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A topology for the set of non-negative integers (**)

A Bianca Manfredi con amicizia e stima

1 - Introduction

As far as I know, topology has not yet influenced number theory extensively.

This short note is only an example, showing that the introduction of a convenient topology for the set N of non-negative integers can offer a new outlook (at least a new language) to some topics of classical number theory.

More explicitly, we define in N the *division topology* $\mathscr{D}$ (Sec. 2) and prove that N is a compact, connected, T_0-topological space (Sec. 3).

We show that continuity of a function $f\colon N \to N$ can be regarded as a *compatibility condition* of f with respect to division (Sec. 4, Proposition 5). Consequently we prove that the Euler function is continuous and remark that every completely multiplicative function f is continuous.

The topology $\mathscr{D}$ of N can be extended to a topology $\tilde{\mathscr{D}}$ for Z in a standard way. We prove that the Moebius function and the Liouville function are continuous (Sec. 5, Proposition 9).

Section 6 deals with arithmetical functions. First we prove two propositions, that follow from the compactness of the subspace $N^* = N \setminus \{0\}$. Then, giving a new form to some results obtained by F. Succi in 1960 ([3], Sec. 1-5), we show that the closed subsets of N^* play an essential role in some topics of the theory.

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2 - The division topology

Let $N = \{0, 1, 2, \ldots\}$ be the set of the *non-negative integers*.

For any $x \in N$ we consider the subset $D(x)$, defined by

$$D(x) = \{y \in N \mid x = yz \text{ for some } z \in N\}. \tag{1}$$

The elements of $D(x)$ are the *divisors* of x. We write $y \mid x$ iff $y \in D(x)$.

Note that, for any $x \in N$, we have $x \in D(x)$ and $1 \in D(x)$. Moreover $D(0) = N$ and $D(1) = \{1\}$.

Let X be a subset of N. If $X \neq \phi$, consider the set

$$\bar{X} = \bigcup_{x \in X} D(x). \tag{2}$$

If $X = \phi$, define $\bar{X} = \phi$.

Proposition 1. *The mapping* $k\colon \mathscr{P}(N) \to \mathscr{P}(N)$ *defined by*

$$k(X) = \bar{X} \tag{3}$$

is a closure operator.

It is elementary to check that we have

$$X \subset \bar{X} \qquad \bar{\phi} = \phi \qquad \bar{\bar{X}} = \bar{X} \tag{4}$$

$$\overline{X_1 \cup X_2} = \bar{X}_1 \cup \bar{X}_2 \tag{5}$$

for any $X, X_1, X_2 \in \mathscr{P}(N)$. This proves Proposition 1 ([2], p. 9).

A known theorem (see for example [2], Proposition 5, p. 9) leads us to define a *topology* in N. More explicitly, a subset X of N is called *closed*, iff $X = \bar{X}$; the complement of X is called *open*.

The topology, we have introduced, will be denoted by $\mathscr{D}$ (*division topology*). From now on N will be regarded as a *topological space* (with the topology $\mathscr{D}$).

From (1) it follows that $\overline{\{x\}} = D(x)$. So (2) can be rewritten in the form

$$\bar{X} = \bigcup_{x \in X} \overline{\{x\}} \tag{6}$$

where X is any non-empty subset of N.

It is immediate that

$$\overline{\{0\}} = N \qquad \overline{\{1\}} = \{1\} \qquad \overline{\{x\}} \supset \{x, 1\} \quad \text{for } x \geq 2.$$

Therefore, any non-empty closed set of N contains the element 1. The only closed set containing exactly one element is $\{1\}$. A non-empty closed set C, different from $\{1\}$, has at least two elements.

We remark also that an element x of N is *prime* iff $\overline{\{x\}}$ has exactly two elements. Moreover, $y, z \in N$ are *relatively prime* iff $\overline{\{y\}} \cap \overline{\{z\}} = \{1\}$.

Consider the following subsets of N

$$P = \{\text{prime numbers}\} \qquad E = \{\text{even numbers}\} \qquad O = \{\text{odd numbers}\}.$$

It is immediate to check that $P \cup \{1\}$ and O are closed sets, while E is an open set.

3 - Some properties of $\mathscr{D}$

For what concerns separation properties, we recall that if $x \geqslant 2$, then $\overline{\{x\}}$ contains at least two elements. Thus $\{x\}$ is not a closed set. So $\mathscr{D}$ is not a T_1-topology ([2], Prop. 13, p. 28). However we have

Proposition 2. *$\mathscr{D}$ is a T_0-topology.*

Let x, y be two points of N and assume $x < y$. If $x > 0$, consider the set

$$I_{xy} = \{z \in N \mid x \leqslant z < y\}$$

and remark that $x \in I_{xy}$ and $y \notin I_{xy}$. Since when $z \neq 0$, $w \in D(z)$ implies $w \leqslant z$, we have $x \in \bar{I}_{xy}$ and $y \notin \bar{I}_{xy}$. Therefore the complement of $\bar{I}_{xy}$ is an open set containing y but not containing x.

If $x = 0$, it is immediate that the complement of $\overline{\{y\}}$ is an open set containing $x = 0$ but not containing y. So Proposition 2 is proved.

Proposition 3. *The set N with the topology $\mathscr{D}$ is a compact topological space.*

We know that any non-empty closed set of N contains 1. Thus, the intersection of any centered system of closed sets is non-empty; so Proposition 3 is proved ([2], Def. 6, p. 11; Prop. 2, p. 57).

Proposition 4. *The topological space N is connected.*

We cannot find two non-empty closed sets, having empty intersection.

4 - Continuous mappings

For a mapping $f: N \to N$ we consider the *condition*

$$\delta: \quad \textit{For any } x \in N, \textit{ if } y \mid x \textit{ then } f(y) \mid f(x).$$

We have

Proposition 5. *The mapping f is continuous, iff f satisfies condition δ.*

In other words, the continuity of f can be regarded as a *compatibility condition* of f with respect to the division.

Let f be a continuous mapping, then for any subset X of N we have $f(\bar{X}) \subset \overline{f(X)}$. In particular, for any $x \in N$ we have $f\overline{\{x\}} \subset \{\overline{f(x)}\}$, i.e. $f(D(x)) \subset D(f(x))$, which is exactly condition δ.

Conversely, if condition δ is satisfied, we immediately have $f\overline{\{x\}} \subset \{\overline{f(x)}\}$ for any $x \in N$. Let X be any subset of N. By using (5), we can write

$$f(\bar{X}) = f(\bigcup_{x \in X} \overline{\{x\}}) = \bigcup_{x \in X} f\overline{\{x\}} \subset \bigcup_{x \in X} \{\overline{f(x)}\} = \overline{\bigcup_{x \in X} \{f(x)\}} = \overline{f(X)}.$$

Thus f is continuous.

We also have

Proposition 6. *If f is completely multiplicative, then f is continuous.*

Note that for a completely multiplicative function f ([**1**], p. 33) we have $f(0) = 0$ or $f(x) = 1$ for any $x \in N$. In the second case, f is continuous and Proposition 6 is proved. In the first case you can check that if $y|0$, then $f(y)|f(0)$.

Now, for any $x \in N \setminus \{0\}$ consider the standard factorization

$$x = p_1^{a_1} \dots p_n^{a_n}$$

where p_j are primes and a_j are positive integers. If $y|x$, then we can write

$$y = p_1^{c_1} \dots p_n^{c_n}$$

where c_j is an integer, $0 \leq c_j \leq a_j$ and $j = 1, \dots, n$.

Since f is completely multiplicative, we have

$$f(x) = (f(p_1))^{a_1} \dots (f(p_n))^{a_n} \qquad f(y) = (f(p_1))^{c_1} \dots (f(p_n))^{c_n}$$

Therefore we get $f(y)|f(x)$. Now, using Proposition 5, we obtain Proposition 6.

Using again Proposition 5, we prove easily:

If f is continuous and injective, then $f(0) = 0$.
If f is continuous and surjective, then $f(1) = 1$.

If f is continuous and $f(0) = x$, then $f(N) \subset \overline{\{x\}}$. In particular, if $x = 1$, then $f(y) = 1$ for any $y \in N$.

Since $\{1\}$ is a closed set, we also have: *If f is continuous and $1 \in f(N)$, then $f(1) = 1$.*

Consider now the mappings

$$s_k\colon N \to N \qquad m_k\colon N \to N \qquad k \in N$$

defined by

$$s_k\colon x \mapsto x + k \qquad m_k\colon x \mapsto kx\,.$$

It is easily seen that *m_k is continuous, while s_k is continuous only if $k = 0$.*

Moreover, *the characteristic function of the set O of the odd numbers is a continuous function.*

We end the section considering the well known Euler function φ ([1], p. 25), assuming also $\varphi(0) = 0$. We obtain

Proposition 7. *The Euler function φ is a continuous function.*

Since $\varphi(N) \subset N$, condition δ is satisfied for $x = 0$. The same condition is true also for $x > 0$ by Theorem 2.5, d of [1]. Therefore f is continuous by Proposition 5.

It is worth remarking that φ is a multiplicative function, but not a completely multiplicative function.

5 - A topology for $\boldsymbol{Z}$

The topology $\mathscr{D}$ of $\boldsymbol{N}$ can be extended to a topology $\tilde{\mathscr{D}}$ of $\boldsymbol{Z}$ in a standard way. Consider the mapping $\alpha\colon \boldsymbol{Z} \to \boldsymbol{N}$ defined by $\alpha\colon \tilde{x} \mapsto |\tilde{x}|$. Then assume as closed sets in $\boldsymbol{Z}$ the inverse images by α of the closed sets of $\boldsymbol{N}$. We remark that a set $\tilde{C}$ is closed in $\boldsymbol{Z}$, iff for any $\tilde{x}$ of $\tilde{C}$ any divisor of $\tilde{x}$ in $\boldsymbol{Z}$ belongs to $\tilde{X}$.

Proposition 8. *The set $\boldsymbol{Z}$ with the topology is a compact, connected topological space, but not a T_0-space.*

The proof is elementary.

Denote now by $\tilde{D}(\tilde{x})$ the set of the divisors of $\tilde{x}$ in $\boldsymbol{Z}$. For a mapping $f\colon \boldsymbol{N} \to \boldsymbol{Z}$ consider the *condition*

$\tilde{\delta}$: *For any x of $\boldsymbol{N}$, if $y \in D(x)$, then $f(y) \in \tilde{D}(f(x))$.*

In analogy with Proposition 5, we can prove that f *is continuous, iff* f *satisfies condition* $\tilde{\delta}$.

Let μ, λ be the classical Moebius function and Liouville function ([1], p. 24, 37), completed by $\mu(0) = \lambda(0) = 0$. Using condition $\tilde{\delta}$, we can prove easily

Proposition 9. *The Moebius function* μ *and the Liouville function* λ *are continuous functions.*

6 - Arithmetical functions

Let f be an *arithmetical function*, i.e.

$$f\colon \boldsymbol{N}^* \to \boldsymbol{C}$$

where $\boldsymbol{N}^*$ is the set of the positive integers ($\boldsymbol{N}^* = \boldsymbol{N} \setminus \{0\}$) and $\boldsymbol{C}$ the set of the complex numbers. We regard $\boldsymbol{N}^*$ and $\boldsymbol{C} = \boldsymbol{R}^2$ as topological spaces, with the topology $\mathscr{D}^*$, induced by the topology $\mathscr{D}$ of Sec. 2, and the euclidean topology, respectively.

Proposition 10. *If* f *is a continuous arithmetical function, then* f *is a closed map. In particular,* $f(D(x))$ *is a closed set of* $\boldsymbol{C}$ *for any* x *of* $\boldsymbol{N}^*$.

Proposition 11. *If* f *is a continuous arithmetical function, then* $f^{-1}(K)$ *is a compact set of* $\boldsymbol{N}^*$ *for any compact subset* K *of* $\boldsymbol{C}$. *In particular, for any* $\zeta \in \boldsymbol{C}$ $f^{-1}\{\zeta\}$ *is a compact of* $\boldsymbol{N}^*$.

To prove Proposition 10, we remark first that $\boldsymbol{N}^*$ is a closed subset of $\boldsymbol{N}$, so $\boldsymbol{N}^*$ is a compact topological space by Proposition 3 ([2], Th. 3, p. 59). Then we use Th. 6 of [2], p. 59. We note also that $D(x)$ is closed in $\boldsymbol{N}^*$.

Since f is continuous, Proposition 11 follows immediately from Th. 4 and Th. 3 of [2], p. 59.

The aim of the last part of the present section is to show that the subsets of $\boldsymbol{N}^*$, closed in the $\mathscr{D}^*$-topology, play an essential role in some topics concerning arithmetical functions.

We begin with some definitions. For more details and significant examples see F. Succi [3], Sec. 2, 3.

Let $\mathscr{A}$ be the set of all arithmetical functions. The subset of all arithmetical functions satisfying a fixed property π will be denoted by $[\pi]$.

A property π is said *compatible with the Dirichlet product* (*D-compatible*) if for any $f, g \in \mathcal{A}$ satisfying π also $f*g$ satisfies π.

Let I, J be subsets of N^* and $J \subset I$. If, for any $x \in I, f(x)$ can be expressed in terms of a finite number of values $f(y_r)$ with $y_r \in J$, then we say that $f(J)$ *generates* $f(I)$ or, equivalently, that f *satisfies a property of type* (I, J).

Further, let f, g be any two arithmetical functions satisfying a property γ of type (I, J). If, for any $x \in I$, $(f*g)(x)$ depends only on the values of f and of g on I, we say that γ is a *property of type* $(I, J)^*$.

Finally, let α be a property of type (I, J). Then, for any $I_1 \subset I$, α determines a set $J_1 \subset I$ with $J_1 \subset I_1$. The property of type (I_1, J_1) induced by α is called the *restriction* of α.

We are able now to give a new form to some results due to F. Succi ([3], p. 460-463).

1. *A D-compatible property γ of type (I, J) is a D-compatible property of type $(I, J)^*$, iff the subset I of N^* is closed in the topology $\mathcal{D}^*$.*

2. *Let γ be a D-compatible property of type (I, J) and γ_1 the restriction of γ to (I_1, J_1). If I_1 is a closed subset of N^*, then also γ_1 is a D-compatible property.*

3. *Let γ be a D-compatible property of type (I, J) and γ_1 a D-compatible restriction of γ of type (I_1, J_1). If $[\gamma]$ is a group with respect to the Dirichlet product and I_1 is a closed subset of N^*, then also $[\gamma_1]$ is a group.*

4. *The set of the arithmetical functions, that are multiplicative on a closed subset of N^*, is a group with respect to the Dirichlet product.*

References

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Sommario

*Questa breve nota mostra come l'introduzione di una opportuna topologia nell'insieme dei naturali **N** possa riuscire utile in alcune questioni di teoria dei numeri.*

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