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On integrability conditions of an almost r-contact manifold (**)

Introduction

Almost r-contact manifolds have been defined and studied by Vanzura [4] and others. Mishra ([1], [2]) studied integrability conditions of an almost contact manifold. The aim of the present paper is to study *integrability conditions of an almost r-contact structure manifold.*

1 - Preliminaries

Let M^{2n+r} be a $(2n+r)$-dimensional differentiable manifold of class C^∞. Suppose there exists on M^{2n+r} a *tensor field* F $(F \neq 0)$ of type (1, 1), r C^∞ *contravariant vectorfields* $\underset{1}{U}, \underset{2}{U}, \dots \underset{r}{U}$ and r C^∞ *1-forms* $\overset{1}{u}, \overset{2}{u}, \dots \overset{r}{u}$ (r some finite integer) satisfying

$$F^2 = -I + \sum_{k=1}^{r} \overset{k}{u} \otimes \underset{k}{U} \tag{1}$$

$$F \underset{k}{U} = 0 \qquad \overset{k}{u} \circ F = 0 \qquad \overset{k}{u}(\underset{m}{U}) = \delta^k_m \tag{2}$$

where k, m take the values $1, 2, \dots, r$ and δ^k_m denotes the Kronecker delta.

The manifold M^{2n+r} satisfying conditions (1), (2) is said to possess an *almost r-contact structure* [4].

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Suppose rank $F = 2n$. Then, for F, there exist n eigenvalues equal to $+i$, n eigenvalues equal to $-i$ and r eigenvalues equal to zero [2].

Let $\underset{x}{P}$, $\underset{x}{Q}$, $\underset{k}{U}$, be the *eigenvectors* corresponding to the eigenvalues $+i$, $-i$ and zero of F respectively. Let $\{\overset{x}{p}, \overset{x}{q}, \overset{k}{u}\}$ be the set of 1-*forms* dual to the set $\{\underset{x}{P}, \underset{x}{Q}, \underset{k}{U}\}$. The necessary and sufficient condition that the manifold M^{2n+r} may admit an almost r-contact structure is that it possesses a tangent bundle π_n of dimension n, a tangent bundle $\widetilde{\pi}_n$, conjugate to π_n, and the product space π_r of r-tuples of real numbers such that [2]

$$\pi_n \cap \widetilde{\pi}_n = \widetilde{\pi}_n \cap \pi_r = \pi_n \cap \pi_r = \emptyset$$

and $\pi_n \cup \widetilde{\pi}_n \cup \pi_r$ be a tangent bundle of dimension $(2n + r)$, the *projections* on π_n, $\widetilde{\pi}_n$ and π_r being given by

$$(3) \qquad L = -\frac{1}{2}(F^2 + iF) \qquad M = -\frac{1}{2}(F^2 - iF) \qquad N = F^2 + I = \sum_{k=1}^{r} \overset{k}{u} \otimes \underset{k}{U}.$$

The *projection operators* L, M and N can be written in the form (see [2])

$$(4) \qquad L = \overset{x}{p} \otimes \underset{x}{P} \qquad M = \overset{x}{q} \otimes \underset{x}{Q} \qquad N = \overset{k}{u} \otimes \underset{k}{U}.$$

2 - Integrability conditions

In this section, we shall prove certain results on *integrability conditions* of almost r-contact manifolds.

Lemma 1. *We have*

$$(5) \qquad \begin{aligned} &LM = ML = MN = NM = LN = NL = 0 \\ &L^2 = L, \qquad M^2 = M, \qquad N^2 = N. \end{aligned}$$

Proof. Since $\underset{x}{P}$, $\underset{x}{Q}$ and $\underset{k}{U}$ are eigenvectors of F, corresponding to the eigenvalues i, $-i$ and zero, we have [2]

$$(6) \qquad L\underset{x}{P} = \underset{x}{P} \qquad L\underset{x}{Q} = 0 \qquad L\underset{k}{U} = 0$$

$$(7) \qquad M\underset{x}{P} = 0 \qquad M\underset{x}{Q} = \underset{x}{Q} \qquad M\underset{k}{U} = 0$$

$$(8) \qquad N\underset{x}{P} = 0 \qquad N\underset{x}{Q} = 0 \qquad N\underset{k}{U} = \underset{k}{U}.$$

Now for arbitrary vector field X, we have

$$(LM)X = L(M(X)) = L(\overset{x}{q}(X)\underset{x}{Q}) = \overset{x}{q}(X)L(\underset{x}{Q}) = 0$$

by virtue of the equation (6). Thus $LM = 0$.

Similarly we can show that $ML = MN = NM = LN = NL = 0$.

To prove $(5)_2$, we note that

$$L^2(X) = L(L(X)) = L(\overset{x}{p}(X)\underset{x}{P}) = \overset{x}{p}(X)L(\underset{x}{P}) = \overset{x}{p}(X)\underset{x}{P} = L(X)$$

by virtue of the equation (6). Thus $L^2 = L$.

The other part of Lemma 1 can be proved in a similar manner.

Lemma 2. *For an almost r-contact structure manifold* M^{2n+r}, *we have*

$$(9)\qquad \begin{aligned} \frac{1}{2}(dL)(NX, NY) &= \overset{h}{u}(X)\overset{k}{u}(Y)\overset{x}{p}[\underset{h}{U},\underset{k}{U}]\underset{x}{P} \\ (dL)(MX, MY) &= -\overset{x}{q}(X)\overset{y}{q}(Y)\{F^2[\underset{x}{Q},\underset{y}{Q}] + iF[\underset{x}{Q},\underset{y}{Q}]\}. \end{aligned}$$

Proof. The *torsion* dL of L is defined by

$$\frac{1}{2}(dL)(X, Y) = [LX, LY] + L^2[X, Y] - L[X, LY] - L[LX, Y].$$

Since $LN = 0$, $L^2 = L$ we get

$$(10)\qquad \frac{1}{2}(dL)(NX, NY) = L[NX, NY].$$

In view of the equation $(3)_3$, the above equation takes the form

$$\frac{1}{2}(dL)(NX, NY) = L[\overset{h}{u}(X)\underset{h}{U}, \overset{k}{u}(Y)\underset{k}{U}],$$

or

$$\frac{1}{2}(dL)(NX, NY) = \overset{h}{u}(X)\overset{k}{u}(Y)L[\underset{h}{U},\underset{k}{U}] = \overset{h}{u}(X)\overset{k}{u}(Y)\overset{x}{p}[\underset{h}{U},\underset{k}{U}]\underset{x}{P}$$

by virtue of $(6)_3$ and $(4)_1$.

In a similar way we have

$$\frac{1}{2}(\mathrm{d}L)(MX,\, MY) = L[MX,\, MY] = L[\overset{x}{q}(X)\,\underset{x}{Q}, \overset{y}{q}(Y)\,\underset{y}{Q}] = \overset{x}{q}(X)\,\overset{y}{q}(Y)\,L[\underset{x}{Q}, \underset{y}{Q}]$$

which in view of the equation $(3)_1$ becomes

$$(\mathrm{d}L)(MX,\, NM) = -\,\overset{x}{q}(X)\,\overset{y}{q}(Y)\{F^2[\underset{x}{Q}, \underset{y}{Q}] + iF[\underset{x}{Q}, \underset{y}{Q}]\}$$

which proves the second part of the lemma.

Theorem 1. *A necessary and sufficient condition that the distribution π_r is completely integrable is that*

$$\overset{h}{u}(X)\,\overset{k}{u}(Y)\,\overset{x}{p}[\underset{h}{U}, \underset{k}{U}]\,\underset{x}{P} = 0 \qquad \overset{h}{u}(X)\,\overset{k}{u}(Y)\,\overset{x}{q}[\underset{h}{U}, \underset{k}{U}]\,\underset{x}{Q} = 0\,. \tag{11}$$

Proof. The distribution π_r is given by

$$LX = 0 \qquad MX = 0 \qquad NX = X\,. \tag{12}$$

Hence, in order that the distribution π_r is completely integrable, it is necessary and sufficient that $LX = 0$ and $MX = 0$ are completely integrable. Hence we have [3]

$$(\mathrm{d}L)(X,\, Y) = 0 \qquad (\mathrm{d}M)(X,\, Y) = 0\,. \tag{13}$$

That in view of equation $(12)_3$ takes the form

$$(\mathrm{d}L)(NX,\, NY) = 0 \qquad (\mathrm{d}M)(NX,\, NY) = 0\,. \tag{14}$$

By virtue of equations $(9)_1$ we prove $(11)_1$.

In a similar way, we can also show that the equation $(14)_2$ is equivalent to $(11)_2$.

Theorem 2. *In order that the distribution π_n be completely integrable, it is necessary and sufficient that*

$$\begin{aligned} &\overset{x}{p}(X)\,\overset{y}{p}(Y)(F^2[\underset{x}{P}, \underset{y}{P}] - iF[\underset{x}{P}, \underset{y}{P}]) = 0 \\ &\overset{x}{p}(X)\,\overset{y}{p}(Y)(F^2[\underset{x}{P}, \underset{y}{P}] + [\underset{x}{P}, \underset{y}{P}]) = 0\,. \end{aligned} \tag{15}$$

Proof. The distribution π_n is given by

$$LX = X \quad MX = 0 \quad NX = 0\,. \tag{16}$$

Therefore in order that π_n is completely integrable it is necessary and sufficient that $MX = 0$ and $NX = 0$ are completely integrable. Hence

$$(\mathrm{d}M)(X, Y) = 0 \quad (\mathrm{d}N)(X, Y) = 0\,. \tag{17}$$

In view of $(16)_1$ equation (17) takes the form

$$(\mathrm{d}M)(LX, LY) = 0 \quad (\mathrm{d}N)(LX, LY) = 0\,. \tag{18}$$

By virtue of equation $(5)_1$, above equations take the form

$$M[LX, LY] = 0 \quad N[LX, LY] = 0\,. \tag{19}$$

In view of (3) and (4) equation (19) is equivalent to equation (15). This proves Theorem 2.

Theorem 3. *The distribution $\tilde{\pi}_n$ is completely integrable if and only if*

$$\overset{x}{q}(X)\,\overset{y}{q}(Y)\,\overset{z}{p}[\underset{x}{Q},\underset{y}{Q}]\,\underset{z}{P} = 0 \quad \overset{x}{q}(X)\,\overset{y}{q}(Y)\,\overset{k}{u}[\underset{x}{Q},\underset{y}{Q}]\,\underset{k}{U} = 0\,. \tag{20}$$

Proof. The proof follows in a way similar to that of the Theorems 1 and 2.

References

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Summary

Some results concerning integrability conditions for almost r-contact manifolds are proved.

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