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An extension of a property of the Fourier transform (**)

1 - A property of the Fourier transform

Let $f(x) \in L(-\infty, \infty)$. Denote by

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx \tag{1.1}$$

the *Fourier transform* of f.

Let $He_k(x)$ be the *Hermite polynomials*, defined by Rodrigues formula

$$He_k(x) = (-1)^k e^{\frac{x^2}{2}} D^k (e^{-\frac{x^2}{2}}) \qquad k \in N_0 = N \cup \{0\}. \tag{1.2}$$

The system $\{He_k(x)\}_{k \in N_0}$ is *orthogonal* in $(-\infty, \infty)$ with respect to the *weight function* $W(x) = e^{-\frac{x^2}{2}}$, and furthermore

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} He_k^2(x)\, dx = \sqrt{2\pi}\, k! \qquad (k \in N_0).$$

Remark. The Hermite polynomials $He_k(x)$, we consider here, are related to Hermite polynomials $H_k(x)$, orthogonal on the real axis with respect to the

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weight function e^{-x^2}, by the following formula (see [2], p. XXXV)

$$He_k(x) = 2^{-\frac{k}{2}} H_k\left(\frac{x}{\sqrt{2}}\right).$$

Let us prove (cf. [1]) the following

Proposition 1. *Let* $x^k f(x) \in L(-\infty, \infty)$, $\forall k \in N_0$, *and* α_k $(k \in N_0)$ *be the Fourier coefficients of the function* $f(x)\, e^{\frac{x^2}{2}}$. *Suppose that the function* $f(x)\, e^{\frac{x^2}{2}}$ *satisfies the hypotheses stated in* [3], *Theorem* 1, *p.* 59, *so that we can write*

$$e^{\frac{x^2}{2}} f(x) = \sum_{k=0}^{\infty} \alpha_k He_k(x), \tag{1.3}$$

the convergence of the series being uniform in every bounded interval of the real axis.

Then the coefficients of the Taylor expansion, in a neighborhood of the origin, of the function $\hat{f}(y)\, e^{\frac{y^2}{2}}$ *are given by* $i^k \alpha_k$.

Proof. Let us write (1.1) in the form

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\, e^{-\frac{x^2}{2}} \left(e^{\frac{x^2}{2}} f(x)\right) dx$$

and expand $e^{\frac{x^2}{2}} f(x)$ by (1.3). Coefficients α_k have the form

$$\alpha_k = \frac{1}{\sqrt{2\pi}\, k!} \int_{-\infty}^{\infty} f(x)\, He_k(x)\, dx. \tag{1.4}$$

Then (1.1) becomes

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\, e^{-\frac{x^2}{2}} \left(\sum_{k=0}^{\infty} \alpha_k He_k(x)\right) dx. \tag{1.5}$$

Integrating term by term and using Rodrigues formula (1.2), we can write

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} (-1)^k \alpha_k \int_{-\infty}^{\infty} e^{ixy} D^k \left(e^{-\frac{x^2}{2}}\right) dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \alpha_k (-i)^k y^k e^{-\frac{y^2}{2}} = e^{-\frac{y^2}{2}} \sum_{k=0}^{\infty} i^k \alpha_k y^k$$

and so we have proved the formula

$$e^{\frac{y^2}{2}} \hat{f}(y) = \sum_{k=0}^{\infty} i^k \alpha_k y^k . \tag{1.6}$$

Remark. In order to extend the above property to different *Orthogonal Polynomial Sets* (shortly OPS), let us note that the same result could be obtained using the *generating function*

$$\psi(x, z) = e^{xz - \frac{z^2}{2}} = \sum_{k=0}^{\infty} He_k(x) \frac{z^k}{k!} . \tag{1.7}$$

As a matter of fact, putting $z = iy$, we have

$$e^{ixy + \frac{y^2}{2}} = \sum_{k=0}^{\infty} \frac{i^k}{k!} He_k(x) y^k .$$

Multiplying the two terms of the above equality by $f(x)$ and integrating term by term we obtain

$$e^{\frac{y^2}{2}} \int_{-\infty}^{\infty} e^{ixy} f(x)\, dx = \sum_{k=0}^{\infty} \frac{i^k}{k!} [\int_{-\infty}^{\infty} f(x) He_k(x)\, dx] y^k$$

that is, by (1.1) and (1.4)

$$e^{\frac{y^2}{2}} \hat{f}(y) = \sum_{k=0}^{\infty} i^k \alpha_k y^k = \sum_{k=0}^{\infty} \alpha_k z^k ,$$

which is the same as (1.6).

2 - Extension of the property to different OPS

Consider a *classical* Orthogonal Polynomial Set (see e.g. [**3**], p. 30) $\{G_k(x)\}_{k \in N_0}$, generated in (a, b) by a *weight function* $W(x)$. Remember that $W(x)$ is assumed to be such that

$W(x) > 0$ in all interior points of (a, b),

$\forall k \in N_0$, $x^k \in L_{W(x)}(a, b)$, i.e. all the moments of the measure associated to the weight are finite.

Put $$\int_a^b G_h(x) G_k(x) W(x)\, dx = h_k \delta_{h,k} .$$

Let $F(x, y)$ be the *Generating Function* of the set $\{G_k(x)\}_{k \in N_0}$, corresponding to the sequence $\{(c_k)\}_{k \in N_0}$, i.e.

$$F(x, y) = \sum_{k=0}^{\infty} c_k G_k(x) y^k . \tag{2.1}$$

Remark. Usually the choice of the sequence $\{c_k\}_{k \in N_0}$ is performed in the following two ways (see e.g. [4], p. 29):

a) $c_k = 1, \quad \forall k \in N_0$ *(ordinary generating function)*

b) $c_k = \dfrac{1}{k!}, \quad \forall k \in N_0$ *(exponential generating function).*

For any $f(x)$ such that: $x^k f(x) \in L(a, b)$, $\forall k \in N_0$, consider now the *Integral Transform* (see [5]):

$$\hat{f}(y) = \int_a^b F(x, y) f(x)\, dx , \tag{2.2}$$

related to the kernel $K(x, y) = F(x, y)$, defined by (2.1).

We prove now the following extension of Proposition 1.

Proposition 2. *Let* $x^k f(x) \in L(a, b)$, $\forall k \in N_0$, *and* α_k $(k \in N_0)$ *be the Fourier coefficients of the function* $W^{-1}(x) f(x)$. *Suppose that the function* $W^{-1}(x) f(x)$ *satisfies the hypotheses stated in* [3], *Theorem* 1, *p.* 59, *so that we can write*

$$W^{-1}(x) f(x) = \sum_{k=0}^{\infty} \alpha_k G_k(x) \tag{2.3}$$

the convergence of the series being uniform in every bounded interval $[x_1, x_2] \subset (a, b)$.

Then the coefficients of the Taylor expansion, in a neighborhood of the origin, of the function $\hat{f}(y)$ *are given by* $\alpha_k c_k h_k$.

Proof. Let us write (2.2) in the form

$$\hat{f}(y) = \int_a^b F(x, y)\, W(x)\, W^{-1}(x) f(x)\, dx = \int_a^b F(x, y)\, W(x) \Big(\sum_{k=0}^{\infty} \alpha_k G_k(x)\Big)\, dx \tag{2.4}$$

where
$$\alpha_k = \frac{1}{h_k} \int_a^b f(x)\, G_k(x)\, dx .$$

By (2.4) we obtain

$$\hat{f}(y) = \sum_{k=0}^{\infty} \alpha_k \int_a^b F(x, y)\, W(x)\, G_k(x)\, \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \alpha_k \int_a^b \sum_{l=0}^{\infty} c_l\, W(x) G_l(x)\, G_k(x)\, y^l\, \mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \alpha_k c_k\, h_k\, y^k,$$

so that

$$\hat{f}(y) = \sum_{k=0}^{\infty} \alpha_k c_k\, h_k\, y^k = \sum_{k=0}^{\infty} c_k \left(\int_a^b f(x)\, G_k(x)\, \mathrm{d}x\right) y^k$$

which proves the proposition.

Example. Jacobi case.

Assume $a=-1$, $b=1$, $W(x)=(1-x)^{\alpha}(1+x)^{\beta}$ and $\alpha>-1$, $\beta>-1$. Then

$$G_k(x) = P_k^{(\alpha,\beta)}(x) \qquad (\textit{Jacobi polynomials}).$$

Put $c_k = 1$, $\forall k \in N_0$; $R = \sqrt{1 - 2xy + y^2}$, then:

$$F(x, y) = \frac{2^{\alpha+\beta}}{R(1 - y + R)^{\alpha}(1 + y + R)^{\beta}}$$

$$\hat{f}(y) = \int_{-1}^{1} F(x, y) f(x)\, \mathrm{d}x$$

$$h_k = \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2k+1} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{k!\,\Gamma(\alpha+\beta+k+1)}.$$

As a consequence of Proposition 2, we can write the following expansion formulae:

$$(1 - x)^{-\alpha}(1 + x)^{-\beta} f(x) = \sum_{k=0}^{\infty} \alpha_k P_k^{(\alpha,\beta)}(x)$$

$$\hat{f}(y) = \sum_{k=0}^{\infty} \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2k+1} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}{k!\,\Gamma(\alpha+\beta+k+1)} \alpha_k\, y^k.$$

Similar expressions can be found for all other classical OPS.

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Summary

In this paper we first recall of a known property of the Fourier transform (Proposition 1), *connected with Hermite polynomials, and then we give an extension to the case of different types of classical orthogonal polynomial sets.*

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