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Multihelicoids in standard spaces of constant curvature (**)

1 - Introduction

In [4] Kowalski and Külich introduced the notion of generalized k-symmetric submanifolds of the Euclidean space $\boldsymbol{R}^n$. In [2] generalized 2-symmetric submanifolds are considered in any space of constant curvature and it is proved that any such submanifold satisfies the condition

$$\nabla \overset{k}{\sigma}_M = 0 \quad \text{for each } k \tag{1.1}$$

where $\overset{k}{\sigma}_M$ is the k-th fundamental form on M.

In this work we consider, more generally, *a nicely curved submanifold M of a standard space $\overline{M} = \overline{M}(c)$ of constant curvature c, satisfying condition* (1.1). For such a submanifold M we put $V_x = \bigoplus_{h=0}^{\bar{l}} \overset{2h+1}{N_x} M$, where $\overset{2h+1}{N_x} M$ is the $(2h+1)$-th normal space of M at x, $\bar{l} = [\frac{1}{2}(l-1)]$ and l the number defined in Sec. 2.

Our main result (see Theorem in Sec. 3) asserts that *any submanifold R of $\overline{M}$, which is a tubular neighbourhood of M in the set $\{\exp_x v\}_{x \in M,\, v \in V_x}$ is minimal.*

We observe that R is a submanifold of $\overline{M}$ foliated by totally geodesic submanifold of $\overline{M}$.

Since, if dim $M = 1$, condition (1.1) implies that M is just a curve with con-

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stant curvatures and R becomes the helicoid, associated to the curve in the sense of [1], we will call R a *multihelicoid.*

2 - Preliminaries

Let $\overline{M} = \overline{M}(c)$ be a *standard space of constant curvature* c. In other words, we assume that $\overline{M}$ is the Euclidean sphere $S^{\overline{m}}(c)$, the Euclidean space $\boldsymbol{R}^{\overline{m}}$ or the hyperbolic space $H^{\overline{m}}(c)$, according to c positive, zero or negative.

For the sake of simplicity we will take $c = 1, 0, -1$. We consider $S^{\overline{m}}(1)$: $\sum_{1}^{\overline{m}+1}{}_i (x^i)^2 = 1$ as an hypersurface of $\boldsymbol{R}^{\overline{m}+1}$, $\boldsymbol{R}^{\overline{m}}$ identified with the hyperplane: $x^{\overline{m}+1} = 0$ of $\boldsymbol{R}^{\overline{m}+1}$ and $H^{\overline{m}}(-1)$: $-(x^1)^2 + \sum_{2}^{\overline{m}+1}{}_i (x^i)^2 = -1$, $x^1 > 0$, as a hypersurface of the Lorentz space $L^{\overline{m}+1}$.

From now on we will indicate both $\boldsymbol{R}^{\overline{m}+1}$, $L^{\overline{m}+1}$ with $\widehat{M}$, so we have, in any case, $\overline{M} \subset \widehat{M}$.

We recall that for the *exponential map* $\exp_x$: $T_x\overline{M} \to \overline{M}$ at a point $x \in X$ we have

$$\exp_x v = \varphi(|v|)\, x + \psi(|v|)\, v \qquad v \in T_x\overline{M} \tag{2.1}$$

where $\varphi(|v|) = \cos |v| \qquad \psi(|v|) = \dfrac{\text{sen}\, |v|}{|v|} \qquad \text{if } c = 1$

$$\varphi(|v|) = 1 \qquad \psi(|v|) = 1 \qquad \text{if } c = 0$$

$$\varphi(|v|) = \cosh |v| \qquad \psi(|v|) = \frac{\text{senh}\, |v|}{|v|} \qquad \text{if } c = -1.$$

We also recall that, if we indicate with $\widehat{\nabla}$ the Levi-Civita connection on $\widehat{M}$ and with $\overline{\nabla}$ the induced connection on $\overline{M}$, then the second fundamental form $\overset{0}{\sigma}_{\overline{M}}$ on $\overline{M}$, defined by the equality

$$\widehat{\nabla}_{X_x} Y = \overline{\nabla}_{X_x} Y + \overset{0}{\sigma}_{\overline{M}}(X_x, Y_x)$$

where X_x, $Y_x \in T_x\overline{M}$ and Y is a tangent field to $\overline{M}$ which extends Y_x, satisfies the equality

$$\overset{0}{\sigma}_{\overline{M}}(X_x, Y_x) = -c\, g(X_x, Y_x)\, x \tag{2.2}$$

where g is the standard Riemannian metric of $\boldsymbol{R}^{\overline{m}+1}$ or the Lorentzian metric of $L^{\overline{m}+1}$ according to $c = 1, 0$ or -1.

We observe that if $c = 1, -1$ the position vector x is orthogonal, with respect to g, to the tangent space $T_x\overline{M}$.

Now let M be a submanifold of $\overline{M}$. We will indicate with ∇ the Levi-Civita connection induced by $\overline{\nabla}$ on M and with $\overset{0}{\sigma}_M$ the second fundamental form of M (in $\overline{M}$), so we have

$$\overline{\nabla}_{X_x} Y = \nabla_{X_x} Y + \overset{0}{\sigma}_M (X_x, Y_x)$$

where X_x, $Y_x \in T_x M$ and Y is a tangent field of M which extends Y_x.

The vector space generated by the values of the second fundamental form $\overset{0}{\sigma}_M$, on a fixed point $x \in M$, is called the *first normal space* of M at x and is denoted by $\overset{1}{N}_x M$.

If $\overset{1}{n}(x) = \dim \overset{1}{N}_x M$ is constant, then we have a vector bundle $\overset{1}{N} M$ on M with fiber $\overset{1}{N}_x M$ on a point $x \in M$. It can be proved that, for $X_x \in T_x M$ and $\overset{1}{\xi} \in \Gamma(\overset{1}{N} M)$, the orthogonal projection $\overline{P}_{(T_x M \oplus \overset{1}{N}_x M)^\perp}$ of $\overline{\nabla}_{X_x} \overset{1}{\xi}$ on $(T_x M \oplus \overset{1}{N}_x M)^\perp$ (subspace of $T_x\overline{M}$) depends only on the value $\overset{1}{\xi}_x$, of $\overset{1}{\xi}$ at x. Then we have a bilinear form

$$\overset{1}{\sigma}_M \colon T_x M \times \overset{1}{N}_x M \to (T_x M \oplus \overset{1}{N}_x M)^\perp$$

defined by
$$\overset{1}{\sigma}_x (X_x, \overset{1}{\xi}_x) = \overline{P}_{(T_x M \oplus \overset{1}{N}_x M)^\perp} (\overline{\nabla}_{X_x} \overset{1}{\xi}) .$$

The vector space $\overset{2}{N}_x M$, generated by the values of $\overset{1}{\sigma}_M$ on a fixed point $x \in M$, is called the *second normal space of* M at x.

If we suppose $\overset{2}{n}(x) = \dim \overset{2}{N}_x M$ is constant, we can consider the vector bundle $\overset{2}{N} M$, and so, step by step, we have a family of vector bundles $\overset{k}{N} M$, with fiber $\overset{k}{N}_x M$ on x. $\overset{k}{N} M$ is called the k-th *normal bundle of* M. As usual we put also $\overset{0}{N} M = TM$ and $\overset{-1}{N} M = 0$.

We point out the fact that the k-th normal space at $x \in M$ is the vector subspace of $T_x\overline{M}$ generated by the values of the form

$$\overset{k-1}{\sigma}_M \colon T_x M \times \overset{k-1}{N}_x M \to (T_x M \oplus \overset{1}{N}_x M \oplus \ldots \oplus \overset{k-1}{N}_x M)^\perp$$

defined by the equality

$$\overset{k-1}{\sigma}_M(X_x, \overset{k-1}{\xi}_x) = \overline{P}_{(T_xM \oplus \overset{1}{N}_xM \oplus \ldots \oplus \overset{k-1}{N}_xM)^\perp}(\overline{\nabla}_{X_x} \overset{k-1}{\xi})$$

where $\overset{k-1}{\xi} \in \Gamma(\overset{k-1}{N} M)$ is an extension of $\overset{k-1}{\xi}_x$.

We call l *the smallest value of k such that* $\overset{k}{\sigma}_M = 0$, and hence the biggest value of k such that $\overset{k}{N}_xM \neq 0$. We observe that for $k \geq l$ we have $\overset{k}{\sigma}_M = 0$, for $k \leq l$ we have $\overset{k}{N}_xM \neq 0$ and that the vector spaces $\overset{k}{N}_xM$ are mutually orthogonal.

It is well known that, for $X_x \in T_xM$ and $\overset{k}{\xi} \in \Gamma(\overset{k}{N}M)$ we have

$$\overline{\nabla}_{X_x}\overset{k}{\xi} \in \overset{k-1}{N}_xM \oplus \overset{k}{N}_xM \oplus \overset{k+1}{N}_xM .$$

Moreover we have $\qquad \overline{P}_{\overset{k+1}{N} M}(\overline{\nabla}_{X_x}\overset{k}{\xi}) = \overset{k}{\sigma}_M(X_x, \overset{k}{\xi}_x) .$

Now we denote by $\overset{k}{\nabla}$ the connection on the vector bundle $\overset{k}{N}M$ induced by $\overline{\nabla}$, that is the connection defined by

$$\overset{k}{\nabla}_{X_x}\overset{k}{\xi} = \overline{P}_{\overset{k}{N}_xM}(\overline{\nabla}_{X_x}\overset{k}{\xi})$$

and by $\overset{k}{A}_M$ the bilinear form on $T_xM \times \overset{k}{N}_xM$ to $\overset{k-1}{N}_xM$, defined by

$$\overset{k}{A}_M(X_x, \overset{k}{\xi}_x) = -\overline{P}_{\overset{k-1}{N}_xM}(\overline{\nabla}_{X_x}\overset{k}{\xi}) .$$

Then we have the Frenet equations of M

$$\overline{\nabla}_{X_x}\overset{k}{\xi} = -\overset{k}{A}_M(X_x, \overset{k}{\xi}_x) + \overset{k}{\nabla}_{X_x}\overset{k}{\xi} + \overset{k}{\sigma}_M(X_x, \overset{k}{\xi}_x) .$$

The derivatives $\nabla\overset{k}{\sigma}_M$ and $\nabla\overset{k}{A}_M$, respectively of $\overset{k}{\sigma}_M$ and of $\overset{k}{A}_M$, are defined by

$$(\nabla_{X_x}\overset{k}{\sigma}_M)(Y_x, \overset{k}{\xi}_x) = \overset{k+1}{\nabla}_{X_x}(\overset{k}{\sigma}_M(Y, \overset{k}{\xi})) - \overset{k}{\sigma}_M(\nabla_{X_x}Y, \overset{k}{\xi}_x) - \overset{k}{\sigma}_M(Y_x, \overset{k}{\nabla}_{X_x}\overset{k}{\xi})$$

$$(\nabla_{X_x}\overset{k}{A}_M)(Y_x, \overset{k}{\xi}_x) = \overset{k+1}{\nabla}_{X_x}(\overset{k}{A}_M(Y, \overset{k}{\xi})) - \overset{k}{A}_M(\nabla_{X_x}Y, \overset{k}{\xi}_x) - \overset{k}{A}_M(Y_x, \overset{k}{\nabla}_{X_x}\overset{k}{\xi})$$

where $Y \in \Gamma(TM)$ is a vector field which extends Y_x and $\overset{k}{\xi} \in \Gamma(\overset{k}{N}M)$ is a section which extends $\overset{k}{\xi}_x$.

We remark explicitly that, for each $x \in M$, $\nabla \overset{k}{\sigma}_M$ and $\nabla \overset{k}{A}_M$ are multilinear forms

$$\nabla \overset{k}{\sigma}_M \colon T_x M \times T_x M \times \overset{k}{N}_x M \to \overset{k+1}{N}_x M \qquad \nabla \overset{k}{A}_M \colon T_x M \times T_x M \times \overset{k}{N}_x M \to \overset{k-1}{N}_x M .$$

For these derivatives the following proposition holds.

Proposition 1. *If M is a nicely curved submanifold of $\overline{M}$, then $\nabla \overset{k}{\sigma}_M = 0$ for any k, if and only if $\nabla \overset{k}{A}_M = 0$ for any k.*

Proof. If $\overline{M} = \boldsymbol{R}^{\overline{m}}$, this proposition is just Lemma 4.1 (iii) of [3]. On the other hand the proof of such a lemma, essentially based on the Frenet equations, is obviously valid also when $\overline{M}$ is any Riemannian manifold. Hence we obtain the thesis.

3 - Minimality of the multihelicoids

Let M be a nicely curved submanifold of $\overline{M}$, satisfying condition (1.1), and let R be a multihelicoid associated to M by the vector bundle $V \to M$, whose fiber on x is V_x, as described in Sec. 1. According to (2.1), each point y of R is given by

$$y = \varphi(|v|)\, x + \psi(|v|)\, v \qquad x \in M, \quad v \in V_x .$$

For each point $y_0 = \varphi(|v_0|)\, x_0 + \psi(|v_0|)\, v_0 \in R$, we can consider two submanifolds of R, $D(v_0)$ and $F(x_0)$, defined only locally. More explicitly

$$D(v_0)\colon\ y = \varphi_0 x + \psi_0 v_x \qquad x \in M \tag{3.1}$$

where $v_x \in V_x$ is the vector obtained by parallel transport of $v_0 = v_{x_0}$ on V, as vector subbundle of $T\overline{M}$, along the geodesic of M joining x_0 with x, $\varphi_0 = \varphi(|v_0|)$, $\psi_0 = \psi(|v_0|)$. The submanifold $F(x_0)$, *the leaf*, is defined by

$$F(x_0)\colon\ y = \varphi(|v|)\, x_0 + \psi(|v|)\, v \qquad v \in V_{x_0} . \tag{3.2}$$

If $\gamma\colon x = x(s)$ is any geodesic of M passing through $x_0 = x(0)$, let $\tilde{\gamma}$ be the curve of $D(v_0)$, passing through y_0, defined by

$$\tilde{x}(s) = \varphi_0\, x(s) + \psi_0\, \xi(s) ,$$

where $\xi(s) = v_{x(s)}$. The generic tangent vector of $\tilde{\gamma}$ will be

$$\tilde{X}(s) = \frac{d\tilde{x}}{ds} = \varphi_0 X(s) + \psi_0 \frac{d\xi}{ds},$$

where $X(s) = \dfrac{dx}{ds}$ is the tangent vector of the geodesic γ.

Now we observe that, if we indicate with $\widehat{P}_{T\overline{M}}$ and $\widehat{P}_{\perp\overline{M}}$ the orthogonal projections respectively on the tangent and on the orthogonal space of the manifold $\overline{M}$, we have

$$\frac{d\xi}{ds} = \widehat{P}_{T\overline{M}}\Big(\frac{d\xi}{ds}\Big) + \widehat{P}_{\perp\overline{M}}\Big(\frac{d\xi}{ds}\Big) = \overline{\nabla}_{X(s)}\xi + \overset{0}{\sigma}_{\overline{M}}((X(s), \xi(s)) = \overline{\nabla}_{X(s)}\xi \qquad (\in T_{x(s)}\overline{M}).$$

In fact, from (2.2), we have $\overset{0}{\sigma}_{\overline{M}}(X(s), \xi(s)) = 0$ trivially if $c = 0$ and because $X(s)$ is orthogonal to $\xi(s) \in V_{x(s)} \subset N_{x(s)}M$ if $c = 1, -1$. Then

$$\tilde{X}(s) = \varphi_0 X(s) + \psi_0 \overline{\nabla}_{X(s)}\xi.$$

Moreover $\xi(s) = v_{x(s)}$ is obtained by parallel transport in V, as vector subbundle of $T\overline{M}$, therefore we have $\overline{\nabla}_{X(s)}\xi \perp V$ and in particular $\overline{\nabla}_{X(s)}\xi \perp V_{x(s)}$. Hence, being also $X(s) \perp V_{x(s)}$, we have

$$\tilde{X}(s) \perp V_{x(s)}. \tag{3.3}$$

Now let δ: $v(t) = v_0 + wt$, $w \in V_x$, be the generic straight line of V_{x_0} passing through v_0. We indicate with $\tilde{\delta}$ the curve of $F(x_0)$ defined by

$$\tilde{y}(t) = \varphi(|v(t)|)\, x_0 + \psi(|v(t)|)\, v(t).$$

The tangent vector of $\tilde{\delta}$ will be

$$\tilde{Y}(t) = \frac{d\tilde{y}}{dt} = \Phi'(t)\, x_0 + \Psi'(t)\, v(t) + \Psi(t)\, w, \tag{3.4}$$

where $\Phi(t) = \varphi(|v(t)|)$ and $\Psi(t) = \psi(|v(t)|)$.

From (3.4) we have immediately $\tilde{Y}(t) \in V_{x_0}$ if $c = 0$ and $\tilde{Y}(t) \in V_{x_0} \oplus \langle x_0 \rangle$ if $c = 1, -1$. Therefore, being $\tilde{X}(s) \in T_{\tilde{x}(s)}D(v_0) \subset T_{\tilde{x}(s)}R \subset T_{\tilde{x}(s)}\overline{M}$, from (3.3) we have $\tilde{X}(0) \perp \tilde{Y}(t)$. Consequently it results

$$T_{y_0}D(v_0) \perp T_{y_0}F(x_0). \tag{3.5}$$

We observe here, explicitly, that we also have

$$T_{y_0}R = T_{y_0}D(v_0) \oplus T_{y_0}F(x_0)\,. \tag{3.6}$$

Moreover the following equality holds

$$T_{y_0}F(x_0) \oplus \langle y_0 \rangle = V_{x_0} \oplus \langle x_0 \rangle. \tag{3.7}$$

Proof of (3.7). From the expression of $y_0 = \varphi_0 x_0 + \psi_0 v_0$ and from that of the generic tangent vector of $F(x_0)$ we immediately have

$$T_{y_0}F(x_0) \oplus \langle y_0 \rangle \subset V_{x_0} \oplus \langle x_0 \rangle\,.$$

In order to prove the reversed inclusion we consider a vector u of $V_{x_0} \oplus \langle x_0 \rangle$. We can write $u = v + \lambda x_0$, where $v \in V_{x_0}$ and $\lambda \in \boldsymbol{R}$. Recalling (3.4), we have to prove that there exists $\rho \in \boldsymbol{R}$ and $w \in V_{x_0}$ such that

$$v + \lambda x_0 = \Phi'(0)\, x_0 + \Psi'(0)\, v_0 + \Psi(0)\, w + \rho y_0\,.$$

But the second member of the previous equality is equal to

$$\varphi'(|v|)\frac{\mathrm{d}}{dt}\,|v_0+wt|\,\Big|_{t=0} x_0 + \psi'(|v_0|)\frac{\mathrm{d}}{\mathrm{d}t}\,|v_0+wt|\,\Big|_{t=0} v_0 + \psi(|v_0|)\, w + \rho\varphi_0\, x_0 + \rho\psi_0\, v_0$$

$$= \varphi_0'\,\frac{2g(v_0\,,\,w)}{|v_0|}\,x_0 + \psi_0'\,\frac{2g(v_0\,,\,w)}{|v_0|}\,v_0 + \rho\varphi_0 x_0 + \rho\psi_0 v_0 + \psi_0 w$$

$$= \Big(\frac{2\varphi_0' g(v_0\,,\,w)}{|v_0|} + \rho\varphi_0\Big)\, x_0 + \Big(\frac{2\psi_0' g(v_0\,,\,w)}{|v_0|} + \rho\psi_0\Big)\, v_0 + \psi_0 w$$

where we have put $\varphi_0' = \varphi'(0)$, $\psi_0' = \psi'(0)$.

Since $\varphi_0 \neq 0$ for $|v_0|$ sufficiently small, the condition $\dfrac{2\varphi_0' g(v_0\,,\,w)}{|v_0|} + \rho\varphi_0 = \lambda$ gives $\rho = \dfrac{\lambda|v_0| - 2\varphi_0' g(v_0\,,\,w)}{\varphi_0|v_0|}$ and the equation in w

$$\Big(\frac{2\psi_0' g(v_0\,,\,w)}{|v_0|} + \frac{\psi_0(\lambda|v_0| - 2\varphi_0' g(v_0\,,\,w))}{\varphi_0|v_0|}\Big)\, v_0 + \psi_0 w = v\,,$$

is solved by $w = \dfrac{v - av_0}{\psi_0}$, where

$$a = \frac{2\psi_0'\varphi_0 g(v_0\,,\,v) - 2\varphi_0'\psi_0 g(v_0\,,\,v) + \psi_0^2|v_0|\lambda}{2\psi_0'\varphi_0|v_0|^2 - 2\varphi_0'\psi_0|v_0|^2 + \varphi_0\psi_0|v_0|}\,.$$

We come now to our main result.

Theorem. *A multihelicoid R, associated as in Sec.* 1 *to a nicely curved submanifold M, satisfying condition* (1.1), *of a standard space $\overline{M}$ of constant curvature is minimal.*

Proof. Because of (3.5) and (3.6) we can prove that the *mean curvature vector H* of R at $y_0 \in R$ is zero. It suffices to compute the trace of the second fundamental form $\overset{0}{\sigma}_R$ on R (in $\overline{M}$) at y_0, by taking orthonormal bases of $T_{y_0}D(v_0)$ and of $T_{y_0}F(x_0)$.

Moreover we prove that

$$\overset{0}{\sigma}_R(\widetilde{X}(0), \widetilde{X}(0)) = 0 \qquad \overset{0}{\sigma}_R(\widetilde{Y}(0), \widetilde{Y}(0)) = 0\,, \tag{3.8}$$

for each $\widetilde{X}(0) \in T_{y_0}D(v_0)$ and $\widetilde{Y}(0) \in T_{y_0}F(x_0)$, so H is at once zero.

In order to prove the first of (3.8), we recall that

$$\overset{0}{\sigma}_R(\widetilde{X}(0), \widetilde{X}(0)) = \overline{P}_{\perp R}(\overline{\nabla}_{\widetilde{X}(0)}\widetilde{X})\,, \tag{3.9}$$

then we compute explicitly $\overline{P}_{\perp R}(\overline{\nabla}_{\widetilde{X}(0)}\widetilde{X})$. We have

$$\overline{P}_{\perp R}(\overline{\nabla}_{\widetilde{X}(0)}\widetilde{X}) = \overline{P}_{\perp R}(\widehat{P}_{T\overline{M}}(\widehat{\nabla}_{\widetilde{X}(0)}\widetilde{X})) = \overline{P}_{\perp R}(\widehat{P}_{T\overline{M}}(\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}s}\,|_{s=0}))$$

$$= \overline{P}_{\perp R}(\widehat{P}_{T\overline{M}}(\varphi_0 \frac{\mathrm{d}X}{\mathrm{d}s}\,|_{s=0} + \psi_0 \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\xi}{\mathrm{d}s}\,|_{s=0}))$$

$$= \overline{P}_{\perp R}(\widehat{P}_{T\overline{M}}(\varphi_0(\overline{\nabla}_{X(0)}X + \overset{0}{\sigma}_{\overline{M}}(X(0), X(0)) + \psi_0 \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\sum_{0}^{\bar{l}}{}_h \overset{2h+1}{\xi}\,|_{s=0}))$$

where $\overset{2h+1}{\xi}(s) \in \overset{2h+1}{N}_{x(s)}M$.

From the fact that $\xi(s) = v_{x(s)} = \sum_0^{\bar{l}}{}_h \overset{2h+1}{\xi}(s)$ is parallel in $V = \bigoplus_0^{\bar{l}}{}_h \overset{2h+1}{N} M$, considered as vector subbundle of $T\overline{M}$, we remark that $\overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} = 0$, i.e. $\overset{2h+1}{\xi}(s)$ *is parallel, in* $\overset{2k+1}{N} M$, *along* γ.

In fact $\overset{V}{\nabla}_{X(s)}\xi = 0$ implies $\overline{\nabla}_{X(s)}\xi \in V^\perp$, being $V^\perp$ the orthogonal bundle of V (in $T\overline{M}$). Furthermore, using the Frenet equations, we obtain

$$\overline{\nabla}_{X(s)}\xi = \overline{\nabla}_{X(s)}\sum_0^{\bar{l}}{}_h \overset{2h+1}{\xi} = \sum_0^{\bar{l}}{}_h \overline{\nabla}_{X(s)} \overset{2h+1}{\xi}$$

$$= \sum_0^{\bar{l}}{}_h[-\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} + \overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))]\,.$$

So, being $-\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) \in \overset{2h}{N}_{x(s)}M \subset V^\perp$, $\overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s)) \in \overset{2h+1}{N}_{x(s)}M \subset V^\perp$ and $\overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} \in \overset{2h+1}{N}_{x(s)}M \subset V$, it must be $\overset{2h+1}{\nabla}_{X(s)} \overset{2h+1}{\xi} = 0$.

Because of such a property of $\overset{2h+1}{\xi}$, using the same argument that leads to $\frac{d}{ds}\xi = \overline{\nabla}_{X(s)}\xi$, we have $\frac{d}{ds}\overset{2h+1}{\xi} = \overline{\nabla}_{X(s)}\overset{2h+1}{\xi}$.

Then applying the Frenet equations, we have

$$\frac{d}{ds}\overset{2h+1}{\xi} = -\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \overset{2h+1}{\nabla}_{X(s)}\overset{2h+1}{\xi} + \overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))$$

$$= -\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s)).$$

So we have

$$\frac{d}{ds}\frac{d\xi}{ds} = \sum_0^{l}{}_h\Big[-\frac{d}{ds}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \frac{d}{ds}\overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))\Big].$$

Now we observe that, being

$$\frac{d}{ds}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s))\big|_{s=0} = \overline{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \overset{0}{\sigma}_{\overline{M}}(X(0), \overset{2h+1}{A}_M(X(0), \overset{2h+1}{\xi}(0)))$$

for each h, when if $c = 0$, and for $h \geqslant 1$, when $c = 1, -1$, we have

$$\frac{d}{ds}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s))\big|_{s=0} = \overline{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)).$$

In fact, if $c = 0$, it is $\overset{0}{\sigma}_{\overline{M}} = 0$ and, if $c = 1, -1$, being $X(0) \in T_{x_0}M$ and $\overset{2h+1}{A}_M(X(0), \overset{2h+1}{\xi}(0)) \in \overset{2h}{N}_{x_0}M$, it is $X(0) \perp \overset{2h+1}{A}_M(X(0), \overset{2h+1}{\xi}(0))$ for $h \geqslant 1$ and hence

$$\overset{0}{\sigma}_{\overline{M}}(X(0), \overset{2h+1}{A}_M(X(0), \overset{2h+1}{\xi}(0))) = -cg(X(0), \overset{2h+1}{A}_M(X(0), \overset{2h+1}{\xi}(0)))x_0 = 0.$$

Analogously it results $\frac{d}{ds}\overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))\big|_{s=0} = \overline{\nabla}_{X(0)}\overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))$ for each $h > 0$.

Then it is

$$\frac{d}{ds}\frac{d\xi}{ds}\big|_{s=0} = \sum_0^{2h+1}{}_h\Big[-\overline{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s), \overset{2h+1}{\xi}(s)) + \overline{\nabla}_{X(0)}\overset{2h+1}{\sigma}_M(X(s), \overset{2h+1}{\xi}(s))\Big]$$

$$- \overset{0}{\sigma}_{\overline{M}}(X(0), \overset{1}{A}_M(X(0), \overset{1}{\xi}(0))).$$

Applying again the Frenet equations, we have

$$\frac{d}{ds}\frac{d\xi}{ds}\Big|_{s=0} = -\overset{0}{\sigma}_{\overline{M}}(X(0),\overset{1}{A}_M(X(0),\overset{1}{\xi}(0)))$$

$$+\sum_0^{\bar{l}}{}_h\,(-\overset{2h}{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s),\overset{2h+1}{\xi}(s)) + \overset{2h+2}{\nabla}_{X(0)}\overset{2h+1}{\sigma}_M(X(s),\overset{2h+1}{\xi}(s))$$

$$+(\overset{2h}{A}_M(X(0),\overset{2h+1}{A}_M(X(0),\overset{2h+1}{\xi}(0))) - \overset{2h}{\sigma}_M(X(0),\overset{2h+1}{A}_M(X(0),\overset{2h+1}{\xi}(0)))$$

$$-\overset{2h+2}{A}_M(X(0),\overset{2h+1}{\sigma}_M(X(0),\overset{2h+1}{\xi}(0))) + \overset{2h+2}{\sigma}_M(X(0),\overset{2h+1}{\sigma}_M(X(0),\overset{2h+1}{\xi}(0)))).$$

Now we use the condition $\nabla\overset{k}{A}_M = 0$ for each k (Proposition 1). Then we have

$$0 = \overset{2h}{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s),\overset{2h+1}{\xi}(s)) - \overset{2h+1}{A}_M(\nabla_{X(0)}X,\overset{2h+1}{\xi}(0)) - \overset{2h+1}{A}_M(X(0),\overset{2h+1}{\nabla}_M\overset{2h+1}{\xi}).$$

But it is $\nabla_{X(0)}X = 0$, because γ is a geodesic of M, and $\overset{2h+1}{\nabla}_{X(0)}\overset{2h+1}{\xi} = 0$, as above observed, so we also have $\overset{2h}{\nabla}_{X(0)}\overset{2h+1}{A}_M(X(s),\overset{2h+1}{\xi}(s)) = 0$.

Analogously, applying the hypothesis $\nabla\overset{k}{\sigma} = 0$, for each k, we have $\overset{2h+2}{\nabla}_M\overset{2h+1}{\sigma}(X(s),\overset{2h+1}{\xi}(s)) = 0$.

Finally we have $\quad \dfrac{d}{ds}\dfrac{d\xi}{ds}\Big|_{s=0} = cg(X(0),\overset{1}{A}_M(X(0),\overset{1}{\xi}(0)))\,x_0$

$$+\sum_0^{\bar{l}}{}_h\,(\overset{2h}{A}_M(X(0),\overset{2h+1}{A}_M(X(0),\overset{2h+1}{\xi}(0))) - \overset{2h}{\sigma}_M(X(0),\overset{2h+1}{A}_M(X(0),\overset{2h+1}{\xi}(0)))$$

$$-\overset{2h+2}{A}_M(X(0),\overset{2h+1}{\sigma}_M(X(0),\overset{2h+1}{\xi}(0))) + \overset{2h+2}{\sigma}_M(X(0),\overset{2h+1}{\sigma}_M(X(0),\overset{2h+1}{\xi}(0)))).$$

So, being $\overset{-1}{N}_{x_0}M = 0$, we have

$$\frac{d}{ds}\frac{d\xi}{ds}\Big|_{s=0} \in \bigoplus_0^{\bar{l}}{}_h\,\overset{2h+1}{N}_{x_0}M \oplus \langle x_0\rangle = V_{x_0}\oplus\langle x_0\rangle.$$

Then since we have

$$\widehat{\nabla}_{\tilde{X}(0)}\widetilde{X} = \varphi_0(\nabla_{X(0)}X + \overset{0}{\sigma}_M(X(0),X(0)) + \overset{0}{\sigma}_{\overline{M}}(X(0),X(0))) + \psi_0\frac{d}{ds}\frac{d\xi}{ds}\Big|_{s=0}$$

$$= \varphi_0\overset{0}{\sigma}_M(X(0),X(0)) - \varphi_0 cg(X(0),X(0))\,x_0 + \psi_0\frac{d}{ds}\frac{d\xi}{ds}\Big|_{s=0}$$

and $\varphi_0 \overset{0}{\sigma}_M (X(0), X(0)) \in \overset{1}{N}_{x_0} M$, we have

$$\widehat{\nabla}_{\widetilde{X}(0)} \widetilde{X} \in V_{x_0} \oplus \langle x_0 \rangle = T_{y_0} F(x_0) \oplus \langle y_0 \rangle \subset T_{y_0} R \oplus \langle y_0 \rangle$$

being more precisely $\widehat{\nabla}_{\widetilde{X}(0)} \widetilde{X} \in V_{x_0} = T_{y_0} F(x_0) \subset T_{y_0} R$ if $c = 0$. Moreover, if $c = 1, -1$, then y_0 is orthogonal to $T_{y_0} \overline{M}$, so

$$\overline{P}_{\perp R} (\overline{\nabla}_{\widetilde{X}(0)} \widetilde{X}) = \overline{P}_{\perp R} (\widehat{P}_{T\overline{M}} (\widehat{\nabla}_{\widetilde{X}(0)} \widetilde{X})) = 0 .$$

Hence using (3.9) we have the first of (3.8).

Now we note that $\overline{P}_{\perp R} ((\overline{\nabla}_{\widetilde{Y}(0)} \widetilde{Y}) = 0$ because the leaf $F(x_0)$ is a totally geodesic submanifold of $\overline{M}$ and then $\overline{\nabla}_{\widetilde{Y}(0)} \widetilde{Y} \in T_{y_0} F(x_0) \subset T_{y_0} R$.

Let M, $\overline{M}$, R satisfy the assumptions of the theorem we have proved. If γ is a geodesic of M, we are interested to the submanifold

$$R_\gamma = \{\exp_x v\}_{x \in \gamma,\, v \in V_x} \cap R$$

obtained by considering the restriction of R to γ.

For R_γ the following proposition holds

Proposition 2. *R_γ is a minimal ruled submanifold of $\overline{M}$. Moreover, locally, R is a part of a generalized helicoid.*

Proof. R_γ is foliated by the leaves of R obtained for $x \in \gamma$. These leaves are codimension 1 totally geodesic submanifolds of $\overline{M}$, so R_γ is just a ruled submanifold of $\overline{M}$ as defined in [1].

In order to prove the minimality of R_γ we remark that, being γ a geodesic of M, if we above replace $D(v_0)$ with $\widetilde{\gamma}$, all the results found for the submanifolds $D(v_0)$ and $F(x_0)$ of R are valid also for $\widetilde{\gamma}$ and for $F(x_0)$, considered as submanifolds of R_γ.

In particular it results $T_{y_0} \widetilde{\gamma} \perp T_{y_0} F(x_0)$ and we have now

$$T_{y_0} R_\gamma = T_{y_0} \widetilde{\gamma} \oplus T_{y_0} F(x_0) .$$

So, if we prove that $\overset{0}{\sigma}_{R_\gamma} (\widetilde{X}(0), \widetilde{X}(0)) = \overline{P}_{\perp R_\gamma} (\widehat{P}_{T\overline{M}} (\widehat{\nabla}_{\widetilde{X}(0)} \widetilde{X})) = 0$ and $\overset{0}{\sigma}_{R_\gamma} (\widetilde{Y}(0), \widetilde{Y}(0)) = \overline{P}_{\perp R_\gamma} (\widehat{P}_{T\overline{M}} (\widehat{\nabla}_{\widetilde{Y}(0)} \widetilde{Y})) = 0$, for $\widetilde{X}(0) \in T_{y_0} \widetilde{\gamma}$ and $\widetilde{Y}(0) \in T_{y_0} F(x_0)$, we have that the mean curvature vector of R_γ at y_0 is equal to zero, that is R_γ is *minimal*.

But we already proved that $\widehat{P}_{T\overline{M}}(\widehat{\nabla}_{\widetilde{X}(0)}\widetilde{X})$ and $\widehat{P}_{T\overline{M}}(\widehat{\nabla}_{\widetilde{Y}(0)}\widetilde{Y})$ are vectors of $T_{y_0}F(x_0)$. Since we have $T_{y_0}F(x_0) \subset T_{y_0}R_\gamma \subset T_{y_0}\overline{M}$, we get $\overline{P}_{\perp R_\gamma}(\widehat{P}_{T\overline{M}}(\widehat{\nabla}_{\widetilde{X}(0)}\widetilde{X})) = 0$ and $\overline{P}_{\perp R_\gamma}(\widehat{P}_{T\overline{M}}(\widehat{\nabla}_{\widetilde{Y}(0)}\widetilde{Y})) = 0$ as desired.

The last assertion of Proposition 2 is an immediate consequence of Theorem 4.1 of [1] applied to R_γ.

References

[1] J. M. BARBOSA, M. DAJCZER and L. P. JORGE, *Minimal ruled submanifolds in spaces of constant curvature*, Indiana Univ. Math. J. 33 (1984), 531-547.

[2] A. CARFAGNA D'ANDREA, R. MAZZOCCO and G. ROMANI, *Some characterizations of 2-symmetric submanifolds of constant curvature* (to appear).

[3] I. CATTANEO GASPARINI and G. ROMANI, *Normal and osculating maps for submanifolds of* $\boldsymbol{R}^N$, Proc. Roy. Soc. Edimburgh **114**A (1990), 39-55.

[4] O. KOWALSKI and I. KÜLICH, *Generalized symmetric submanifolds of euclidean spaces*, Math. Ann. 277 (1987), 67-78.

[5] M. SPIVAK, *A comprehensive introduction to differential geometry*, Publish or perish, Boston 1975.

Summary

We begin from a nicely curved submanifold M of a standard space $\overline{M}$. *We suppose the derivatives of all fundamental forms on M be equal to zero and we obtain a minimal submanifold R of* $\overline{M}$, *foliated by totally geodesic submanifolds of* $\overline{M}$. *Since, if* $\dim M = 1$, *M is just a curve with constant curvatures and R becomes the helicoid, associated to the curvature in the sense of* [1], *we call R a multihelicoid.*

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