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Extension theorems and the problem of measure (**)

1 - Introduction

The Hahn-Banach theorem is a well-known extension theorem which is fundamental in the study of topological vector spaces. H. Hahn proved the following extension theorem in 1927 for bounded linear functionals in a normed linear space [4].

Theorem 1. *Let M be a subspace of the real normed linear space X, and f a bounded linear functional defined on M. Then there exists a bounded linear functional F defined on X such that $F(x) = f(x)$ for all $x \in M$ and $\|F\| = \|f\|$.*

S. Banach proved the same theorem in 1929 [3] and published the following generalization in 1932 [**1**].

Theorem 2. *Let M be a subspace of the real vector space X, p a sublinear functional defined on X, and f a linear functional defined on M with $f(x) \leqslant p(x)$ for all $x \in M$. Then there exists a linear functional F defined on X such that $F(x) = f(x)$ for all $x \in M$ and $F(x) \leqslant p(x)$ for all $x \in X$.*

Also in this 1932 text Banach gave a solution to the *problem of measure* using Theorem 2 ([**1**], p. 32).

Most texts which refer to the problem of measure state that it is a consequence of the Hahn-Banach theorem (Theorem 2). However, Banach's original

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solution of the problem appeared in 1923 [2]. Although Banach did not use the Hahn-Banach theorem initially to solve the problem, he did use an extension theorem for linear functionals. An anlysis of [2] demonstrates that the extension theorem used in the solution of the problem of measure has some interesting rami-
fications and is actually related to a result due to M. Krein which appeared in 1948 [6].

2 - The problem of measure

The problem of measure as solved by Banach in [2] may be formulated as follows.

Problem of measure. *Is it possible to assign to every bounded subset E of real numbers a nonnegative number $m(E)$ such that*

a $m(E) \geqslant 0$

b $m([0, 1]) = 1$

c $m(E_1 \cup E_2) = m(E_1) + m(E_2)$, *where* $E_1 \cap E_2 = \emptyset$

d $m(E_1) = m(E_2)$ *for all sets* E_1 *and* E_2 *which are congruent (isometric)?*

The original statement of the problem was due to H. Lebesgue [7] and encompassed both finite and countable unions in condition **c**. He restricted his investigation to a certain class of sets without any mention of the solvability or insolvability of the general problem. Banach refers to F. Hausdorff's work on this problem in Grundzüge der Mengenlehre which was originally published in 1914 [5]. In this work Hausdorff generalizes the problem to sets in n-dimensional space. He shows that the question is answered in the negative for countable unions in any space and also in the negative for finite unions in spaces with three or more dimensions. The question for finite unions in space with one or two dimensions was answered by Banach in the affirmative in 1923 [2].

3 - Banach's solution

To solve the problem of measure, Banach begins with the set $\mathcal{F}$ of all bounded real valued functions $f(x)$ of a real variable which are periodic with period 1. Using an idea from Hausdorff's work ([5], p. 401), Banach considers these functions to be defined on the circumference of the circle with centre at the origin and radius $\frac{1}{2\pi}$ where x denotes the arclength. He creates a structure on this set of functions by starting out with the following definition ([2], p. 9).

Definition 1. We write $f(x) \sim 0$, if for each $\varepsilon > 0$ there is a finite set of real numbers $a_1, a_2, \ldots, a_n$ such that for each real number x

$$\frac{1}{n} \left| \sum_{k=1}^{n} f(x + a_k) \right| < \varepsilon .$$

This definition leads to a decomposition of $\mathcal{F}$ into equivalence classes in the usual way i.e., $f(x) \sim g(x)$ if $f(x) - g(x) \sim 0$. Banach calls these equivalence classes *hyperfunctions* and defines addition and scalar multiplication of classes again in the usual way.

An order relation is defined on $\mathcal{F}$ as follows ([2], p.11).

Definition 2. We write $f(x) \gg 0$, if there is a $c > 0$ and a finite set of real numbers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{n} \left| \sum_{k=1}^{n} f(x + a_k) \right| \geqslant c$$

for all x.

This leads to an ordering on $\mathcal{F}$ by defining $f(x) \gg g(x)$ if $f(x) - g(x) \gg 0$. In turn this order relation on $\mathcal{F}$ is used to define an order relation on the hyperfunctions ([2], p.14).

Definition 3. Let F_1 and F_2 be any two equivalence classes, i.e. hyperfunctions. Then $F_1 > F_2$ if $f(x) \gg 0$ for each $f \in F_1 - F_2$; $F_1 \geqslant F_2$ if either $F_1 = F_2$ or $F_1 > F_2$.

It can easily be shown that the set of all hyperfunctions with the above definitions forms an *ordered vector space*. Furthermore, this space has an *order-unit*. The hyperfunction containing the constant function $f(x) = 1$ is an order-unit and is denoted by 1.

Banach then discusses the concept and properties of nonnegative linear functionals on the space of hyperfunctions. After some preliminaries he proves the following theorems ([2], p. 16, 19).

Theorem 3. *Let $\mathfrak{M}$ be a subspace of hyperfunctions and F_1 a hyperfunc-*

tion not in $\mathfrak{M}$. *Let* $F_2, F_3 \in \mathfrak{M}$ *with* $F_3 > F_1 > F_2$ *and* A *a nonnegative linear functional defined on* $\mathfrak{M}$. *Then there exists a nonnegative linear functional* $\widehat{A}$ *defined on the space spanned by* $\mathfrak{M} \cup \{F_1\}$ *such that* $\widehat{A}(F) = A(F)$ *for all* $F \in \mathfrak{M}$.

Theorem 4. Let $\mathfrak{M}$ *be a subspace of hyperfunctions containing the hyperfunction* 1 *and* A *a nonnegative linear functional defined on* $\mathfrak{M}$. *Then there exists a nonnegative linear functional* $\widehat{A}$ *defined on the space of all hyperfunctions such that* $\widehat{A}(F) = A(F)$ *for all* $F \in \mathfrak{M}$.

Therem 4 enables Banach to establish the following result ([2], p. 23).

Theorem 5. *There exists a functional* H *defined for all bounded functions with period* 1 *satisfying the following conditions*:

1 $H(c_1 f_1(x) + c_2 f_2(x)) = c_1 H(f_1(x)) + c_2 H(f_2(x))$ *for any bounded functions* f_1, f_2 *with period* 1 *and any real numbers* c_1, c_2.
2 $H(f_1(x)) \geqslant 0$ *if* $f_1(x) \geqslant 0$.
3 $H(f_1(x)) = c$ *if* $f_1(x) = c$ *for all* x, *i.e.,* $H(c) = c$ *for any real number* c.
4 $H(f(x)) = H(f(\pm x + \alpha))$ *for every real number* α.
5 *If* f *is Lebesgue integrable, then*

$$H(f(x)) = (L)\int_0^1 f(x)\,dx.$$

Theorem 5 follows from Theorem 4 by the following considerations. The subspace $\mathfrak{M}$ consists of all hyperfunctions which contain Lebesgue integrable functions. The nonnegative linear functional A is given by $A(F) = (L)\int_0^1 f(x)\,dx$ where $F \in \mathfrak{M}$ and $f \in F$ is Lebesgue integrable. Theorem 4 guarantees the existence of $\widehat{A}$, i.e., an extension of A to the space of all hyperfunctions. The functional $\widehat{A}$ in turn gives rise to a functional G on $\mathcal{F}$ defined by $G(f(x)) = \widehat{A}(F)$ where $f \in F$. The required functional H for Theorem 5 is defined by

$$H(f(x)) = \frac{1}{2}[G(f(x)) + G(f(-x))].$$

By setting $m(E) = H(\chi_E(x))$, where χ_E is the characteristic function on $E \subset [0, 1)$, the problem of measure is solved in the affirmative for the real line. Banach notes ([2], p. 31) that an analogous procedure leads to an affirmative answer to the problem of measure in two dimensions.

4 - Extension theorems

Clearly Theorem 3 and Theorem 4 are extension theorems for nonnegative linear functionals. Moreover, with these two theorems Banach established a prototype for the procedure used in proving the Hahn-Banach theorem. Specifically he shows that the functional can be extended from $\mathcal{M}$ to $\mathcal{M} \cup \{F_1\}$ (Theorem 3). Then he uses transfinite induction to extend the functional from $\mathcal{M}$ to the whole space (Theorem 4). This is precisely the procedure used by Hahn in 1927 [4] and Banach in 1929 [3] and in 1932 [1] to prove the Hahn-Banach theorem.

The works of Hahn [4] and Banach [3] cited in the preceding paragraph contain the original version of the Hahn-Banach theorem, i.e., the extension theorem for normed linear spaces. It is easy to see that the space of hyperfunctions is actually an archimedean ordered vector space. There is a natural norm which can be assigned to such a space. If e denotes an order-unit in the space, then this norm is given by

$$\|x\| = \inf\{\lambda > 0 \,|\, x \in [-\lambda e, \lambda e]\}.$$

This norm gives rise to a corresponding norm for linear functionals, i.e.,

$$\|A\| = \sup_{\|x\|=1} |A(x)|.$$

Using these norms in Theorem 4, it can be shown that the norm of the original functional and its extension are the same. With these modifications Theorem 4 becomes a special case of the Hahn-Banach theorem for normed linear spaces (Theorem 1).

Also, the proofs of Theorem 3 and Theorem 4 as given by Banach do not rely on any special properties of hyperfunctions. With appropriate changes in terminology and notation, Banach's work can be applied to prove the following extension theorem.

Theorem 6. *Let X be an ordered vector space with an order-unit e, M a subspace of X with $e \in M$, and f a nonnegative linear functional defined on M. Then there exists a nonnegative linear functional F defined on X such that $F(x) = f(x)$ for all $x \in M$.*

An extension theorem related to Theorem 6 can be found in the 1948 monograph *Linear operators leaving invariant a cone in a Banach space* by M. Krein and M. Rutman. In this monograph they prove an extension theorem for positive linear functionals. Before stating this theorem, some definitions are necessary.

Definition 4. Let X be a normed linear space and $K \subset X$. Then K is called a *(linear) semi-group* if

a $x \in K \Rightarrow \lambda x \in K \quad \forall \lambda \geqslant 0$
b $x, y \in K \Rightarrow x + y \in K$.

A semi-group gives rise to a partial ordering on X.

Definition 5. Let $x, y \in X$. Then $x \leqslant y$ if $y - x \in K$.

Definition 6. A closed semi-group $K \subset X$ is called a *cone* if for each element $x \in K$ with $x \neq 0$ we have, $-x \notin K$.

Definition 7. A linear functional f is said to be *positive (with respect to K)* if $f(x) \geqslant 0 \ \forall x \in K$ and there exists at least one $x_0 \in K$ such that $f(x_0) > 0$.

The Krein-Rutman extension theorem may be given as follows ([7], p. 13).

Theorem 7. *Let $K \neq X$ be a semi-group with interior and let the subspace $M \subset X$ contain at least one strictly positive element in the interior of K. Then each positive linear functional $f(x)$ defined on M can be extended to a positive linear functional $F(x)$ defined on all of X.*

In Theorem 7 the semi-group provides the space with an order relation. Also, order units are always positive interior points. Hence the relation between Theorem 6 and Theorem 7 is clear, viz., Theorem 6 is a special case of Theorem 7.

It should be mentioned that in the introduction to their monograph, Krein and Rutman state that Theorem 7, i.e., the extension theorem for positive linear functionals, is due to Krein. It was part of an unpublished manuscript *written on the eve of the war* ([6], p. 9). Whatever the case may be it is clear that Theorem 7 has a precursor in the works of Banach, specifically in his 1923 paper in which he solves the problem of measure.

5 - Conclusion

Although the Hahn-Banach theorem can be used to solve the problem of measure, Banach's solution to the problem predates that theorem. In fact, Banach's solution to the problem of measure furnishes more than a solution to that problem. The techniques used to solve it provide a prototype for the proof of the Hahn-Banach theorem. Also, the extension theorems proven by Banach are a precursor to the result published by M. Krein in 1948 on the extension of positive linear functionals.

References

[1] S. BANACH, *Opérationes linéaires*, Chelsea Publishing Company, New York 1971.

[2] S. BANACH, *Sur le problème de la mesure*, Fund. Math. 4 (1923), 7-33.

[3] S. BANACH, *Sur les fonctionells linéaires*, Studia Math. **1** (1929), 211-218.

[4] H. HAHN, *Über lineare Gleichungssysteme in linearen Raümen*, J. Reine Angew. Math. **157** (1927), 214-229.

[5] F. HAUSDORFF, *Grundzüge der Mengenlehre*, Chelsea Publishing Company, New York 1978.

[6] M. KREIN and M. RUTMAN, *Linear operators leaving invariant a cone in a Banach space*, Uspekhi Mat. Nauk 3, **1** (23) (1948), 3-95 (Trans. Amer. Math. Soc. **26** (1950)).

[7] H. LEBESGUE, *Leçons sur l'intégration et la recherche des fonctions primitives*, Gauthier Villars, Paris 1904.

Sommario

Nella sua soluzione originale al problema di misura, S. Banach adoperò un teorema di estensione che precede il teorema di Hahn-Banach. Un'analisi del suo lavoro dimostra che questo teorema d'estensione può essere modificato per gli spazi vettoriali ordinati e porta ad un caso speciale del teorema di Krein-Rutman.

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