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Perfect systems in G_2 (**)

1 - Introduction

There are well known constructions of the irreducible representations and of the irreducible modules, called Specht modules, for the symmetric groups S_n which are based on combinatorial concepts connected with Young tableaux and tabloids (see, e.g., [4]).

In [5] Morris described a possible extension of this work to Weyl groups in general. An alternative and improved approach was described by Halicioğlu and Morris [2]. Later on Halicioğlu [1] develops the theory and shows how a K-basis of Specht modules can be constructed in terms of standard tableaux and tabloids.

In this paper we show in detail how the theory works in the special case of the Weyl group of type G_2.

2 - Some general results on Weyl groups

We now state some results on Weyl groups which are required later. Any unexplained notation may be found in J. E. Humphreys [3], Halicioğlu and Morris [2], Halicioğlu [1].

Let V be l-dimensional Euclidean space over the real field $\boldsymbol{R}$ equipped with a positive definite inner product (,) for $\alpha \in V$, $\alpha \neq 0$, let τ_α be the *reflection* in the

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hyperplane orthogonal to α, that is, τ_α is the linear transformation on V defined by

$$\tau_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha$$

for all $v \in V$. Let Φ be a root system in V and π a simple system in Φ with corresponding positive system Φ^+ and negative system Φ^-. Then, the *Weyl group* of Φ is the finite reflection group $\mathcal{W} = \mathcal{W}(\Phi)$, which is generated by the τ_α, $\alpha \in \Phi$.

We now give some of the basic facts presented in [2].

Let Ψ be a subsystem of Φ with simple system $J \subset \Phi^+$ and Dynkin diagram Δ and $\Psi = \bigcup_{i=1}^{r} \Psi_i$, where Ψ_i are the indecomposable components of Ψ, then let J_i be a simple system in Ψ_i $(i = 1, 2, \ldots, r)$ and $J = \bigcup_{i=1}^{r} J_i$. Let $\Psi^\perp$ be the largest subsystem in Φ orthogonal to Ψ and let $J^\perp \subset \Phi^+$ be the simple system of $\Psi^\perp$.

Let Ψ' be a subsystem of Φ which is contained in $\Phi \setminus \Psi$, with simple system $J' \subset \Phi^+$ and Dynkin diagram Δ', $\Psi' = \bigcup_{i=1}^{s} \Psi_i'$, where Ψ_i' are the indecomposable components of Ψ' then let J_i' be a simple system in Ψ_i' $(i = 1, 2, \ldots, s)$ and $J' = \bigcup_{i=1}^{s} J_i'$. Let $\Psi'^\perp$ be the largest subsystem in Φ orthogonal to Ψ' and let $J'^\perp \subset \Phi^+$ be the simple system of $\Psi'^\perp$.

Let $\bar{J}$ stand for the ordered set $\{J_1, J_2, \ldots, J_r; J_1', J_2', \ldots, J_s'\}$, where in addition the elements in each J_i and J_i' are also ordered, then let $\mathcal{C}_\Delta = \{w\bar{J} \mid w \in \mathcal{W}\}$. The pair $\{J, J'\}$ is called a *useful system* in Φ if $\mathcal{W}(J) \cap \mathcal{W}(J') = \langle e \rangle$ and $\mathcal{W}(J^\perp) \cap \mathcal{W}(J'^\perp) = \langle e \rangle$.

The elements of $\mathcal{C}_\Delta$ are called *Δ-tableaux*, the J_i and J_i' are called the *rows* and the *columns* of $\{J, J'\}$ respectively. Two Δ-tableaux $\bar{J}$ and $\bar{K}$ are *row-equivalent*, written $\bar{J} \sim \bar{K}$, if there exists $w \in W(J)$ such that $\bar{K} = w\bar{J}$. The equivalence class which contains the Δ-tableau $\bar{J}$ is denoted by $\{\bar{J}\}$ and is called a *Δ-tabloid.*

Let τ_Δ be set of all Δ-tabloids. Then $\tau_\Delta = \{\{d\bar{J}\} \mid d \in D_\Psi\}$, where $D_\Psi = \{w \in \mathcal{W} \mid w(j) \in \Phi^+ \text{ for all } j \in J\}$ is a distinguished set of coset representatives of $\mathcal{W}(\Psi)$ in $\mathcal{W}$. The group $\mathcal{W}$ acts on τ_Δ as $\sigma\{\overline{wJ}\} = \{\overline{\sigma w J}\}$ for all $\sigma \in \mathcal{W}$.

Let K be an arbitrary field, let M^Δ be the K-space whose basis elements are the Δ-tabloids. Extend the action of $\mathcal{W}$ on τ_Δ linearly on M^Δ, then M^Δ becomes a $K\mathcal{W}$-module. Let

$$\kappa_{J'} = \sum_{\sigma \in W(J')} s(\sigma)\sigma \qquad e_{J,J'} = \kappa_{J'}\{\bar{J}\}$$

where $s(\sigma) = (-1)^{l(\sigma)}$ is the sign function and $l(\sigma)$ is the length of σ. Then $e_{J,J'}$ is called the generalized *Δ-polytabloid* associated with $\bar{J}$. Let $S^{J,J'}$ be the subspace of M^{Δ} generated by $e_{wJ,wJ'}$ where $w \in \mathcal{W}$. Then $S^{J,J'}$ is called a *generalized Specht module.* A useful system $\{J, J'\}$ in Φ is called a *good system* if $d\Psi \cap \Psi' = \emptyset$ for $d \in D_{\Psi}$ then $\{\overline{dJ}\}$ appears with non-zero coefficient in $e_{J,J'}$. If $\{J, J'\}$ is a good system, then $S^{J,J'}$ is irreducible.

Proposition 1. *If $\{J, J'\}$ is a useful system in Φ, then we have the isomorphisms*:

$$\text{If } w \in \mathcal{W}, \quad \text{then} \quad S^{J,J'} \cong S^{wJ,wJ'}$$
$$\text{If } w \in \mathcal{W}(J), \quad \text{then} \quad S^{J,J'} \cong S^{J,wJ'}$$
$$\text{If } w \in \mathcal{W}(J'), \quad \text{then} \quad S^{J,J'} \cong S^{wJ,J'}.$$

Proposition 2. *If $\{J, J_1'\}$ and $\{J, J_2'\}$ are useful systems in Φ and $\Psi_1' \subseteq \Psi_2'$, then $S^{J,J_2'}$ is a $K\mathcal{W}$-submodule of $S^{J,J_1'}$, where J_1' and J_2' are simple systems for Ψ_1' and Ψ_2' respectively.*

The following are given in [1].

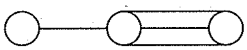
Let $d \in D_{\Psi} \cap D_{\Psi'}$ and let $d' \in D_{\Psi}$. A good system $\{J, J'\}$ is called a *very good system* in Φ if $d \leq d'$ for all $d \in D_{\Psi} \cap D_{\Psi'}$, $d' \in D_{\Psi}$ such that $d' = d\sigma\rho$, where $\rho \in \mathcal{W}(J)$, $\sigma \in \mathcal{W}(J')$ and $\leq$ is Bruhat order.

Proposition 3. *If $\{J, J'\}$ is a very good system in Φ, then the set $\{e_{dJ,dJ'} \mid d \in D_{\Psi} \cap D_{\Psi'}\}$ is linearly independent over K.*

A very good system $\{J, J'\}$ is called a *perfect system* in Φ if the set $\{e_{dJ,dJ'} \mid d \in D_{\Psi} \cap D_{\Psi'}\}$ is a basis for $S^{J,J'}$.

3 - Perfect systems in G_2

Let $\Phi = G_2$ with simple system $\pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$. The extended Dynkin diagram for G_2 is



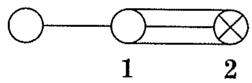
As usual $a_1\alpha_1 + a_2\alpha_2$ is denoted by $a_1 a_2$ and τ_{α_1}, τ_{α_2} are denoted by τ_1, τ_2 respectively. Let $g_1 = e$, $g_2 = \tau_2$, $g_3 = \tau_1\tau_2$, $g_4 = (\tau_1\tau_2)^2$, $g_5 = (\tau_1\tau_2)^3$, $g_6 = \tau_1$ be representatives of conjugate classes $C_1, C_2, C_3, C_4, C_5, C_6$ respectively of $\mathcal{W}(\boldsymbol{G}_2)$. Then the character table of $\mathcal{W}(\boldsymbol{G}_2)$ is

	C_1	C_2	C_3	C_4	C_5	C_6
χ_1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	-1
χ_3	1	-1	-1	1	-1	1
χ_4	1	1	-1	1	-1	-1
χ_5	2	0	-1	-1	2	0
χ_6	2	0	1	-1	-2	0

The non-conjugate subsystems of $\boldsymbol{G}_2$ are:

(1) $\Psi_1 = \boldsymbol{A}_2$ with simple system $J_1 = \{01, 31\}$
(2) $\Psi_2 = \boldsymbol{A}_1 + \widetilde{\boldsymbol{A}}_1$ with simple system $J_2 = \{10, 32\}$
(3) $\Psi_3 = \boldsymbol{A}_1$ with simple system $J_3 = \{10\}$
(4) $\Psi_4 = \widetilde{\boldsymbol{A}}_1$ with simple system $J_4 = \{01\}$
(5) $\Psi_5 = \emptyset$ with simple system $J_5 = \emptyset$
(6) $\Psi_6 = \boldsymbol{G}_2$ with simple system $J_6 = \{10, 01\}$.

(1) Let $\Psi_1 = \boldsymbol{A}_2$ be the subsystem of Φ with simple system $J_1 = \{01, 31\}$. Then $\Psi_1^{\perp} = \emptyset$ with simple system $J_1^{\perp} = \emptyset$. The Dynkin diagram for $\boldsymbol{A}_2$ is



In this case the possible useful systems in Φ are:

(i) $\{J_1, J_1'\}$, where $\Psi_1' = \boldsymbol{A}_1$ with simple system $J_1' = \{10\}$
(ii) $\{J_1, J_2'\}$, where $\Psi_2' = \boldsymbol{A}_1$ with simple system $J_2' = \{11\}$
(iii) $\{J_1, J_3'\}$, where $\Psi_3' = \boldsymbol{A}_1$ with simple system $J_3' = \{21\}$
(iv) $\{J_1, J_4'\}$, where $\Psi_4' = \emptyset$ with simple system $J_4' = \emptyset$.

In the case (i) the Δ_1-tabloids are:

$$\{\bar{J}_1\} = \{01, 31;\ 10\} \qquad \{\overline{\tau_1 J_1}\} = \{31, 01;\ -10\}.$$

If $d = \tau_1$, then $d\Psi_1 \cap \Psi_4' = \emptyset$, but $e_{J_1, J_4'} = \{\bar{J}_1\}$, that is, $\{\overline{dJ_1}\}$ does not appear in $e_{J_1, J_4'}$. By definition of good system $\{J_1, J_4'\}$ is not a good system in Φ. Also if $d = \tau_1$ then $d\Psi_1 \cap \Psi_1' = \emptyset$. Since $e_{J_1, J_1'} = \{\bar{J}_1\} - \{\overline{\tau_1 J_1}\}$, then $\{J_1, J_1'\}$ is a good system in Φ. Similarly it can be verified that $\{J_1, J_2'\}$ and $\{J_1, J_3'\}$ are good systems in Φ. Since

$$\Psi_1' = \tau_2 \Psi_2' = \tau_1 \tau_2 \tau_1 \Psi_3'$$

then by Proposition 1 we have the following isomorphisms

$$S^{J_1, J_1'} \cong S^{J_1, J_2'} \cong S^{J_1, J_3'}$$

Now let K be a field and Char $K = 0$. Let M^{Δ_1} be a K-space. From the definition of M^{Δ_1} we have

$$M^{\Delta_1} = \mathrm{Sp}\,\{\{\overline{eJ_1}\}, \{\overline{\tau_1 J_1}\}\}.$$

Let T_1 be the matrix representation of $\mathcal{W}$ afforded by M^{Δ_1} with character ψ_1. Now we can compute $T_1(g_i)$ for each $g_i \in C_i$ $(i = 1, 2, 3, 4, 5, 6)$. We have

$$e(\{\overline{eJ_1}\}) = \{\overline{eJ_1}\} \qquad \tau_2(\{\overline{eJ_1}\}) = \{\overline{eJ_1}\}$$

$$e(\{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\} \qquad \tau_2(\{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\}$$

$$(\tau_1\tau_2)(\{\overline{eJ_1}\}) = \{\overline{\tau_1 J_1}\} \qquad (\tau_1\tau_2)^2(\{\overline{eJ_1}\}) = \{\overline{eJ_1}\}$$

$$(\tau_1\tau_2)(\{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\} \qquad (\tau_1\tau_2)^2(\{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\}$$

$$(\tau_1\tau_2)^3(\{\overline{eJ_1}\}) = \{\overline{\tau_1 J_1}\} \qquad \tau_1(\{\overline{eJ_1}\}) = \{\overline{\tau_1 J_1}\}$$

$$(\tau_1\tau_2)^3(\{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\} \qquad \tau_1(\{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\}.$$

Thus we have

$$T_1(g_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T_1(g_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T_1(g_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T_1(g_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T_1(g_5) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T_1(g_6) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the corresponding character is

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_1	2	2	0	2	0	0

Now let $S^{J_1, J_1'}$ be the $K\mathcal{W}$-submodule of M^{Δ_1} and $T_1^{(1)}$ be the corresponding representation of $\mathcal{W}$ afforded by $S^{J_1, J_1'}$ with character $\psi_1^{(1)}$. By definition of the

Specht module we have

$$S^{J_1, J_1'} = \mathrm{Sp}\{\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}\}.$$

Now we can compute $T_1^{(1)}(g_i)$ $(i = 1, 2, 3, 4, 5, 6)$

$$e(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}$$

$$\tau_2(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}$$

$$(\tau_1\tau_2)(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\} - \{\overline{eJ_1}\}$$

$$(\tau_1\tau_2)^2(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}$$

$$(\tau_1\tau_2)^3(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\} - \{\overline{eJ_1}\}$$

$$\tau_1(\{\overline{eJ_1}\} - \{\overline{\tau_1 J_1}\}) = \{\overline{\tau_1 J_1}\} - \{\overline{eJ_1}\}.$$

Thus we have

$$T_1^{(1)}(g_1) = 1 \qquad T_1^{(1)}(g_2) = 1 \qquad T_1^{(1)}(g_3) = -1$$
$$T_1^{(1)}(g_4) = 1 \qquad T_1^{(1)}(g_5) = -1 \qquad T_1^{(1)}(g_6) = -1$$

and

	C_1	C_2	C_3	C_4	C_5	C_6
$\psi_1^{(1)}$	1	1	−1	1	−1	−1

that is, $\psi_1^{(1)}$ is the character χ_4.

In case (i) $D_{\Psi_1} \cap D_{\Psi_1'} = \{e\}$ and

$$\mathcal{W}(J_1) = \{e, \tau_2, \tau_1\tau_2\tau_1, \tau_1\tau_2\tau_1\tau_2, \tau_2\tau_1\tau_2\tau_1, \tau_2\tau_1\tau_2\tau_1\tau_2\}, \qquad \mathcal{W}(J_1') = \{e, \tau_1\}.$$

Now let $d = e \in D_{\Psi_1} \cap D_{\Psi_1'}$ and let $d' = \tau_1 \in D_{\Psi_1}$. Then there exist $\sigma = \tau_1 \in \mathcal{W}(J_1')$ and $\rho = e \in \mathcal{W}(J_1)$ such that $d' = d\sigma\rho$. Then $e < \tau_1$. Hence $\{J_1, J_1'\}$ is a very good system in Φ.

In case (ii) $D_{\Psi_1} \cap D_{\Psi_2'} = \{e, \tau_1\}$ and $\mathcal{W}(J_2') = \{e, \tau_2\tau_1\tau_2\}$. Let $d = \tau_1 \in D_{\Psi_1} \cap D_{\Psi_2'}$ and let $d' = e \in D_{\Psi_1}$. Then there exist $\sigma = \tau_2\tau_1\tau_2 \in \mathcal{W}(J_2')$ and $\rho = \tau_2\tau_1\tau_2\tau_1 \in \mathcal{W}(J_1)$ such that $d' = d\sigma\rho$. But $\tau_1 > e$. Hence $\{J_1, J_2'\}$ is not a very good system in Φ. Similarly it can be verified that $\{J_1, J_3'\}$ is not a very good system in Φ. By Proposition 3 the set $\{e_{dJ, dJ'} \mid d \in D_{\Psi_1} \cap D_{\Psi_1'}\}$ is linearly independent over K. Since $\{e_{J_1, J_1'}\}$ is a basis for $S^{J_1, J_1'}$, then $\{J_1 J_1'\}$ is a perfect system in $\boldsymbol{G}_2$.

(2) Let $\Psi_2 = A_1 + \widetilde{A}_1$ be the subsystem of Φ with simple system $J_2 = \{10, 32\}$. Then $\Psi_2^{\perp} = \emptyset$ with simple system $J_2^{\perp} = \emptyset$. The Dynkin diagram for $A_1 + \widetilde{A}_1$ is

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In this case the possible useful systems in Φ are:

(i) $\{J_2, J_1'\}$, where $\Psi_1' = \widetilde{A}_1$ with simple system $J_1' = \{01\}$
(ii) $\{J_2, J_2'\}$, where $\Psi_2' = \widetilde{A}_1$ with simple system $J_2' = \{31\}$
(iii) $\{J_2, J_3'\}$, where $\Psi_3' = A_1$ with simple system $J_3' = \{11\}$
(iv) $\{J_2, J_4'\}$, where $\Psi_4' = A_1$ with simple system $J_4' = \{21\}$
(v) $\{J_2, J_5'\}$, where $\Psi_5' = \emptyset$ with simple system $J_5' = \emptyset$.

In case (i) the Δ_2-tabloids are:

$$\{\overline{eJ_2}\} = \{10, 32;\ 11\} \qquad \{\overline{\tau_2 J_2}\} = \{11, 31;\ 10\} \qquad \{\overline{\tau_1\tau_2 J_2}\} = \{21, 01;\ -10\}.$$

Hence, by definition of good system, $\{J_2, J_5'\}$ is not a good system in Φ. By the same method as in case (1) it can be verified that the remaining systems are good systems in Φ. By Proposition 1 we have the following isomorphisms

$$S^{J_2, J_1'} \cong S^{J_2, J_2'} \cong S^{J_2, J_3'} \cong S^{J_2, J_4'}.$$

By a similar calculation, it can be showed that

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_2	3	1	0	0	3	1
$\psi_2^{(1)}$	2	0	-1	-1	2	0

that is, $\psi_2^{(1)}$ is the character χ_5.

In case (ii) $D_{\Psi_2} \cap D_{\Psi_2'} = \{e, \tau_2, \tau_1\tau_2\}$,

$$\mathcal{W}(J_2) = \{e, \tau_1, \tau_2\tau_1\tau_2\tau_1\tau_2, \tau_1\tau_2\tau_1\tau_2\tau_1\tau_2\}$$

and $\mathcal{W}(J_2') = \{e, \tau_1\tau_2\tau_1\}$. Now let $d = \tau_1\tau_2 \in D_{\Psi_2} \cap D_{\Psi_2'}$ and let $d' = \tau_2 \in D_{\Psi_2}$. Then there exist $\sigma = \tau_1\tau_2\tau_1 \in \mathcal{W}(J_2')$ and $\rho = \tau_1\tau_2\tau_1\tau_2\tau_1\tau_2 \in \mathcal{W}(J_2)$ such that $d' = d\sigma\rho$. But $\tau_1\tau_2 > \tau_2$. Hence $\{J_2, J_2'\}$ is not a very good system in G_2. Similarly it can be verified that also $\{J_2, J_4'\}$ is not a very good system in G_2.

In case (iii) $D_{\Psi_2} \cap D_{\Psi_3'} = \{e, \tau_2\}$ and $\mathcal{W}(J_3') = \{e, \tau_2\tau_1\tau_2\}$. Now let $d = \tau_2 \in D_{\Psi_2} \cap D_{\Psi_3'}$ and let $d' = \tau_2 \in D_{\Psi_2}$. Then there exist $\sigma = e \in \mathcal{W}(J_3')$ and

$\rho = e \in \mathcal{W}(J_2)$ such that $d' = d\sigma\rho$. Then $d' = d$. Let $d = \tau_2 \in D_{\Psi_2} \cap D_{\Psi_3'}$ and let $d' = \tau_1\tau_2 \in D_{\Psi_2}$. Then there exist $\sigma = \tau_2\tau_1\tau_2 \in \mathcal{W}(J_3')$ and $\rho = e \in \mathcal{W}(J_2)$ such that $d' = d\sigma\rho$. Then $\tau_2 < \tau_1\tau_2$. Hence $\{J_2, J_3'\}$ is a very good system in $\boldsymbol{G}_2$. Similarly it can be verified that $\{J_2, J_1'\}$ is a very good system in $\boldsymbol{G}_2$. By Proposition 3 the sets $\{e_{dJ_2, dJ_1'} \mid d \in D_{\Psi_2} \cap D_{\Psi_1'}\}$ and $\{e_{dJ_2, dJ_3'} \mid d \in D_{\Psi_2} \cap D_{\Psi_3'}\}$ are linearly independent over K. Since $\{e_{J_2, J_3'}, e_{\tau_2 J_2, \tau_2 J_3'}\}$ is a basis for $S^{J_2, J_3'}$, then $\{J_2, J_3'\}$ is a perfect system in $\boldsymbol{G}_2$.

(3) Let $\Psi_3 = \boldsymbol{A}_1$ be the subsystem of Φ with simple system $J_3 = \{10\}$. Then $\Psi_3^{\perp} = \tilde{\boldsymbol{A}}_1$ with simple system $J_3^{\perp} = \{32\}$. The Dynkin diagram for $\boldsymbol{A}_1$ is

In this case the possible useful systems in Φ are:

(i)	$\{J_3, J_1'\}$,	where $\Psi_1' = \boldsymbol{A}_2$	with simple system $J_1' = \{01, 31\}$
(ii)	$\{J_3, J_2'\}$,	where $\Psi_2' = \boldsymbol{A}_1 + \tilde{\boldsymbol{A}}_1$	with simple system $J_2' = \{11, 31\}$
(iii)	$\{J_3, J_3'\}$,	where $\Psi_3' = \boldsymbol{A}_1 + \tilde{\boldsymbol{A}}_1$	with simple system $J_3' = \{01, 21\}$
(iv)	$\{J_3, J_4'\}$,	where $\Psi_4' = \tilde{\boldsymbol{A}}_1$	with simple system $J_4' = \{01\}$
(v)	$\{J_3, J_5'\}$,	where $\Psi_5' = \tilde{\boldsymbol{A}}_1$	with simple system $J_5' = \{31\}$
(vi)	$\{J_3, J_6'\}$,	where $\Psi_6' = \tilde{\boldsymbol{A}}_1$	with simple system $J_6' = \{32\}$
(vii)	$\{J_3, J_7'\}$,	where $\Psi_7' = \boldsymbol{A}_1$	with simple system $J_7' = \{11\}$
(viii)	$\{J_3, J_8'\}$,	where $\Psi_8' = \boldsymbol{A}_1$	with simple system $J_8' = \{21\}$.

For case (vi) the Δ_3-tabloids are:

$$\{\overline{eJ_3}\} = \{10;\, 32\} \qquad \{\overline{\tau_2 J_3}\} = \{11;\, 31\} \qquad \{\overline{\tau_1\tau_2 J_3}\} = \{21;\, 01\}$$

$$\{\overline{\tau_2\tau_1\tau_2 J_3}\} = \{21;\, -01\} \qquad \{\overline{(\tau_2\tau_1)^2\tau_2 J_3}\} = \{10;\, -32\}$$

$$\{\overline{(\tau_1\tau_2)^2 J_3}\} = \{11;\, -31\}.$$

Hence by the same method as in case (1) it can be verified that good systems in Φ are $\{J_3, J_1'\}$, $\{J_3, J_2'\}$, $\{J_3, J_3'\}$. By Proposition 1 we have the following isomorphisms

$$S^{J_3, J_2'} \cong S^{J_3, J_3'} \qquad S^{J_3, J_4'} \cong S^{J_3, J_5'} \qquad S^{J_3, J_7'} \cong S^{J_3, J_8'}.$$

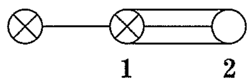
However $\Psi_7' \subset \Psi_2'$ and by Proposition 2 $S^{J_3, J_2'}$ is a $K\mathcal{W}$-submodule of $S^{J_3, J_7'}$. If $T_3^{(7,2)}$ is the corresponding representation of $\mathcal{W}$ afforded by $S^{J_3, J_7'}/S^{J_3, J_2'}$ with character $\psi_3^{(7,2)}$, then we have

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_3	6	0	0	0	0	2
$\psi_3^{(1)}$	1	-1	-1	1	-1	1
$\psi_3^{(2)}$	2	0	-1	-1	2	0
$\psi_3^{(7)}$	4	0	0	-2	0	0
$\psi_3^{(7,2)}$	2	0	1	-1	-2	0

that is, $\psi_3^{(1)}$ is the character χ_3, $\psi_3^{(2)}$ is the character χ_5 and $\psi_3^{(7,2)}$ is the character χ_6.

By the same method as in case (1) it can be verified that $\{J_3, J_1'\}$ and $\{J_3, J_2'\}$ are very good systems in Φ. By Proposition 3 the sets $\{e_{dJ_3, dJ_1'} \mid d \in D_{\Psi_3} \cap D_{\Psi_1'}\}$ and $\{e_{dJ_3, dJ_2'} \mid d \in D_{\Psi_3} \cap D_{\Psi_2'}\}$ are linearly independent over K. Since $\{e_{J_3, J_1'}\}$ is a basis for $S^{J_3, J_1'}$ and $\{e_{J_3, J_2'}, e_{\tau_2 J_3, \tau_2 J_2'}\}$ is a basis for $S^{J_3, J_2'}$, then $\{J_3, J_1'\}$ and $\{J_3, J_2'\}$ are perfect systems in $\boldsymbol{G}_2$.

(4) Let $\Psi_4 = \widetilde{\boldsymbol{A}}_1$ be the subsystem of Φ with simple system $J_4 = \{01\}$. Then $\Psi_4^{\perp} = \boldsymbol{A}_1$ with simple system $J_4^{\perp} = \{21\}$. The Dynkin diagram for $\widetilde{\boldsymbol{A}}_1$ is



In this case the following are the possible useful systems in Φ:

(i) $\{J_4, J_1'\}$, where $\Psi_1' = \boldsymbol{A}_1 + \widetilde{\boldsymbol{A}}_1$ with simple system $J_1' = \{10, 32\}$
(ii) $\{J_4, J_2'\}$, where $\Psi_2' = \boldsymbol{A}_1 + \widetilde{\boldsymbol{A}}_1$ with simple system $J_2' = \{11, 31\}$
(iii) $\{J_4, J_3'\}$, where $\Psi_3' = \widetilde{\boldsymbol{A}}_1$ with simple system $J_3' = \{31\}$
(iv) $\{J_4, J_4'\}$, where $\Psi_4' = \widetilde{\boldsymbol{A}}_1$ with simple system $J_4' = \{32\}$
(v) $\{J_4, J_5'\}$, where $\Psi_5' = \boldsymbol{A}_1$ with simple system $J_5' = \{21\}$
(vi) $\{J_4, J_6'\}$, where $\Psi_6' = \boldsymbol{A}_1$ with simple system $J_6' = \{11\}$
(vii) $\{J_4, J_7'\}$, where $\Psi_7' = \boldsymbol{A}_1$ with simple system $J_7' = \{10\}$.

For case (v) the Δ_4-tabloids are:

$$\{\overline{eJ_4}\} = \{01;\ 21\} \qquad \{\overline{\tau_1 J_4}\} = \{31;\ 11\} \qquad \{\overline{\tau_2\tau_1 J_4}\} = \{32;\ 10\}$$

$$\{\overline{\tau_1\tau_2\tau_1 J_4}\} = \{32;\ -10\} \qquad \{\overline{(\tau_2\tau_1)^2\tau_2 J_4}\} = \{31;\ -11\}$$

$$\{\overline{(\tau_1\tau_2)^2\tau_1 J_4}\} = \{01;\ -21\}.$$

The good systems in Φ are $\{J_4, J_1'\}$, $\{J_4, J_2'\}$. By Proposition 2 we have the following isomorphisms

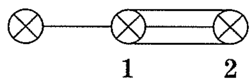
$$S^{J_4, J_1'} \cong S^{J_4, J_2'} \qquad S^{J_4, J_3'} \cong S^{J_4, J_4'} \qquad S^{J_4, J_6'} \cong S^{J_4, J_7'} .$$

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_4	6	2	0	0	0	0
$\psi_4^{(1)}$	2	0	-1	-1	2	0

that is, $\psi_4^{(1)}$ is the character χ_5.

By the same method as in case (1) it can be verified that $\{J_4, J_2'\}$ is a very good system in Φ. Since $\{e_{J_4, J_2'}, e_{\tau_1 J_4, \tau_1 J_2'}\}$ is a basis for $S^{J_4, J_2'}$, then $\{J_4, J_2'\}$ is a perfect system in $\boldsymbol{G}_2$.

(5) Let $\Psi_5 = \emptyset$ be the subsystem of Φ with simple system $J_5 = \emptyset$. Then $\Psi_5^{\perp} = \boldsymbol{G}_2$. The Dynkin diagram for Ψ_5 is

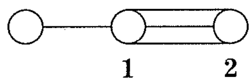


If $\Psi_1' = \boldsymbol{G}_2$, then $\{J_5, J_1'\}$ is a perfect system in $\boldsymbol{G}_2$. Then

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_5	12	0	0	0	0	0
$\psi_5^{(1)}$	1	-1	1	1	1	-1

that is, $\psi_5^{(1)}$ is the character χ_2.

(6) Let $\Psi_6 = \boldsymbol{G}_2$ be the subsystem of Φ. Then $\Psi_6^{\perp} = \emptyset$. The Dynkin diagram for $\boldsymbol{G}_2$ is



If $\Psi_1' = \emptyset$, then $\{J_6, J_1'\}$ is a perfect system in $\boldsymbol{G}_2$. Then

	C_1	C_2	C_3	C_4	C_5	C_6
ψ_6	1	1	1	1	1	1
$\psi_6^{(1)}$	1	1	1	1	1	1

that is, $\psi_6^{(1)}$ is the character χ_1.

Thus we have obtained a complete set of irreducible modules for G_2 and perfect systems in G_2.

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Summary

See Introduction.

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