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A necessary condition for global Stein immersion of compact CR-manifolds (**)

1 - Abstract CR-manifolds and CR-complexes

An *abstract CR-manifold* is a triple (M, H, J) where M is a paracompact smooth real manifolds, H is an even dimensional subbundle of the tangent bundle TM and J a partial pseudocomplex structure on H, i.e. a fiber preserving bundle isomorphism $J: H \to H$ with $J^2 = -1$. We also require that J be formally integrable, i.e. that, for

$$\tau^{0,1}M = \{X + iJX \mid X \in \Gamma(M, H)\} \subset \Gamma(M, CTM) \tag{1.1}$$

we have

$$[\tau^{0,1}M, \tau^{0,1}M] \subset \tau^{0,1}M. \tag{1.2}$$

Let m be the real dimension of M and $2n$ the real dimension of the fibers of H. Then n is called the CR-dimension of M and $k = m - 2n$ its CR-codimension. In this case we say that M is of type (n, k). For a general reference on CR-manifolds see G. Taiani [3].

Let $\Omega(M)$ denote the *exterior algebra* of smooth complex valued alternating

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forms on M and $\Omega^p(M)$ the subspace of forms of degree p. We consider the ideal

$$\mathscr{I}(M) = \mathscr{I}^1(M) = \{\eta \in \bigoplus_{p \geq 1} \Omega^p(M) \,|\, \eta|_{T^{0,1}M} = 0\} \tag{1.3}$$

and its exterior powers

$$\mathscr{I}^0(M) = \Omega(M) \qquad \mathscr{I}^p(M) = \mathscr{I}^{p-1}(M) \wedge \mathscr{I}(M) \quad (p = 1, 2, \ldots). \tag{1.4}$$

By integrability condition (1.2) we have $\mathrm{d}\mathscr{I}^p(M) \subset \mathscr{I}^p(M)$ for every $p \geq 0$.

Note that $\mathscr{I}^{n+k+1}(M) = 0$ by reasons of degree and that we obtain a decreasing sequence of ideals of $\Omega(M)$

$$\mathscr{I}^0(M) = \Omega(M) \supset \mathscr{I}^1(M) \supset \mathscr{I}^2(M) \supset \ldots \supset \mathscr{I}^{n+k}(M) \supset \{0\}. \tag{1.5}$$

For integers $p, q \geq 0$ we set $\mathscr{I}^{p,q}(M) = \mathscr{I}^p(M) \cap \Omega^{p+q}(M)$. Then clearly we have

$$\mathrm{d}\mathscr{I}^{p,q}(M) \subset \mathscr{I}^{p,q+1}(M) \qquad \mathscr{I}^{p+1,q}(M) \subset \mathscr{I}^{p,q}(M). \tag{1.6}$$

Now we define, for $p, q \geq 0$

$$Q^{p,q}(M) = \mathscr{I}^{p,p+q}(M) / \mathscr{I}^{p+1,p+q}(M) \tag{1.7}$$

so that, passing to the quotient we obtain from the De Rham complex the *Cauchy-Riemann complexes* on M

$$0 \longrightarrow Q^{p,0}(M) \xrightarrow{\bar\partial_M} Q^{p,1}(M) \xrightarrow{\bar\partial_M} \ldots \longrightarrow Q^{p,n} \longrightarrow 0. \tag{1.8}$$

(Notice that by definition $Q^{p,q}(M) = 0$ for $q > n$). Setting $Q^{p,-1}(M) = 0$, we define the *cohomology groups of the CR-complexes*

$$\mathscr{H}^{p,q}(M) = \mathrm{Ker}\,(\bar\partial_M\colon Q^{p,q}(M) \to Q^{p,q+1}(M)) / \mathrm{Im}\,(\bar\partial_M\colon Q^{p,q-1}(M) \to Q^{p,q}(M)).$$

An open subset of a CR-manifold is in an obvious way itself a CR-manifold and thus we can define, via the natural restriction maps, the *local CR-cohomology* at a point $P \in M$

$$\mathscr{H}^{p,q}_P(M) = \varinjlim_{U \text{ open} \ni P} \mathscr{H}^{p,q}(U). \tag{1.9}$$

Remark 1. *For every* $p, q \geqslant 0$ *we have natural restriction maps* $\mathscr{H}^{p,q}(M) \to \mathscr{H}_P^{p,q}(M)$. *They are surjective when* $q = n$.

A *CR-map* of a CR-manifold (M_1, H_1, J_1) into a CR-manifold (M_2, H_2, J_2) is a differentiable map φ: $M_1 \to M_2$ such that

$$\varphi_*(H_1) \subset H_2 \qquad \varphi_*(J_1 X) = J_2 \varphi_*(X) \qquad \text{for } X \in H_1 . \tag{1.10}$$

If φ is a diffeomorphism and φ^{-1} is CR, we say that φ is a *CR-isomorphism.*

The CR-map φ induces natural homomorphisms of the CR-cohomology groups

$$\varphi^* : \mathscr{H}^{p,q}(M_2) \to \mathscr{H}^{p,q}(M_1) \qquad \forall p, q \geqslant 0 . \tag{1.11}$$

They are isomorphisms if φ is a CR-isomorphism. Clearly a CR-isomorphism φ of a neighborhood U of $P \in M_1$ onto a neighborhood V of $\varphi(P) = Q \in M_2$ induces isomorphisms $\varphi^* : \mathscr{H}_Q^{p,q}(M_2) \to \mathscr{H}_P^{p,q}(M_1)$.

A CR-manifold of the form (M, TM, J) is a complex manifold by the Newlander-Nirenberg theorem and the CR-complexes on M are (modulo sheaf isomorphisms) the usual Dolbeault complexes on M.

Let M be a real submanifold of a complex manifold $\widetilde{M}$, with complex structure $\widetilde{J}$. If for every $P \in M$ we set $H_P M = T_P M \cap \widetilde{J} T_P M$, then we realize that $(M, HM, \widetilde{J}_{|HM})$ is a CR-manifold provided that the spaces $H_P M$ have a constant dimension. In this case the embedding map i: $M \to \widetilde{M}$ is a CR-map of $(M, HM, \widetilde{J}_{|HM})$ into $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$.

An *embedding* (resp. *immersion*) φ of a CR-manifold (M, H, J) in a complex manifold $(\widetilde{M}, T\widetilde{M}, \widetilde{J})$ is a CR-map φ: $M \to \widetilde{M}$ which is an embedding (resp. immersion).

The CR-maps of (M, H, J) into $\boldsymbol{C}$ (with the usual complex structure) are called CR-functions and correspond in a natural way to the elements of the group $\mathscr{H}^{0,0}(M)$.

Remark. *A necessary condition in order that a CR-manifold* (M, H, J) *of type* (n, k) *could be inmersed in a Stein manifold* $\widetilde{M}$ *is that* $\mathscr{H}^{p,0}(M)$ *be infinite dimensional for* $0 \leqslant p \leqslant n + k$.

An immersion (resp. embedding) of a CR-manifold (M, H, J) of type (n, k) into a complex manifold $\widetilde{M}$ of dimension $n + k$ is said to be *generic.*

2 - Obstructions to global immersions of CR-manifolds

a) The Levi form.

Let (M, H, J) be a CR-manifold of type (n, k). Let $H^0 \subset T^*M$ be the annihilator bundle of the bundle H (i.e. the characteristic bundle of the CR-complexes on M). We consider the bundle $T^{0,1}M = \{X + iJX | X \in H)$.

Then the Levi form of (M, H, J) at $\omega \in H_P^0$ is the Hermitean form on $T_P^{0,1}M$

$$L(\omega, Z) = i d\tilde{\omega}(Z, \overline{Z}) = -i\omega[\underset{\sim}{Z}, \underset{\sim}{\overline{Z}}] \tag{2.1}$$

where $\tilde{\omega} \in \Gamma(M, H^0)$ satisfies $\tilde{\omega}(P) = \omega$ and $\underset{\sim}{Z} \in \tau^{0,1}M$ satisfies $\underset{\sim}{Z}(P) = Z$.

The equality of the last two expressions shows that they do not depend on the choice of $\tilde{\omega}$ and $\underset{\sim}{Z}$ and therefore L is a function defined on the direct sum of the bundles H^0 and $T^{0,1}M$.

Let $\sigma(\omega) = (\sigma_1(\omega), \sigma_2(\omega))$ denote the signature of the Hermitean form $Z \mapsto L(\omega, Z)$. In [1] the following statement is proved

Proposition 1. *Assume that, for a point $P \in M$ we can find a generic immersion of an open neighborhood of P in M into some complex manifold. Then, if for some $\omega \in H_P^0$ we have $\sigma_1(\omega) + \sigma_2(\omega) = n$, then the groups $\mathscr{H}_P^{p,\sigma_1(\omega)}(M)$ and $\mathscr{H}_P^{p,\sigma_2(\omega)}$ are infinite dimensional for $0 \leq p \leq n + k$.*

If we drop the assumption that there is a generic immersion of a neighborhood of P in some complex manifold, then we have a weak version of the above result (cf. [2]).

Proposition 2. *If for some $\omega \in H_P^0$ we have $\sigma_1(\omega) + \sigma_2(\omega) = n$, then the groups $\mathscr{H}_P^{p,\sigma_1(\omega)}(M) \oplus \mathscr{H}_P^{p,\sigma_1(\omega)+1}(M)$ and $\mathscr{H}_P^{p,\sigma_2(\omega)}(M) \oplus \mathscr{H}_P^{p,\sigma_2(\omega)+1}(M)$ are infinite dimensional for $0 \leq p \leq n + k$.*

b) In this subsection we state and prove the main result of this paper.

Theorem 1. *Let (M, H, J) be a compact CR-manifold of type (n, k). A necessary condition in order that M could be immersed in a Stein manifold $\tilde{M}$ is that the following two conditions hold*

(i) *$\mathscr{H}^{p,n}(M)$ is infinite dimensional for every $0 \leq p \leq n + k$;*

(ii) *We can find $\omega \in H^0$ such taht $\sigma(\omega) = (n, 0)$.*

The same conclusions (i), (ii) *hold, if we substitute for the assumption that* $\tilde{M}$ *be Stein the weaker assumption that there exists on* $\tilde{M}$ *a globally defined smooth strictly plurisubharmonic function* $\psi\colon \tilde{M} \to \boldsymbol{R}$.

We note that, in view of the results on the embedding of CR-strictly pseudoconvex manifolds of hypersurface type (i.e. of type $(N, 1)$ for some N), conditions (i) and (ii) of Theorem 1 are also necessary for the existence of a CR-immersion of a compact CR-manifold (M, H, J) into a strictly pseudoconvex CR-manifold $(\tilde{M}, \tilde{H}, \tilde{J})$ of type $(N, 1)$ when $2N + 1 \neq 5$ or $\tilde{M}$ is compact. In the case $N = 1$, where in general $\tilde{M}$ is not embeddable into a complex manifold, the conditions are an easy consequence of [2]. For $N > 1$, we reduce to the second statement in Theorem 1.

Let us prove first the following weaker result

Proposition 3. *Let* (M, H, J) *be a compact CR-submanifold of type* (n, k) *of a Euclidean complex space* $\boldsymbol{C}^N$ *(with* $N \geqslant n + k$*). Then the conclusions* (i) *and* (ii) *of Theorem* 1 *hold.*

Proof. Let $B(R) = \{z \in \boldsymbol{C}^N \mid |z| \leqslant R\}$ for $R = \max_M |z|$. Let us fix a point θ in M with $|\theta| = R$.

If $N = n + k$, i.e. (M, H, J) is generic in $\boldsymbol{C}^N$, then we can represent M near θ as the set of common zeros of k real valued functions $\rho_1, \ldots, \rho_k$ with

$$\partial\rho_1(\theta) \wedge \ldots \wedge \partial\rho_k(\theta) \neq 0 . \tag{2.2}$$

We can choose ρ_1 in such a way that $\nabla\rho_1(\theta) = \theta$ and that the hypersurface $\rho_1 = 0$ is contained in $B(R)$ near θ.

It follows that the restriction of ρ_1 to the boundary $bB(R)$ has a local minimum at θ and hence its real Hessian is non-negative at θ on $T_\theta bB(R)$. But this implies that the real Hessian of ρ_1 is positive definite on the tangent space to $bB(R)$ at θ. Then (ii) follows with

$$\omega = d^c\rho_1(\theta) = \frac{\partial\rho_1 - \bar{\partial}\rho_1}{2i}(\theta) \in H^0_\theta M \tag{2.3}$$

and (i) follows from Proposition 1 and Remark 1.

The general statement of Proposition 3 is then an easy consequence of the following

Lemma 1. *Let M be a compact CR-submanifold of* $\boldsymbol{C}^N$, *of CR-dimension n and CR-codimension k. Let* $\theta \in M$ *and assume that, for a closed Euclidean* *ball B in $\boldsymbol{C}^N$ we have*

(i) $M \subset B$ (ii) $\theta \in M \cap bB$.

Then we can find a neighborhood U of θ in $\boldsymbol{C}^N$ and an open holomorphic map Φ: $U \to \boldsymbol{C}^{n+k}$, such that $\Phi(M \cap U)$ is a generic CR-submanifold of $\Phi(U)$, $\Phi_{|M}$: $M \cap U \to \Phi(M \cap U)$ is a CR-isomorphism and for a closed Euclidean ball B' in $\boldsymbol{C}^{n+k}$ we have

(i′) $\Phi(M \cap U) \subset B'$ (ii′) $\Phi(\theta) \in \Phi(M \cap U) \cap bB'$.

Proof. With no loss of generality, we can assume that $|\theta| = 1$ and that

$$M \subset \{z \in \boldsymbol{C}^N \,|\, |z - \frac{\theta}{2}| \leq \frac{1}{2}\} \quad \text{so that} \tag{2.4}$$

$$|z|^2 \leq 1 - |z - \theta|^2 \quad \text{for } z \in M. \tag{2.5}$$

Let $\widetilde{H}_\theta M = T_\theta M + JT_\theta M$ denote the smallest complex subspace of $T_\theta \boldsymbol{C}^N$ containing $T_\theta M$. It has dimension $n + k$ and, if $n + k = N$, the embedding of M is generic and there is nothing to prove.

We assume therefore that $n + k < N$ and set $l = N - n - k > 0$. Let $S = \{|z| = 1\}$ be the unit Euclidean sphere centered at 0 in $\boldsymbol{C}^N$ and

$$H_\theta S = \{X \in T_\theta \boldsymbol{C}^N \,|\, (X|\theta) = (JX|\theta) = 0\} \tag{2.6}$$

be *the analytic tangent space to S at θ*, where by $(\,|\,)$ we denote the real part of the canonical Hermitean scalar product in $\boldsymbol{C}^N$.

If $\widetilde{H}_\theta M$ is not contained in $H_\theta S$, then we can find in $H_\theta S$ an l-dimensional complex subspace W such that $\widetilde{H}_\theta M \cap W = \{0\}$.

If V is the orthogonal $(n + k)$-dimensional linear subspace to W through the origin of $\boldsymbol{C}^N$, then we can take as Φ the restriction to a neighborhood U of θ of the orthogonal projection π: $\boldsymbol{C}^N \to V$.

Indeed, π maps the unit ball centered at 0 of $\boldsymbol{C}^N$ into the unit ball B' centered at 0 of V. As W is tangent to the unit ball of $\boldsymbol{C}^N$ at θ, then $\pi(\theta)$ is in the boundary of B'. Moreover, as W is transversal to $\widetilde{H}_\theta M$, the projection π defines a CR-isomorphism of $M \cap U$ into $\pi(M \cap U)$ for some small open neighborhood U of θ in $\boldsymbol{C}^N$.

So, to prove the lemma we only need to get rid of the case where $\widetilde{H}_\theta M \subset H_\theta S$. If this happens, let us fix $X \in T_\theta M - H_\theta M$ with $|X| = 1$. By assumption, $(\theta|JX) = 0$ and then the ball

$$B_\varepsilon = \{z \in \boldsymbol{C}^N \,|\, |z - \varepsilon JX|^2 \leq 1 + \varepsilon^2\}$$

contains θ in its boundary for every $\varepsilon \in \boldsymbol{R}$. We claim that $M \subset B_\varepsilon$ if ε is sufficiently small. Indeed, for $z \in M$ we have

$$|z - \varepsilon JX|^2 = |z|^2 + \varepsilon^2 |JX|^2 - 2\varepsilon(z|JX) = |z|^2 + \varepsilon^2 - 2\varepsilon(z|JX).$$

We note now that JX is orthogonal to $T_\theta M$ for the real scalar product. It follows that, for some constant $c > 0$, we have $|(z|JX)| \leqslant c|z - \theta|^2$ for $z \in M$.

Then we obtain by (2.5), for $z \in M$

$$|z - \varepsilon JX|^2 \leqslant 1 - |z - \theta|^2 + \varepsilon^2 + 2\varepsilon c|z - \theta|^2 \tag{2.7}$$

and therefore $M \subset B_\varepsilon$ for $|\varepsilon| \leqslant (2c)^{-1}$.

With $S_\varepsilon = bB_\varepsilon$ we have

$$H_\theta S_\varepsilon = \{Y \in T_\theta \boldsymbol{C}^N \,|\, (\theta - \varepsilon JX|Y) = (\theta - \varepsilon JX|JY) = 0\} \tag{2.8}$$

and hence X does not belong to $H_\theta S_\varepsilon$ if $0 < |\varepsilon| \leqslant (2c)^{-1}$, so that $\widetilde{H}_\theta M$ is not contained in $H_\theta S_\varepsilon$ and we can conclude arguing as above.

To complete the proof of Theorem 1, we first notice that the conclusion of Proposition 3 still holds if M is the image of a CR-manifold under a CR-immersion. Indeed, once we used the fact that M is a compact subset of $\boldsymbol{C}^N$, the remaining arguments have only a local character.

If we have an immersion φ of the CR-manifold (M, H, J) into a 0-convex complex manifold $\widetilde{M}$, we choose a strictly plurisubharmonic function $\psi\colon \widetilde{M} \to \boldsymbol{R}$ and a point θ in $\varphi(M)$ such that $\psi(\theta) = \max_{\varphi(M)} \psi$.

Choosing coordinates on a neighborhood of θ in $\widetilde{M}$ in such a way that φ becomes strictly convex in the new coordinates, we conclude by the arguments of Proposition 3 and of Lemma 1.

References

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Summary

In this paper we derive necessary conditions for the global immersion of a compact CR-manifold, of any type, in C^N *for some* N*; or more generally, in a Stein manifold. These conditions are stated in terms of the cohomology groups of the CR-tangential complexes.*

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