifolds known as *trans-Sasakian manifolds*, which generalize both α-Sasakian and β-Kenmotsu structures.

The purpose of this note is to study the class of the semi-invariant submanifolds, normal to the structure vector field ξ of a trans-Sasakian manifold.

1 - Preliminaries

Let $\bar{M}$ be an $(2n+1)$-dimensional *almost contact metric manifold* with almost contact metric structure (ϕ, ξ, η, g). Then we have by definition [4]

$$(1.1) \qquad \phi^2 = -I + \eta \otimes \xi \qquad \phi\xi = 0 \qquad \eta \circ \phi = 0 \qquad \eta(\xi) = 1$$

$$(1.2) \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad \eta(X) = g(X, \xi)$$

for any vector field X, Y on $\bar{M}$.

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An almost contact structure (ϕ, ξ, η) is said to be *normal* if the almost complex structure J on $\bar{M} \times R$ given by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t})$$

where f is a C^∞ function on $\bar{M} \times R$, is *integrable*, which is equivalent to the condition $[\phi, \phi] + 2\,\mathrm{d}\eta \otimes \xi = 0$ where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

Let M be an m-dimensional *Riemannian manifold isometrically immersed in* $\bar{M}$. We say that the submanifold M is a $\xi^\perp$*-submanifold*, if the structure vector field ξ of $\bar{M}$ is normal to the submanifold.

Definition. The $\xi^\perp$-submanifold M of $\bar{M}$ is called a *semi-invariant* $\xi^\perp$*-submanifold*, if there exist on M two differentiable orthogonal distributions D and $D^\perp$ such that the following conditions are satisfied.

i. $TM = D \oplus D^\perp$
ii. the distribution D is invariant under ϕ, i.e. $\phi D_x = D_x$ for each $x \in M$
iii. the distribution $D^\perp$ is anti-invariant under ϕ, i.e. $\phi D_x^\perp \subset T_x^\perp M$ for each $x \in M$ where $T_x^\perp M$ is the normal space of M.

D and $D^\perp$ are called respectively the *invariant distribution* and the *anti-invariant distribution* of M.

A semi-invariant $\xi^\perp$-submanifold is said to be an *invariant* (resp. *anti-invariant*) $\xi^\perp$-submanifold, if we have $D_x^\perp = \{0\}$ (resp. $D_x = \{0\}$) for each $x \in M$. A semi-invariant $\xi^\perp$-submanifold is said to be *proper*, if it is neither an invariant nor an anti-invariant $\xi^\perp$-submanifold.

For a vector field X tangent to M, we put

$$X = PX + QX \tag{1.3}$$

where PX and QX belong to the distributions D and $D^\perp$, respectively. Also for a vector field N normal to M, we put

$$\phi N = BN + CN \tag{1.4}$$

where $BN \in D^\perp$ and $CN \in T^\perp M$ (cf. [3], p. 166).

Let M be a semi-invariant $\xi^\perp$-submanifold of a trans-Sasakian manifold $\bar{M}$. We denote by μ the complementary orthogonal vector bundle of $\phi D^\perp$ i.e. $T^\perp M = \phi D^\perp \oplus \mu$. Then it is easy to see that μ is invariant by ϕ.

Now the formulas of Gauss and Weingarten are given respectively by

$$(1.5)\qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

where $\bar{\nabla}$ is the Riemannian connection of $\bar{M}$, ∇ the Riemannian connection determined by the induced metric g on M, $\nabla^{\perp}_X$ the metric connection in the normal bundle of M, h is the second fundamental form and A is defined by

$$(1.6)\qquad g(h(X, Y), N) = g(A_N X, Y).$$

M is called *totally umbilical* if $h(X, Y) = g(X, Y)H$, where H is the *mean curvature vector*. If $H = 0$ then M is said to be *minimal*. If $h = 0$ identically, then M is said *totally geodesic*.

In the classification of Gray and Harvella [8] of almost Hermitian manifolds, there appears a class of Hermitian manifold named W_4 which contains locally conformal Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called *trans-Sasakian* if $(\bar{M} \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$. This may be expressed by condition ([5], p. 201)

$$(1.7)\qquad (\bar{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}$$

where α and β are functions on $\bar{M}$ (*trans-Sasakian structure of type* (α, β)). In particular, $\bar{M}$ is *normal*. From the formula, one easily obtain

$$(1.8)\qquad \bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

2 - Integrability of distributions

First we prove

Lemma 1. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then*

$$(2.1)\qquad P\nabla_X \phi PY - PA_{\phi QY}X = \phi P\nabla_X Y$$

$$(2.2)\qquad Q\nabla_X \phi PY - QA_{\phi QY}X = Bh(X, Y)$$

$$(2.3)\qquad h(X, \phi PY) + \nabla^{\perp}_X \phi QY = \phi Q\nabla_X Y + Ch(X, Y) + \alpha g(X, Y)\xi + \beta g(\phi PX, Y)\xi$$

for any vector field $X, Y \in TM$.

Proof. Since $\xi \in T^{\perp}M$, so $\eta(Y)=0$ for any $Y \in TM$, and equation (1.7) reduces to

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi .$$

Using (1.3) we have

$$\bar{\nabla}_X \phi(PY + QY) - \phi \bar{\nabla}_X Y = \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi .$$

As P (resp. Q) is a projection on D (resp. $D^{\perp}$), so $\phi PY \in TM$ and $\phi QY \in T^{\perp}M$ for any $Y \in TM$. Thus by virtue of Gauss and Weingarten equations we get

$$\nabla_X \phi PY + h(X, \phi PY) - A_{\phi QY}X + \nabla_X^{\perp} \phi QY - \phi(\nabla_X Y + h(X, Y))$$
$$= \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi$$

for any $X, Y \in TM$.

Again, using (1.3), (1.4) we get

$$\begin{aligned} &\nabla_X PY + h(X, \phi PY) - A_{\phi QY}X + \nabla_X^{\perp} \phi QY - \phi P \nabla_X Y - \phi Q \nabla_X Y \\ &-Bh(X, Y) - Ch(X, Y) - \alpha g(X, Y)\xi - \beta g(\phi X, Y)\xi = 0 . \end{aligned} \tag{2.4}$$

Now $BN \in D^{\perp}$, $CN \in T^{\perp}M$ for any vector field N normal to M and $\phi QY \in T^{\perp}M$ for any vector field Y tangent to M. Thus, using (1.3) and comparing the component of D, $D^{\perp}$ and $T^{\perp}M$ in (2.4), we complete the proof.

Lemma 2. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then*

$$\nabla_X BN - A_{CN}X - \phi PA_N X - B\nabla_X^{\perp} N + \alpha\eta(N)X + \beta\eta(N)\phi PX = 0 \tag{2.5}$$

$$h(X, BN) - \phi QA_N X - C\nabla_X^{\perp} N + \nabla_X^{\perp} CN - \beta g(\phi X, N) + \beta\eta(N)\phi QX = 0 \tag{2.6}$$

for any vector field $X \in TM$ and $N \in T^{\perp}M$.

Proof. By using (1.4), (1.5) and (1.7) we obtain

$$\begin{aligned} &\nabla_X BN + h(X, BN) - A_{CN}X + \nabla_X^{\perp} CN = \phi PA_N X + \phi QA_N X + B\nabla_X^{\perp} N \\ &+ C\nabla_X^{\perp} N - \alpha\eta(N)X + \beta g(\phi X, N)\xi - \beta\eta(N)\phi PX - \beta\eta(N)\phi QX . \end{aligned} \tag{2.7}$$

Thus the assertion of the lemma follows by taking the tangent and normal component in (2.7).

Now we study the integrability of the distributions D and $D^{\perp}$ involved in the definition of a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold.

Proposition 1. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the invariant distribution D is integrable if and only if*

$$h(X, \phi Y) - h(\phi X, Y) = 2\beta g(\phi X, Y)\xi$$

for any $X, Y \in D$.

Proof. From (2.3) and by the fact that $\phi Y = 0$ for $Y \in D$, we get

$$h(X, \phi Y) = \phi Q \nabla_x Y + Ch(X, Y) + \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi$$

for any $X, Y \in D$.

Thus interchanging X and Y and subtracting we get

$$h(X, \phi Y) - h(\phi X, Y) = \phi Q[X, Y] + 2\beta g(\phi X, Y)\xi$$

from which we have our assertion.

For the integrability of $D^{\perp}$, first we have

Lemma 3. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of trans-Sasakian manifold $\bar{M}$. Then*

$$A_{\phi X} Y = A_{\phi Y} X \tag{2.8}$$

for any $X, Y \in D^{\perp}$.

Proof. From (1.7) using (1.6) we get

$$g(A_{\phi X} Y, Z) = h(h(Y, Z), \phi X) = g(\bar{\nabla}_Z Y, \phi X) = -g(\bar{\nabla}_Z \phi Y, X) = g(A_{\phi Y} X, Z)$$

for any $X, Y \in D^{\perp}$ and $Z \in TM$.

Now we have

Proposition 2. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-*

Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable.

Proof. Since P is a projection on D so $PY = 0$ for $Y \in D^{\perp}$ and (2.1) gives $\phi P \nabla_X Y = -PA_{\phi QY} X$ for any $X, Y \in D^{\perp}$. Applying ϕ to the above equation and using the fact that $\xi \in T^{\perp} M$, we obtain $P\nabla_X Y = \phi PA_{\phi Y} X$ for any $X, Y \in D^{\perp}$. Thus we get $P[X, Y] = 0$, for any $X, Y \in D^{\perp}$ by virtue of (2.8), which proves our assertion.

Further using Weingarten formula in (1.8) we easily have the following

Lemma 4. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then*

$$A_{\xi} X = \alpha \phi PX - \beta X \qquad \nabla^{\perp}_X \xi = -\alpha \phi QX \tag{2.9}$$

for any X tangent to M.

Next, suppose $\dim TM = m = 2p + q$ where $\dim D = 2p$, $\dim D^{\perp} = q$. Let $(e_1, \ldots, e_p, \phi e_1, \ldots, \phi e_p, e_{2p+1}, \ldots, e_{2p+q})$ be the local field of orthogonal frames of M, where $e_i \in D$ $(i = 1, \ldots, p)$ and $e_{2p+a} \in D^{\perp}$ $(a = 1, \ldots, q)$. We have

Proposition 3. *There do not exist minimal semi-invariant $\xi^{\perp}$-submanifolds of a trans-Sasakian manifold $\bar{M}$ with $\beta \neq 0$.*

Proof. For any $X, Y \in TM$ we have

$$\eta(h(X, Y)) = g(h(X, Y), \xi) = g(A_{\xi} X, Y)$$

using $(2.9)_1$, the above equation yields

$$\eta(h(X, Y)) = -\alpha g(\phi X, Y) + \beta g(X, Y).$$

Thus
$$\eta(H) = \frac{1}{m} \operatorname{trace} A_{\xi} = \beta$$

where H is the mean curvature vector of M.

3 - Totally umbilical semi-invariant $\xi^{\perp}$-submanifold

The aim of this section is to give a complete characterization of totally umbilical semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold.

First we have

Proposition 4. *Let M be an invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then M is a totally umbilical $\xi^{\perp}$-submanifold of $\bar{M}$, if and only if we have*

$$h(X, Y) = -\beta g(X, Y)\xi \tag{3.1}$$

for any $X, Y \in TM$.

Proof. As M is totally umbilical we have

$$h(X, Y) = g(X, Y)H \tag{3.2}$$

for any $X, Y \in TM$.

From (1.7) we have

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = \alpha g(X, Y) + \beta g(\phi X, Y)\xi .$$

Using (1.5) in (1.7) we get

$$\nabla_X \phi Y + h(X, \phi Y) - \phi(\nabla_X Y + h(X, Y)) = \alpha g(X, Y) + \beta g(\phi X, Y)\xi .$$

Comparing normal component, we get

$$h(X, \phi Y) = \phi h(X, Y) + \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi$$

for any $X, Y \in TM$.

Since M is a totally umbilical invariant $\xi^{\perp}$-submanifold of $\bar{M}$, so for any $X, Y \in TM$, (3.2) yields

$$g(X, \phi Y)H = g(X, Y)\phi H + \alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi$$

from which we obtain

$$g(X, \phi Y)\eta(H) = \alpha g(X, Y)\xi + \beta g(\phi X, Y) . \tag{3.3}$$

Now by taking $Y = \phi X$ in (3.3) it follows that $\eta(H) = -\beta$. Thus we have $H = -\beta\xi$ and hence from (3.2) we get the (3.1).

Conversely, suppose $h(X, Y) = -\beta g(X, Y)\xi$. Then $H = -\beta\xi$ and M is a totally umbilical invariant $\xi^{\perp}$-submanifold of $\bar{M}$. This completes the proof.

Proposition 5. *Let M be a proper semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$ such that* $\dim D^{\perp} = q > 1$. *Then M is a totally umbilical submanifold of $\bar{M}$, if and only if* (3.1) *is satisfied for any $X, Y \in TM$.*

Proof. From (2.2) and using (3.2) we get $g(X, X)BH = -QA_{\phi X}X$ for any $X \in D^{\perp}$, from which we get $g(X, X)g(BH, BH) = g^2(X, BH)$.

The above equation gives $BH = 0$, as $\dim D^{\perp} > 1$. Thus from $\phi H = BH + CH$, we get $\phi H = CH$.

Next from (2.3) we obtain

$$h(X, Y) = h(X, \phi Y) - \phi Q \nabla_X Y - \alpha g(X, Y)\xi - \beta g(\phi X, Y)\xi \tag{3.4}$$

for any X tangent to M and $Y \in D$.

Taking $X = Y \in D$ in (3.4) and using (3.2) and $BH = 0$ we get $g(X, X)g(\phi H, \phi H) = 0$ for any $X \in D$, which gives $\phi H = 0$.

Now, since $CH = \phi H = 0$, using the fact that $\eta(H) = -\beta$, we get $H = -\beta\xi$. Thus (3.1) follows from (3.2).

Conversely, suppose (3.1) holds. Then $H = -\beta\xi$ and M is a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of $\bar{M}$. This completes the proof.

Finally we prove

Proposition 6. *Let M be a semi-invariant $\xi^{\perp}$-submanifold of a trans-Sasakian manifold $\bar{M}$. Then the curvature tensor $R^{\perp}(X, Y)$ of the normal bundle annihilates ξ for all $X, Y \in D^{\perp}$.*

Proof. Using $\nabla^{\perp}_X \xi = -\alpha QX$ we have

$$\nabla^{\perp}_Y(\nabla^{\perp}_X \xi) = \nabla^{\perp}_Y(-\alpha QX) = -\alpha \nabla^{\perp}_Y(QX) \qquad \forall X, Y \in D^{\perp}.$$

From (2.3) we have

$$\nabla^{\perp}_Y(\nabla^{\perp}_X \xi) = -\alpha(Q\nabla_Y X + Ch(X, Y) + g(X, Y)\xi).$$

Now by definition

$$R^{\perp}(X, Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X, Y]}\xi$$

$$= -\alpha\nabla^{\perp}_X(\phi Y) + \alpha\nabla^{\perp}_Y(\phi X) + \alpha\phi(X, Y)$$

$$= -\alpha(\nabla^{\perp}_X(\phi Y) - \nabla^{\perp}_Y(\phi X) - \phi[X, Y])$$

$$= -\alpha\{Q\nabla_X Y + Ch(X, Y) + g(X, Y)\xi - \phi\nabla_y X - Ch(X, Y) - g(X, Y)\xi - \phi[X, Y]\}$$

$$= -\alpha(\phi[X, Y] - \phi[X, Y]) = 0$$

which completes the proof.

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Sommario

Scopo del lavoro è lo studio delle condizioni di integrabilità delle distribuzioni D e $D^{\perp}$ di una $\xi^{\perp}$-sottovarietà semi-invariante di una varietà trans-sasakiana. Sono anche caratterizzate le $\xi^{\perp}$-sottovarietà semi-invarianti totalmente ombelicali.

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