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Modules with projective socles (**)

According to Nicholson and Watters [4], a module M is called a PS-*module* if its socle $\mathrm{Soc}(M)$ is projective, and a ring R is called a (left) PS-*ring* if ${}_RR$ is a PS-module. They proved, among other things, that the notion of PS-rings is a Morita invariant. Using a direct proof which avoids the Morita context machinery used in [4], we show that the notion of PS-modules (in particular, PS-rings) is also a Morita invariant. If R is a PS-ring then so is the power series ring $R[\![x]\!]$ by [4], Theorem 3.1. Here we note that $R[\![x]\!]$ is always a PS-ring for any ring R. The notion of PS-rings is not left-right symmetric [4]. We note that even a left semihereditary ring or a local PS-ring need not be a right PS-ring, but we do show that a duo PS-ring must be a right PS-ring. Let S be an excellent extension of a ring R. Steward [8] showed that S is a PS-ring if and only if R is a PS-ring. Using a different approach, we generalize this by proving that an S-module ${}_SM$ is a PS-module if and only if the R-module ${}_RM$ is a PS-module.

Throughout the paper, all rings have a unity and all modules are unitary. We freely use the terminologies and notations of [1].

Modifying the proof of [4], Theorem 2.4, we first have

Proposition 1. *The following are equivalent for an R-module* ${}_RM$:

(1) ${}_RM$ *is a* PS-*module*

(2) *If* L *is a maximal left ideal of* R *then either* $r_M(L) = 0$ *or* $L = Re$ *where* $e^2 = e \in R$

(3) *Every simple module* ${}_RK$ *is either projective or* $\mathrm{Hom}_R(K, M) = 0$.

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Proof. (1)$\Rightarrow$(2). Let L be a maximal left ideal of R and $r_M(L) \neq 0$. Let $0 \neq m \in r_M(L)$. Since $L \subseteq l_R(m) \neq R$ and L is a maximal left ideal, we have $L = l_R(m)$. Now $R/L \cong Rm$ is projective by hypothesis and so L is a summand of ${}_RR$.

(2)$\Rightarrow$(3). Let ${}_RK = Rk$ be simple. Then $K = Rk \cong R/L$ where $L = l_R(k)$ is a maximal left ideal of R. If $L = Re$, $e^2 = e$, then $K \cong R(1-e)$ is projective. If $r_M(L) = 0$ let $f \in \mathrm{Hom}_R(K, M)$. If $f(k) = m \in M$ then $Lm = f(Lk) = f(0) = 0$, so $m = 0$ and $f = 0$.

(3)$\Rightarrow$(1). If K is a simple submodule of ${}_RM$ then $\mathrm{Hom}_R(K, M) \neq 0$ and so K is projective by hypothesis.

Theorem 2. *Let F: R-Mod $\rightarrow$ S-Mod define a Morita equivalence. Then an R-module ${}_RM$ is a* PS-*module if and only if the S-module $F(M)$ is a* PS-*module.*

Proof (second part). Let ${}_RK$ be a simple module which is not projective. By [1] Propositions 21.8 and 21.6, $F(K)$ is a simple S-module which is not projective. Then $\mathrm{Hom}_S(F(K), F(M)) = 0$ by Proposition 1. By [1], Proposition 21.2, $\mathrm{Hom}_R(K, M) = 0$. Hence M is a PS-module by Proposition 1 again.

The direct part can be proved similarly.

Corollary 3 [4]. *If F: R-Mod $\rightarrow$ S-Mod define a Morita equivalence, then R is a* PS-*ring if and only if S is a* PS-*ring.*

Proof. If R is a PS-ring then the faithful S-module $F({}_RR)$ is a PS-module by Theorem 2, hence S is a PS-ring by [4] Theorem 2.4.

Nicholson and Watters [4] proved that the class of PS-rings is closed under the formation of power series extension. The next easier proof shows something more.

Proposition 4. *For each R-module ${}_RM$, the power series module $M[\![x]\!]$ is a* PS-*module over the power series ring $R[\![x]\!]$. In particular, $R[\![x]\!]$ is a* PS-*ring for any ring R.*

Proof. We know that each maximal left ideal of $R[\![x]\!]$ is of the form $I + R[\![x]\!]x$, where I is a maximal left ideal of R. Hence if L is a maximal left ideal of $R[\![x]\!]$ then $R[\![x]\!]x \subseteq L$. It follows that $r_{M[\![x]\!]}(L) \subseteq r_{M[\![x]\!]}(R[\![x]\!]x) = 0$, so the $R[\![x]\!]$-module $M[\![x]\!]$ is a PS-module by Proposition 1.

The notion of PS-rings is not left-right symmetric [4]. The following example shows that even a left semihereditary ring need not be a right PS-ring.

Example 5. Let B be a commutative regular ring that is not a semisimple ring. Then there is a maximal ideal I of B such that B/I is not projective. Let $R = \begin{pmatrix} B/I & B/I \\ 0 & B \end{pmatrix}$. By [5] Corollary 2.3, R is a left semihereditary ring. But R is not a right PS-ring, since R has a minimal right ideal $\begin{pmatrix} 0 & B/I \\ 0 & 0 \end{pmatrix}$ that is not projective.

A ring R is a *left* (*right*) PP *ring* if each principal left (right) ideal is projective. Clearly, left PP rings are PS-rings. There is a left hereditary ring which is not a right PP ring (see [7]). Since the ring R in Example 5 is not left hereditary, it will be interesting to know whether a left hereditary ring must be a right PS-ring. A ring R is *normal* if each idempotent lies in the center. Although a left PP ring need not be right PP (e.g., Example 5 or [7]), a normal left PP ring is right PP [2]. But a normal PS-ring need not be a right PS-ring as shown by the following example.

Example 6. Let D be a division ring. Let R denote the ring of all countably infinite lower triangular matrices over D with constant entries on the main diagonal and having nonzero entries in only finitely many rows below the main diagonal. Then R is a local ring (hence a normal ring). Since $\operatorname{Soc}({}_R R)=0$ and $\operatorname{Soc}(R_R)\neq 0$, R is a PS-ring that is not a right PS-ring.

A ring R is *duo* if each one-sided ideal of R is a two-sided ideal. Thierrin [9] noted that a duo ring is a normal ring. In view of the above example, we prove

Proposition 7. *A duo ring R is a* PS-*ring if and only if it is a right* PS-*ring.*

Proof. Let R be a duo PS-ring. If rR is a minimal right ideal then $Rr = rR$ is also a minimal left ideal. Hence Rr is projective and $l_R(r) = Re$ for some $e^2 = e \in R$. Since e lies in the center of R, we see $e \in r_R(r)$. Since Re is a maximal left ideal, $eR = Re$ is also a maximal right ideal. Now $eR \subseteq r_R(r) \neq R$, so $eR = r_R(r)$ and rR is a projective right R-module.

Let R and S be rings with the same unity, $R \subseteq S$. The ring S is an *excellent extension* of R if

(i) there is a finite set $\{1 = s_1, \ldots, s_n\} \subseteq S$ such that S is free left and right R-module with basis $\{s_1, \ldots, s_n\}$ and $Rs_i = s_iR$ for all $i = 1, \ldots, n$.

(ii) if ${}_SM$ is an S-module with an S-submodule ${}_SN$ and N is a direct summand of M as an R-module, then N is a direct summand of M as an S-module.

See [6] for further information about excellent extensions.

Recently, Stewart [8] has proved that if S is an excellent extension of R, then S is a PS-ring if and only if R is a PS-ring. Using a direct and different proof, we generalize this as follows

Theorem 8. *Let S be an excellent extension of R. If ${}_SM$ is an S-module then*

(1) *${}_SM$ is projective if and only if ${}_RM$ is projective.*
(2) *${}_SM$ is a* PS-*module if and only if ${}_RM$ is a* PS-*module.*

Proof. (1) (direct part). Let ${}_SN$ be an S-module such that $M \oplus N$ is a free S-module. Since ${}_RS$ is a free R-module, $M \oplus N$ is also a free R-module. Hence ${}_RM$ is projective.

To prove the converse part of (1), let ${}_SF$ be a free S-module and $f{:}\,{}_SF \to {}_SM$ be an S-module epimorphism. We have an exact sequence of S-modules

$$0 \to K \to F \to M \to 0$$

where $K = \operatorname{Ker}(f)$. Since this sequence is also an exact sequence of R-modules and ${}_RM$ is projective, we see that ${}_RK$ is a summand of ${}_RF$. Hence ${}_SK$ is also a summand of ${}_SF$. Let ${}_SF = {}_SK \oplus {}_SN$. Then ${}_SN$ is projective and ${}_SN \cong {}_SM$.

(2) By [6] Corollary 1.2, $\operatorname{Soc}({}_SM) = \operatorname{Soc}({}_RM)$. It follows from (1) that ${}_S\operatorname{Soc}({}_SM)$ is projective if and only if ${}_R\operatorname{Soc}({}_RM)$ is projective.

Corollary 9 [8]. *If S is an excellent extension of R then S is a* PS-*ring if and only if R is a* PS-*ring.*

Proof. If S is a PS-ring then the free R-module ${}_RS$ is a PS-module by Theorem 8. Hence ${}_RR$ is also a PS-module. Conversely, if R is a PS-ring then the free R-module ${}_RS$ is a PS-module. Hence ${}_SS$ is also a PS-module by Theorem 8.

Note Added. Our Theorem 8 (1) was proved by Professor Hongjin Fang as Proposition 8 in his paper *Normalizing extensions and modules* appeared in J. Math. Res. Exposition **12**, 3 (Aug. 1992), 401-406.

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Summary

Modules (resp., rings) with projective socles are called PS-*modules (resp.* PS-*rings) which are preserved by Morita equivalences and excellent extensions. although a left semihereditary ring or a local* PS-*ring need not be a right* PS-*ring, we show that a duo* PS-*ring must be a right* PS-*ring.*

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