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Some applications of fully-(λ, $\sigma\mu$)-bases (**)

Introduction

We obtain a characterization for the topological dual of a locally convex space having a fully-λ-base or a fully-λ^{μ}-base, wherein the more general $\sigma\mu$-topology takes the place of the traditional normal topology.

Sufficient attention has been paid to establish the strong impact of different types of nuclearities of μ on the locally convex space admitting a fully-(λ, $\sigma\mu$)-base (or fully-λ^{μ}-base). A few examples concerning this aspect of the study also appear in our discussion.

Most of the results are motivated by their corresponding analogues developed when λ is equipped with the conventional normal topology. Some of the present results extend known propositions regarding λ-bases and λ-nuclearity.

In order to appreciate the subject matter of this paper one is assumed to have a rudimentary familiarity with the *theory of nuclear spaces* (associated nuclearities), *sequence spaces and Schauder basis* as presented in [9], [10] and [7]. However, to have a glimpse into $\lambda(P_0)$-*nuclearity* we turn to [12]. Lastly for a detailed discussion regarding $\sigma\mu$-*topology* and related *fully-λ-base* aspects we adhere to [3] and [5], while for fully-λ-bases (Q-fully-λ-bases) we refer to [8] in the case of the normal topology.

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1 - Duals

This Section is devoted to the representation of members of the topological dual of an l.c.TVS having a fully-λ-base or a fully-λ^μ-base.

We begin the discussion with the following result which makes use of famous Cook's theorem [1].

Proposition 1. *Let μ be perfect and λ be a μ-perfect sequence space. If an* l.c.TVS *X has a shrinking fully-λ-base $\{x_i, f_i\}$, then X is semi-reflexive provided that there exist $y \in \lambda^\mu$ and $z \in \mu^\times$ with $y_i \geqslant \varepsilon > 0$, $z_i \geqslant l > 0$ for all i, for some ε and l.*

Proof. We show that $\{x_i, f_i\}$ is boundedly complete. Then by Cook's theorem X will be semi-reflexive.

Suppose that the sequence $(\sum_{i=1}^{n} \alpha_i x_i)_n$ is bounded in X; take $p \in D_X$, $a \in \lambda^\mu$ and $b \in \mu^\times$. Then there exists $g \in D_X$ such that

$$\sum_{i \geqslant 1} |f_i(x)|\, p(x_i)\, |a_i b_i| \leqslant g(x) \qquad x \in X.$$

Moreover, there exists $K > 0$ such that

$$g(\sum_{i=1}^{n} \alpha_i x_i) \leqslant K \qquad n = 1, 2, \dots .$$

Hence $\sum_{i \geqslant 1} |a_i b_i \alpha_i| p(x_i) < \infty$ for all $a \in \lambda^\mu$ and $b \in \mu^\times$ so $\{\alpha_i\, p(x_i)\, a_i\} \in \mu$, for all $a \in \lambda^\mu$. Thus $\{\alpha_i\, p(x_i)\} \in \lambda$ as λ is μ-perfect. But $\lambda \subseteq l^1$ because for $x \in \lambda$ $\sum_{i \geqslant 1} |x_i| \leqslant \frac{1}{\varepsilon l} \sum_{i \geqslant 1} |x_i\, y_i z_i| < \infty$. Thus the series $\sum_{i \geqslant 1} \alpha_i x_i$ converges in X. Consequently the base $\{x_i, f_i\}$ is boundedly complete.

Note. Notice that this result includes Proposition 3.1 [2].

An inspection of the proof of Proposition 1 suggests.

Proposition 2. *Let μ be a perfect sequence. Suppose there exist $y \in \lambda$ and $z \in \mu^\times$ with $y_i \geqslant \varepsilon > 0$, $z_i \geqslant l > 0$ for all i, for some ε and l. If an* l.c.TVS *X has a shrinking fully-λ^μ-base $\{x_i, f_i\}$ then it is semi-reflexive.*

Note. From this result one can immediately infer that an l.c.TVS X with a shrinking fully-$\lambda^{\times}$-base is semi-reflexive provided that there exists some $y \in \lambda$ with $y_i \geqslant \varepsilon > 0$ $\forall i$, for some ε.

The following result describes the structure of continuous linear functionals on the l.c.TVS possessing a fully-λ-base. Precisely, we have,

Proposition 3. *Let $(\mu, \eta(\mu, \mu^{\times}))$ be a perfect sequence space with K-property (i.e. there exists $y \in \mu^{\times}$ with $y_n \geqslant \varepsilon > 0$, for all n, for some ε) and λ be such that $\lambda\mu^{\times} \subseteq l^1$. Suppose X is an* l.c.TVS *having a fully-λ-base $\{x_n, f_n\}$. Then the topological dual X^* of X consists precisely of those linear functionals f on X which can be written as*

$$f(x) = \sum_{n \geqslant 1} \alpha_n f_n(x) \tag{1}$$

where for some $p \in D_X$, the sequence $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda^{\mu}$.

Proof. Suppose f is a linear functional on X for which (1) is true. Then there exists some $\beta \in \lambda^{\mu}$ such that

$$|f(x)| \leqslant \sum_{n \geqslant 1} |f_n(x)\beta_n| \, p(x_n) \leqslant \varepsilon^{-1} \sum_{n \geqslant 1} |f_n(x)| \, |\beta_n| \, p(x_n) y_n$$

which suggests that $f \in X^*$ as $\{x_n, f_n\}$ is a fully-λ-base.

Conversely, if $f \in X^*$, there exists $p \in D_X$ and $K > 0$ such that $|f(x_n)| \leqslant Kp(x_n)$, $\forall n \geqslant 1$. But $f(x) = \sum_{n \geqslant 1} \alpha_n f_n(x)$ where $\alpha_n = f(x_n)$. The conclusion that $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda^{\mu}$ now follows from the inequality

$$\sum_{n \geqslant 1} \left| \frac{\alpha_n}{p(x_n)} b_n c_n \right| \leqslant K \sum_{n \geqslant 1} |b_n c_n| < \infty \qquad c \in \lambda, \; b \in \mu^{\times}$$

as $\lambda\mu^{\times} \subseteq l^1$, because $\{\frac{\alpha_n c_n}{p(x_n)}\} \in \mu$, $\forall c \in \lambda$.

Similar to above analysis is the proof of the following which characterizes the dual of an l.c.TVS with a fully-λ^{μ}-base.

Proposition 4. *Let μ be a perfect sequence space and λ be a μ-perfect sequence space with $\lambda^{\mu}\mu^{\times} \subseteq l^1$. Suppose X is an* l.c.TVS *with a fully-λ^{μ}-base $\{x_n, f_n\}$. Then the topological dual X^* consists precisely of those linear functionals f which admit the representation* (1), *where for some $p \in D_X$, $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda$.*

Remark 1. Note that the conditions $\lambda\mu^{\times} \subseteq l^1$ in Proposition 2 and $\lambda^{\mu}\mu^{\times} \subseteq l^1$ in Proposition 3 can be replaced respectively by $\lambda\mu^{\times} \subseteq l^{\infty}$ and $\lambda^{\mu}\mu^{\times} \subseteq l^{\infty}$ provided $(\mu, \eta(\mu, \mu^{\times}))$ is nuclear.

2 - Impact of nuclearities of μ

From now on $\lambda(P_0)$ will be a fixed *nuclear* G_∞*-space*. For notions and terminology on $\lambda(P_0)$-nuclearity we refer to [12].

From Proposition 3.4 [2] we know that nuclearity of $(\mu, \eta(\mu, \mu^{\times}))$ implies the nuclearity of a sequentially complete space having a fully-λ-base. Now one would like to expect the $\lambda(P_0)$-nuclearity of X from the $\lambda(P_0)$-nuclearity of $(\mu, \eta(\mu, \mu^{\times}))$.

Proposition 5. *Let X be a sequentially complete space with a fully-λ-base. Suppose that there exist $a \in \lambda^{\mu}$ and $b \in \mu^{\times}$ such that $a_n \geqslant \varepsilon > 0$, $b_n \geqslant l > 0$ for all n for some ε and l. Then X is $\lambda(P_0)$-nuclear provided that $(\mu, \eta(\mu, \mu^{\times}))$ is $\lambda(P_0)$-nuclear.*

For the sake of completeness we give the proof that runs on lines similar to that of Proposition 3.4 [3].

Proof. Invoking Proposition 3.1 [3] X can be topologically identified with $\lambda(Q)$ where

$$Q = \{ p(x_n)\, a_n\, b_n \mid p \in D_X,\ \ a \in \lambda^{\mu}_{+},\ \ b \in \mu^{\times}_{+} \}.$$

Thus X is $\lambda(P_0)$-nuclear iff $\lambda(Q)$ is $\lambda(P_0)$-nuclear. Since $(\mu, \eta(\mu, \mu^{\times}))$ is $\lambda(P_0)$-nuclear, inview of Proposition 3.7 [8] to each $b \in \mu^{\times}_{+}$, there correspond, a $c \in \mu^{\times}_{+}$ and a permutation π such that $\{ \dfrac{b_{\pi(n)}}{c_{\pi(n)}} \} \in \lambda(P_0)$. Consequently, X is $\lambda(P_0)$-nuclear as for any $p \in D_X$, $a \in \lambda^{\mu}_{+}$ and $b \in \mu^{\times}_{+}$ we have

$$\{ \frac{p(x_{\pi(n)})\, a_{\pi(n)}\, b_{\pi(n)}}{p(x_{\pi(n)})\, a_{\pi(n)}\, c_{\pi(n)}} \} \in \lambda(P_0).$$

A close look at the analysis of the proof of Proposition 5 suggests also

Proposition 6. *Let X be a sequentially complete space with a fully-λ-base such that for some $a \in \lambda^{\mu}$ and $b \in \mu^{\times}$ we have $a_i \geqslant \varepsilon > 0$, $b_i \geqslant l > 0$, $\forall i \geqslant 1$ for* *some ε and l. Suppose that, given any $b \in \lambda^{\mu}$, there exists $c \in \lambda^{\mu}$ and a permutation π with $\{\frac{b_{\pi(i)}}{c_{\pi(i)}}\} \in \lambda(P_0)$. Then X is $\lambda(P_0)$-nuclear.*

Note. For the above two results neither the perfectness of μ nor the μ-perfectness of λ is necessary.

The following result shows that *sequential completeness on X is redundant, when μ is choosen to be a nuclear G_{∞}-space.*

Proposition 7. *Let $\lambda(P)$ be a nuclear G_{∞}-space. Suppose S is a sequence space such that its $\lambda(P)$-dual contains an element y with $y_n \geqslant \varepsilon > 0$, for all n and for some ε. Then an* l.c.TVS *X having a fully-S-base is $\lambda(P_0)$-nuclear, whenever $\lambda(P)$ is $\lambda(P_0)$-nuclear.*

Proof. Since $\lambda(P)$ is $\lambda(P_0)$-nuclear, invoking Proposition 3.6.12 [12] we get an $a \in P$ such that $\{a_n^{-1}\} \in \lambda(P_0)$. Also for any $p \in D_X$, $y \in S^{\lambda(P)}$ and $a \in P$ there exists $q \in D_X$ such that

$$\sum_{n \geqslant 1} p(x_n)|f_n(x)|a_n y_n| \leqslant q(x) \qquad x \in X. \tag{2}$$

In particular, choosing the y from the hypothesis and the above a with $\{a_n^{-1}\} \in \lambda(P_0) \subseteq l^1$ in (2) we find that the base is equicontinuous and $\frac{p(x_n)}{q(x_n)} \leqslant (\varepsilon a_n)^{-1}$. Therefore $\{\frac{p(x_n)}{q(x_n)}\} \in \lambda(P_0)$ which concludes that X is $\lambda(P_0)$-nuclear in view of Proposition 3.10 [8].

The foregoing result yields in particular

Corollary 1. *Let $\lambda(P)$ be a nuclear G_{∞}-space. Suppose S is a sequence space, whose $\lambda(P)$-dual contains an element y with $y_i \geqslant \varepsilon > 0$, for all i and for some ε. Then an* l.c.TVS *X with a fully-S-base is nuclear.*

The analysis involved in the proof of Proposition 7 also reveals that

Proposition 8. *Let $\lambda(P)$ be a Schwartz G_{∞}-space. Suppose S is a sequence space, whose $\lambda(P)$-dual has an element z with $\{z_i^{-1}\} \in \lambda(P_0)$. Then an* l.c.TVS *having a fully-S-base is $\lambda(P_0)$-nuclear.*

In particular, this implies

Corollary 2. *Let $\lambda(P)$ be a Schwartz G_∞-space and S be a sequence space, whose $\lambda(P)$-dual has an element z with $\{z_i^{-1}\} \in l^1$. Then an* l.c.TVS *having a fully-S-base is nuclear.*

The following result, which is a variation of Proposition 7, shows the impact of the sequential completness of the dual E^* with respect to the weak topology.

Proposition 9. *Let $\lambda(P)$ be a nuclear G_∞-space and $y \in S^{\lambda(P)}$ be such that $y_i \geqslant \varepsilon > 0$, for all i and for some $\varepsilon > 0$. Suppose E is an* l.c.TVS *with an equicontinuous semi-S-base $\{x_i, f_i\}$ and E^* is weakly sequentially complete. If S is $\lambda(P)$-perfect, then E is $\lambda(P_0)$-nuclear, provided that $\lambda(P)$ is $\lambda(P_0)$-nuclear.*

Proof. Since $\{x_i, f_i\}$ is a semi-S-base we have

(3) $$\sum_{i \geqslant 1} |f_i(x)|\, p(x_i)\, |y_i|\, a_i < \infty \qquad \forall p \in D_E,\ \forall a \in P,\ \forall y \in S^{\lambda(P)}.$$

Now identify E with the sequence space $\Delta = \{(f_i(x)) \mid x \in E\}$. Then, modifying the proof of Proposition 2.3 [4], E^* can be identified with

$$\Delta^\beta = \{(\alpha_i) \mid \sum_i \alpha_i r_i \text{ converges for all } r \in \Delta\}$$

where-in the identification is given by

$$f \in E^* \leftrightarrow \{f(x_i)\} \in \Delta^\beta.$$

Now (3) means that $\{y_i\, p(x_i)\, a_i\} \in \Delta^\beta$. Thus, what we have proved is, for all $y \in S^{\lambda(P)}$, $p \in D_E$ and $a \in P$ there exists $f \in E^*$ with $f(x_i) = p(x_i)\, a_i\, y_i$. Due to the continuity of f we get some $q \in D_E$ and $k > 0$ such that

(4) $$p(x_i)\, a_i\, |y_i| \leqslant k q(x_i).$$

As $\lambda(P)$ is $\lambda(P_0)$-nuclear, in view of Proposition 3.6.12 [12] there exists $a \in P$ with $\{a_i^{-1}\} \in \lambda(P_0)$. So taking this a and y, from the hypothesis in (4) we get

$$\frac{p(x_i)}{q(x_i)} \leqslant \frac{k}{\varepsilon} \frac{1}{a_i} \qquad \text{which implies} \qquad \{\frac{p(x_i)}{q(x_i)}\} \in \lambda(P_0).$$

Now Proposition 3.10 [8] applies and the desired conclusion follows.

Note. Proposition 9 can also be derived by using Proposition 7. It will be enough to show that the base $\{x_n, f_n\}$ is a fully-S-base.

To achieve this let us take any $p \in D_E$, $y \in S^{\lambda(P)}$ and $a \in P$. Then we find that (4) is satisfied for some $q \in D_E$ and $k > 0$. From the inequality

$$\sup_{n \geqslant 1}\{|f_n(x)|p(x_n)|y_n|a_n\} \leqslant k \sup_{n \geqslant 1}\{|f_n(x)|q(x_n)\}$$

it follows that $\{x_n, f_n\}$ is a fully-S-base in view of Proposition 2.1 [3].

From the afore mentioned result it becomes clear that the following is also true.

Corollary 3. *Let $\lambda(P)$ be a nuclear G_∞-space and $y \in S^{\lambda(P)}$ be such that $y_i \geqslant \varepsilon > 0$ for all i and for some ε. Suppose E is an* l.c.TVS *with an equicontinuous semi-S-base $\{x_i, f_i\}$ and E^* is weakly sequentially complete. If S is $\lambda(P)$-perfect, then E is nuclear.*

The method adopted in the proof of Proposition 9 leads to

Proposition 10. *Let $\lambda(P)$ be a nuclear G_∞-space. Suppose S is a $\lambda(P)$-perfect sequence space whose $\lambda(P)$-dual has an element a with $\{a_i^{-1}\} \in \lambda(P_0)$. If E is an* l.c.TVS *with an equicontinuous semi-S-base and E^* is weakly sequentially complete, then E is $\lambda(P_0)$-nuclear.*

This in turn yields

Corollary 4. *Let $\lambda(P)$ be a nuclear G_∞-space. Suppose S is a $\lambda(P)$-perfect sequence space whose $\lambda(P)$-dual has an element a with $\{a_i^{-1}\} \in l^1$. If E is an* l.c.TVS *with an equicontinuous semi-S-base and E^* is weakly sequentially complete, then E is nuclear.*

We conclude this article with the following

Remark 2. Propositions 5, 6, 7 and Corollary 1 remain valid when *fully-λ-bases* are replaced by *Q-fully-λ-bases.*

Example. Let $\lambda(P)$ be a $\lambda(P_0)$-nuclear G_∞-space and S be a $\lambda(P_0)$-perfect sequence space with $S^{\lambda(P_0)} \subseteq l^\infty$. Then by Proposition 3.6.12 [12] there exists $c \in P$ with $\{c_i^{-1}\} \in \lambda(P_0)$. Take any $a \in P$, $y \in S^{\lambda(P_0)}$ and $b \in P_0$. Then for any

$x \in \lambda(P)$ we have the inequality

$$\sum_{i \geq 1} p_a(e_i) |\langle e_i, x\rangle| \, |y_i| b_i \leq \sup_{i \geq 1} |y_i| \cdot \sum_{i \geq 1} |x_i| a_i c_i \cdot \sum_{i \geq 1} \frac{1}{c_i} b_i .$$

Now, since $\lambda(P)$ is a G_∞-space, we conclude that $\{e_i, e_i\}$ is a fully-S-base for $\lambda(P)$ as S is $\lambda(P_0)$-perfect.

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Summary

See Introduction.

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