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Asymptotic and oscillatory behaviour of solutions of a second order nonlinear neutral delay difference equation (**)

1 - Introduction

Consider the *neutral difference equation*

$$\Delta^2(y_n + p_n y_{n-k}) - q_n f(y_{n-l}) = 0 \qquad n = 0, 1, 2, \dots \tag{1}$$

where $\{p_n\}$, $\{q_n\}$ are real numbers, k and l are nonnegative integers and Δ denote the *forward difference operator* $\Delta x_n = x_{n+1} - x_n$. The following conditions will be assumed without further mention

c_1 $\quad q_n \geq 0$ for $n \geq n_0 \geq 0$

c_2 $\quad f: \boldsymbol{R} \to \boldsymbol{R}$ continuous and $u f(u) > 0$ for $u \neq 0$.

Let $m = \max\{k, l\}$. Then by a *solution of* (1), we mean a sequence $\{y_n\}$ of real numbers, which is defined for $n \geq -m$ and which satisfies (1) for $n = 0, 1, 2, \dots$. A solution $\{y_n\}$ of (1) is said to be *nonoscillatory* if the terms y_n are either eventually positive or eventually negative. Otherwise the solution is called *oscillatory*.

In this paper we study asymptotic properties of nonoscillatory solutions of (1) and obtain sufficient conditions for all bounded solutions of (1) to be oscillatory. The results in this paper have been motivated by the results in [1], [2]. For general background on difference equations see [3].

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Further, when (9) holds $z_n^1 = y_n^1 - \frac{1}{2} e^{ak} y_{n-k}^1 = e^{ak}$ satisfies (2) and $z_n^2 = y_n^2 - \frac{1}{2} e^{ak} y_{n-k}^2 = - e^{ak}$ satisfies (4); whereas z_n^1 satisfies (3) and z_n^2 satisfies (5), when (10) holds.

Now we study the behaviour of the solutions of (1).

Theorem 2. *Let* c_3 *and* c_4 *hold. If there exist a constant* B *such that* $B \leqslant p_n \leqslant -1$, *then every nonscillatory solution* $\{y_n\}$ *of* (1) *satisfies* $|y_n| \to \infty$ *as* $n \to \infty$.

Proof. If $\{y_n\}$ is an eventually positive solution of (1) such that y_n does not tend to ∞ as $n \to \infty$, then (2) cannot hold since $y_n \geqslant z_n$. Thus by Lemma 1, (3) holds. Further from the proof of (3) we have (6) holding. But

$$0 < z_n = y_n + p_n y_{n-k} \leqslant y_n - y_{n-k}$$

so $y_n > y_{n-k}$ which contradicts (6). This completes the proof when $\{y_n\}$ is eventually positive. The proof is similar when $\{y_n\}$ is negative.

Remark. Equation E_1 also illustrates Theorem 2. A necessary condition that the assumption about p_n be satisfied for E_1 is that $a > 0$, which implies that $y_n^1 \to \infty$ and $y_n^2 \to -\infty$ as $n \to \infty$.

Theorem 3. *Assume* $-1 < A \leqslant p_n \leqslant 0$. *Assume then that* f *is increasing and sublinear, in the sense that for every constant* $\alpha > 0$ *we have* $\int_0^{\pm\alpha} \frac{du}{f(u)} < \infty$. *Assume also that for* $l \geqslant 1$ *we have* $\sum_{s=n_0}^{\infty} \left(\sum_{\eta=s-l}^{s} q_\eta \right) = \infty$. *Then every nonoscillatory solution* $\{y_n\}$ *of* (1) *satisfies either* $|y_n| \to \infty$ *or* $y_n \to 0$ *as* $n \to \infty$.

Proof. Suppose that the solution $\{y_n\}$ of (1) is eventually positive and that y_n does not tend to zero or increase without bound as $n \to \infty$. Since $\Delta^2 z_n = q_n f(y_{n-l}) \geqslant 0$, $\{\Delta z_n\}$ is increasing and $\{z_n\}$ is monotonic. Now if $y_n + p_n y_{n-k} = z_n \leqslant 0$ for $n \geqslant n_1$, then the assumption concerning p_n implies $y_n \leqslant -A y_{n-k}$ so $y_{n+k} \leqslant -A y_n$. It then follows by induction that for all $n \geqslant n_1$ we have $y_{n+mk} \leqslant (-A)^m y_n$ for every positive integer m. But the last inequality implies that $y_n \to 0$ as $m \to \infty$, a contradiction to our assumption. Thus we have $z_n > 0$ for $n \geqslant n_1$. Note also that, if there exists $n_2 \geqslant n_1$ such that $\Delta z_{n_2} \geqslant 0$, then there exists $n_3 \geqslant n_2$ such that $\Delta z_n \geqslant \Delta z_{n_3} > 0$ for $n \geqslant n_3$, which implies that $y_n \geqslant z_n \to \infty$ as $n \to \infty$, again a contradiction to our assumption.

Therefore we have $z_n > 0$ and $\Delta z_n < 0$ for $n \geqslant n_1$. Summing (1) we obtain

$$-\Delta z_{n-l} > \Delta z_{n+1} - \Delta z_{n-l} = \sum_{s=n-l}^{n} q_s f(y_{s-l}).$$

Since f is an increasing function, we have

$$f(z_{n-l}) \leqslant f(z_{s-l}) \leqslant f(y_{s-l})$$

for $n \geqslant s-l$, so $$-\Delta z_{n-l} \geqslant f(z_{n-l}) \sum_{s=n-l}^{n} q_s$$ or

$$\frac{-\Delta z_{n-l}}{f(z_{n-l})} \geqslant \sum_{s=n-l}^{n} q_s . \tag{11}$$

Let us notice that

$$\frac{z_{n-l} - z_{n+1-l}}{f(z_{n-l})} = \int_{z_{n+1-l}}^{z_{n-l}} \frac{ds}{f(z_{n-l})} \leqslant \int_{z_{n+1-l}}^{z_{n-l}} \frac{ds}{f(s)} .$$

Using the above inequality in (11) and summing the resulting inequality from N to n leads to

$$\int_{z_{n+1-l}}^{z_{N-l}} \frac{dx}{f(x)} > \sum_{s=N}^{n} \Big(\sum_{\eta=s-l}^{s} q_\eta \Big) \to \infty$$

as $n \to \infty$ by the last assumption of our theorem and this contradicts the sublinearity of f. This completes the proof for $\{y_n\}$ eventually positive. The argument when $\{y_n\}$ is eventually negative is similar.

The following result is an immediate consequence of Theorem 3.

Corollary. *Under the assumptions of Theorem* 3 *any bounded solution of* (1) *is either oscillatory or converges to zero as* $n \to \infty$.

In the next two theorems we discuss the behaviour of the unbounded solutions of (1), when p_n satisfies either the inequality of Theorem 3 or the inequality

$$0 \leqslant p_n \leqslant C < 1 \tag{12}$$

where C is a constant.

Theorem 4. *If p_n satisfies the inequality of Theorem 3, then every unbounded solution $\{y_n\}$ of* (1) *is either oscillatory or satisfies* $|y_n| \to \infty$ *as* $n \to \infty$.

Proof. Let $\{y_n\}$ be an unbounded solution of (1) that is eventually positive and let $n_1 \geq n_0$ be such that $y_{n-k-l} > 0$ for $n \geq n_1$. Since $\Delta^2 z_n \geq 0$ for $n \geq n_1$, then $\{\Delta z_n\}$ is increasing and $\{z_n\}$ is monotonic. It then follows that $z_n > 0$ for $n \geq n_2 \geq n_1$. Otherwise there exists $n_3 \geq n_2$ such that $y_n + p_n y_{n-k} = z_n \leq 0$ for $n \geq n_3$ and our assumption implies $y_n \leq -Ay_{n-k} \leq y_{n-k}$. This implies that $\{y_n\}$ is bounded, a contradiction.

Now $z_n > 0$ for $n \geq n_2$ and further $\{\Delta z_n)$ is eventually positive. Otherwise $\{z_n)$ is decreasing and hence bounded from above, say $0 < z_n < K$ for some constant K.

Therefore $y_n = z_n - p_n y_{n-k} \leq K - A y_{n-k}$. Since $\{y_n\}$ is unbounded there is an increasing sequence $\{n_m\}$ such that $n_m \to \infty$ and $y_{n_m} \to \infty$ as $m \to \infty$ and $y_{n_m} \geq \max_{n_2 \leq n \leq n_m} y_n$. Hence we have

$$y_{n_m} \leq K - A y_{n_m - k} \leq K - A y_{n_m}.$$

So $(1 + A) y_{n_m} \leq K$ for all m, which is impossible in view of the first assumption of Theorem 3.

Finally observe that $\{\Delta z_n\}$ increasing and eventually positive implies that $z_n \to \infty$ as $n \to \infty$ and hence $y_n \to \infty$ as $n \to \infty$ since $y_n \geq z_n$. This completes the proof when $\{y_n\}$ is eventually positive. The proof for $\{y_n\}$ eventually negative is similar.

Remark. If $0 < ak < \log 2$, then E_1 satisfies all the conditions of Theorem 4 and has the nonoscillatory solutions $y_n = \pm 2 e^{an}$ which all satisfy $|y_n| \to \infty$ as $n \to \infty$.

Theorem 5. *If* (12) *holds, then every unbounded solution $\{y_n\}$ of* (1) *is either oscillatory or satisfies* $|y_n| \to \infty$ *as* $n \to \infty$.

Proof. Let $\{y_n\}$ be an eventually positive solution of (1), say $y_{n-k-l} > 0$ for $n \geq n_1 \geq n_0$. Then $z_n = y_n + p_n y_{n-k} > 0$ and $\Delta^2 z_n \geq 0$ for $n \geq n_1$, so $\{\Delta z_n\}$ is increasing and $\{z_n\}$ is monotonic.

If $\{\Delta z_n\}$ is not eventually positive, then $\Delta z_n < 0$ for $n \geq n_1$. Then $\{z_n\}$ is bounded from above, which contradicts the hypothesis that $\{y_n\}$ is unbounded. Hence we conclude that $\{\Delta z_n\}$ is eventually positive, which together with the fact that $\{\Delta z_n\}$ is increasing implies that $z_n \to \infty$ as $n \to \infty$. Since

$$z_n \leqslant y_n + C y_{n-k} \leqslant y_n + C z_{n-k} \leqslant C z_n + y_n$$

we have $(1 - C) z_n \leqslant y_n$, which, in view of (12), implies $y_n \to \infty$ as $n \to \infty$. The proof when $\{y_n\}$ is eventually negative is similar.

Remark. The equation

$$\mathrm{E}_2 \qquad \Delta^2 [y_n + \frac{1}{2} y_{n-k}] - \frac{(e^a - 1)^2 e^{3al - ak}(1 + 2e^{ak})}{2 e^{2an}} y_{n-l}^3 = 0$$

satisfies all conditions of Theorem 5 for any positive integers k and l and constant $a \neq 0$. Note that $y_n = e^{an}$ is a nonoscillatory solution of E_2 satisfying the conclusion of Theorem 5 for $a > 0$. It is also interesting to observe that $\{y_n\}$ is a bounded solution of E_2 for $a < 0$.

Finally we give sufficient conditions to ensure that all bounded solutions of (1) are oscillatory.

Theorem 6. *If f is an increasing function, $l \geqslant k$, $\sum^{\infty} (\sum_{\eta = s - k}^{\infty} q_\eta) = \infty$, there are constants D and E such that $D \leqslant p_n \leqslant E < -1$ and we have*

$$(13) \qquad \int_{\alpha}^{\infty} \frac{du}{f(u)} < \infty, \qquad \int_{-\alpha}^{-\infty} \frac{du}{f(u)} < \infty$$

for every positive constant α, then all bounded solutions of (1) *are oscillatory.*

Proof. For the sake of contradiction assume (1) has a bounded solution $\{y_n\}$ such that $|y_{n-k-l}| > 0$ for $n \geqslant n_1 \geqslant n_0$. If $y_{n-k-l} > 0$ for $n \geqslant n_1$ then $\Delta^2 z_n > 0$. Hence $\{\Delta z_n\}$ is increasing and $\{z_n\}$ is monotonic. We first show that $\{z_n)$ is eventually negative. If there exist $n_2 \geqslant n_1$ such that $z_{n_2} > 0$, then by the inequality concerning p_n we get

$$y_{n_2} = z_{n_2} - p_{n_2} y_{n_2 - k} \geqslant -E y_{n_2 - k} .$$

If follows then by induction that $y_{n_2 + mk} \geqslant (-E)^m y_{n_2}$, which implies $y_{n_2 + mk} \to \infty$ as $m \to \infty$, a contradiction. Therefore we conclude that $z_{n-k-l} < 0$ for $n \geqslant N \geqslant n_1 - k - l$, which implies that $\Delta z_{n-k-l} < 0$ for $n \geqslant N$. We then have $0 > z_n > D y_{n-k}$, from which it follows that $y_n > z_{n+k} D^{-1} > 0$. Since f is increasing we see that $\Delta^2 z_n \geqslant q_n f(z_{n+k-l} D^{-1})$ and summing we obtain

$$-\Delta z_{n-k} \geqslant \Delta z_{M+1} - \Delta z_{n-k} \geqslant \sum_{s=n-k}^{M} q_s f(z_{s+k-l} D^{-1}). \tag{14}$$

Observe

$$\sum_{s=n-k}^{M} q_s f(z_{s+k-l} D^{-1}) \geqslant f(z_{n-l} D^{-1}) \sum_{s=n-k}^{M} q_s \geqslant f(z_{n+1-l} D^{-1}) \sum_{s=n-k}^{M} q_s$$

and since $l \geqslant k$, $-\Delta z_{n-l} \geqslant -\Delta z_{n-k}$. This together with (14) implies

$$\frac{-\Delta z_{n-1}}{f(z_{n+1-l} D^{-1})} \geqslant \sum_{s=n-k}^{\infty} q_s . \tag{15}$$

As in Theorem 3, from (15) we have

$$-D \int_{z_{n-1} D^{-1}}^{z_{n+1-l} D^{-1}} \frac{dx}{f(x)} \geqslant \sum_{s=n-k}^{\infty} q_s .$$

Summing the last inequality from N to n, we are led to

$$-D \int_{z_{N-1} D^{-1}}^{z_{n+1-l} D^{-1}} \frac{dx}{f(x)} \geqslant \sum_{s=N}^{n} \Big(\sum_{\eta=s-k}^{\infty} q_\eta \Big)$$

which in view of (13), contradicts an assumption of our theorem and the proof is complete.

Theorem 7. *Under the assumptions of Theorem* 2 *all bounded solutions of* (1) *are oscillatory.*

Proof. The proof follows from Theorem 2.

Remark. The equation

$$\Delta^2 (y_n - (9 e^{-2}) y_{n-2}) + \frac{8(e+1)^2}{e^9} e^{2n} y_{n-3}^3 = 0 \tag{E_3}$$

satisfies all conditions of Theorem 6 and 7, and hence all bounded solutions of E_3 are oscillatory. Note that $y_n = e^{-n} \cos n\pi$ is a bounded oscillatory solution of E_3.

References

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[2] J. R. GRAEF, M. K. GRAMMATIKOPOULOS and P. W. SPIKES, *On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equations,* J. Math. Anal. Appl. **156** (1991), 23-39.

[3] V. LAKSHMIKANTHAM and D. TRIGIANTE, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, New York 1988.

Summary

Asymptotic and oscillatory behaviour of solutions of a second order nonlinear neutral delay difference equation is studied. Some illustrative examples are also included.

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