Riv. Mat. Univ. Parma (4) **17*** (1991), 67-89

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About M. Gromov's conjectures on minimal volume and minimal entropy (**)

1 - Statement of the problems

In this paper M is a *compact smooth manifold* whose *dimension*, denoted n, is supposed to be at least 2.

a Rigidity of flow conjugacy and isospectral problems

To each compact riemannian manifold (M, g) (viewed as the configuration space of mechanics), one associates the *dynamical system* given by the corresponding unit tangent bundle

$$U_g(M) = \{v: v \in TM \text{ and } g(v, v) = 1\}$$

and the geodesic flow ${}^g\phi_t$, which is given by

$${}^g\phi_t(v) = c_v'(t)$$

where c_v is the geodesic whose speed at time t is $c_v'(t)$ and wose initial data is $c_v'(0) = v$. So the geometry gives the dynamics.

We are interested in the inverse problem: does the knowledge of the dynamics (up to coniugacy) give the geometry (up to isometry)? The main problem comes from the fact that, given two metrics g_1 and g_2 such that the total spaces

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(**) Received September 21, 1992. AMS classification 58 E 35. The results, I present here, have been obtained by the author on a joint work with G. Besson and G. Courtois.

of their unit tangent bundles are globally identical as sets and such that their flows coincide, we cannot deduce from these assumptions that the unit tangent fibrations are the same maps. (In other words, a *flow conjugacy* is a homeomorphism φ from the first unit tangent bundle $U_{g_1}(M)$ to the second one $U_{g_2}(M)$ which satisfies

$$ {}^{g_2}\phi_t \circ \varphi = \varphi \circ {}^{g_1}\phi_t . $$

The problem is then to decide if such a flow conjugacy is the differential of some diffeomorphism from M into itself (and then it satisfies a linearity condition on each fiber).

Obviously, the knowledge of the dynamics gives some very strong information about the global geometry. For instance, the time-periods of the periodic movements are the geometric lengths of the periodic geodesics. We then define the *marked length spectrum* as the application which associates to each homotopy class the corresponding time-period, i.e. the length of the minimizing geodesic in the class. When the sectional curvature of (M, g) is negative, the marked length spectrum conversely gives all the time-periods. We also define the *non-marked length spectrum* as the image set of this application. A theorem of Y. Colin de Verdière **[9]** proves that the spectrum of the Laplace-operator of (M, g) gives the non-marked length spectrum when the sectional curvature is supposed to be negative.

To be more precise we are concerned with the three following conjectures:

$\mathbf{Q}_1$. *Does the existence of a C^k flow conjugacy imply that the corresponding metrics are isometric? What minimal value of k must be assumed?*

$\mathbf{Q}_2$. *If two metrics have the same marked length spectrum, are they isometric?*

$\mathbf{Q}_3$. *If the two Laplace operators corresponding to the two metrics have the same spectra, are the two metrics isometric?*

Conjectures $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are obviously false in general; counterexamples are given by Zoll's manifolds whose geodesics are all periodic with same length (see [4]). For this reason, and also because ergodic theory only applies in this case, one usually assumes the curvature to be strictly negative. Under this assumptions, U. Hamenstädt **[17]** proved that two metrics which have the same marked length spectrum have C^0-conjugate flows, so conjectures $\mathbf{Q}_1$ (for a C^0 flow conjugacy) and $\mathbf{Q}_2$ are equivalent.

In dimension 2, C. B. Croke [10] and J. P. Otal [22] proved simultaneously that conjectures $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are true if the curvatures are assumed to be strictly negative. This result was generalized to the case where the curvature is assumed to be nonpositive by C. B. Croke, A. Fathi and J. Feldman [11].

In higher dimensions, conjectures $\mathbf{Q}_1$ and $\mathbf{Q}_2$ remain unproved, except if one assumes either that the two flows come from two conformally equivalent metrics (A. Katok [19]) or that one of the two manifolds is of the type $M = S^1 \times N$ endowed with the product metric (in fact C. B. Croke and B. Kleiner ([12]) proved this in the more general case where there exists a parallel vector field). One of the main problems about these conjectures (see for instance [12]) is the following.

$\mathbf{Q}_4$. *On a given compact manifold M, let g be any metric whose geodesic flow is conjugate to that of a locally symmetric metric* g_0. *Is g isometric to* g_0?

We are proving this conjecture here when g_0 is supposed to be *hyperbolic* (i.e. with constant sectional curvature equal to -1) and when g lies in an explicit $C^{0,\alpha}$ neighbourhood $\mathcal{V}$ of g_0.

Conjecture $\mathbf{Q}_3$ is false in general. Many examples of compact Riemannian manifolds whose Laplace operators have the same spectra and which are not isometric have been given since the first ones, due to J. Milnor (cf. [3]). For our purpose, let us underline that two hyperbolic metrics on the same surface may be isospectral while not isometric (M. F. Vigneras) and that, in general dimension, there exist one-parameter families of metrics which are all isospectral while not isometric (C. Gordon and Wilson). However there are no such examples with negative curvature operator: in fact, V. Guillemin and D. Kazhdan proved that such C^1 one-parameter families do not exist if the initial metric is assumed to be hyperbolic ([16]). Generalizations of this result where then given when the curvatures of the metrics of the family are assumed to be pinched (V. Guillemin and D. Kazhdan [16]) and when their curvature operator is negative (M. Min Oo [21]). Let us remark that this does not prove the non-existence of isospectral sequences converging to a hyperbolic metric; so it is a natural question to decide if, for any hyperbolic metric, there exists a whole neighbourhood (if possible for the C^0-topology, and if possible explicit) containing no metric isospectral to the hyperbolic one (except the isometric ones). An answer to this question will come as a consequence of our method.

In fact, these results are by-products of a stronger result which consists in giving a positive answer, in a neighbourhood $\mathcal{G}$ of any given hyperbolic metric, to M. Gromov's minimal entropy conjecture (see [**14**], p. 58).

b Minimal entropy conjecture

We are interested in fact by two notions of entropy:

A geometric one, the *volume entropy* h_{vol}, which is defined as

$$h_{vol}(M, g) = \lim_{R \to \infty} R^{-1} \operatorname{Log} \operatorname{Vol}(B'(x, R))$$

where the geodesic ball $B'(x, R)$ is taken in the Riemannian universal covering (M', g') of (M, g), endowed with the corresponding distance (by the compactness of M, the limit exists and does not depend on the particular choice of the center x). If (M, g_0) is hyperbolic, the volume of the corresponding balls $B'(x, R)$ is computable and gives

$$h_{vol}(M, g_0) = n - 1. \tag{1.1}$$

A dynamical one, the *topological entropy* h_{top}, whose definition is the following. Let us consider any metric h on $U_g(M)$ (as the one induced by g) whose associated distance is called d and let us call (ε, T)*-net* any finite set $\{v_i\}_{i \in I}$ in $U_g(M)$, satisfying the condition

For every v in $U_g(M)$, there exists some $i \in I$ which satisfies $d({}^g\phi_t(v), {}^g\phi_t(v_i)) < \varepsilon$ for every t in $[0, T]$.

Let us call $r(T, \varepsilon)$ the minimal number of elements of a (ε, T)-net; the topological entropy is defined as

$$h_{top}({}^g\phi) = \lim_{\varepsilon \to 0} \operatorname*{Lim\,Sup}_{T \to \infty} (T^{-1} \operatorname{Log}(r(T, \varepsilon))).$$

By the compactness of M the limit does not depend on the particular choice of the metric h, but only on the flow ${}^g\phi$. Roughly speaking, it gives an estimate of the asymptotic number of informations which are necessary to approximate any movement up to a given precision. The relations between these two notions of entropy were given by Dinaburg and Manning (see [**20**]) who proved the following

Proposition 1 *In general, h_{vol} is less than or equal to h_{top}. If moreover the sectional curvature of (M, g) is nonpositive, then they are equal.*

One wants to consider these two entropies as functionals on the space $\mathfrak{R}_M$ of all metrics on a given manifold M. As the entropies are not invariant under the trivial homothetic changes of the metric (they are in fact homogeneous of degree -1), we must either rescale the entropy by considering the homogeneous functionals

$$g \to h_{vol}(g)^n \operatorname{Vol}(g) \qquad \text{and} \qquad g \to h_{top}(g)^n \operatorname{Vol}(g)$$

(where n is the dimension of M) or replace $\mathfrak{R}_M$ by its quotient by homotheties identified with $\mathfrak{R}_M(g_0) = \{g\colon \operatorname{Vol}(g) = \operatorname{Vol}(g_0)\}$, where g_0 is a fixed reference metric.

We are now able to recall the two versions of the minimal entropy conjecture:

Gromov's minimal entropy conjecture ([**14**], p. 58). *Let M be a manifold which admits a hyperbolic metric g_0, then the volume entropy (defined on $\mathfrak{R}_M(g_0)$) attains its minimum at g_0. If yes, in dimension at least 3, is it the unique minimum (up to isometries)?*

Another weaker version of this conjecture was previously formulated, that is:

Katok's minimal entropy conjecture. *Let M be a manifold which admits a hyperbolic metric g_0, then the topological entropy (defined on $\mathfrak{R}_M(g_0)$ attains its minimum at g_0. If yes, in dimension at least 3, is it the unique minimum (up to isometries)?*

From Proposition 1, it is obvious that Gromov's minimal entropy conjecture would imply Katok's minimal entropy conjecture.

Solving one of these two conjectures and proving the uniqueness of the minimum (up to isometries) would immediately prove conjectures $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_4$ when one of the two metrics is hyperbolic. In fact, it would prove that, if a metric g is such that

$$\operatorname{Vol}(g) = \operatorname{Vol}(g_0) \qquad \text{and} \qquad h_{top}(M, g) = h_{top}(M, g_0),$$

then g is isometric to g_0.

In order to conclude, we only have to notice that the volume and the entropy are dynamical invariants (i.e. invariant by flow conjugacy). This is, by definition, obvious for the topological entropy; moreover Margulis (see [8]) proved that, when the sectional curvature of (M, g) is strictly negative, the topological entropy only depends on the asymptotic behaviour of the non-marked length spectrum.

The invariance of the volume by C^1-conjugacy has been proved by C. B. Croke and B. Kleiner [12], the invariance by C^0-conjugacy when the curvature is negative is announced by U. Hamenstädt [18]; this last result proves (by [17]) that the volume only depends on the marked length spectrum in this case.

2 - Some results

a Gromov's approach (see [13])

The first attempt in order to solve Gromov's minimal entropy conjecture is due to M. Gromov himself ([13], p. 245) who proved that $h_{vol}(g)^n \operatorname{Vol}(g)$ is bounded from below by a topological invariant, the so-called simplicial volume, which is defined as follows.

Let us call $[M]$ the fundamental class of the manifold M. As there are several manners of representing this class as a linear span $\Sigma_i r_i \cdot \sigma_i$ of simplices with real coefficients r_i, one defines the *simplicial volume* $\operatorname{Simpl Vol}(M)$ as the lower bound of $\Sigma_i |r_i|$ for all the choices of a chain $\Sigma_i r_i \cdot \sigma_i$ representing $[M]$.

Using an argument of Thurston, M. Gromov proved that, in the above definition, the simplices may be chosen as *ideal regular totally geodesic hyperbolic simplices* (i.e. simplices which are limits of simplices whose vertices go to infinity, whose edges all have the same length and whose k-dimensional faces are totally geodesic) when the manifold admits a hyperbolic metric.

This gave him, in this case, a computation of the value of the simplicial volume in terms of the volume of the hyperbolic metric (cf. [13], p. 219), this computation is however theoretical, because one doesn't know what is the volume of such simplices in dimension greater than 3). In this case, Gromov's previous inequality may be written

Propositon 2 (from M. Gromov [13], p. 245 and 219). *Let M be a compact manifold which admits a hyperbolic metric g_0, then any other metric g satisfies*

$$h_{vol}(g)^n \operatorname{Vol}(g) \geqslant C(n) \cdot h_{vol}(g_0)^n \operatorname{Vol}(g_0).$$

Unfortunately, the constant $C(n)$ is far from its conjectured optimal value which, by Gromov's minimal entropy conjecture, is 1. This is the reason why M. Gromov asked if it was possible to establish a similar inequality where the simplicial volume would be replaced by another topological invariant whose computation in terms of $h_{vol}(g_0)^n \operatorname{Vol}(g_0)$ would lead to a sharp inequality.

b The present approach (for complete proofs, see [6] and [7])

We partially answered M. Gromov's question by defining a new topological invariant, the *spherical volume*, as follows.

Let us consider the universal covering M' of the manifold M and the fundamental group Γ of M. Let us choose a Hilbert space H, on which Γ acts isometrically, and call S its unit sphere.

Considering the set E of all the immersions ϕ from M' to S which are equivariant under the two actions of Γ on M' and S (i.e. $\gamma \circ \phi = \phi \circ \gamma$), we call g_ϕ the metric defined on M by pull-back of the canonical metric of H (the equivariance implies that the metric g_ϕ is Γ-invariant on M' and is then defined on M).

These metrics g_ϕ define a subset of the set $\mathfrak{R}_M$ of the riemannian metrics on M and we are going to compare the minimal value of a riemannian functional on $\mathfrak{R}_M$ (eventually rescaled) to its minimal value on the subset whose elements are the metrics g_ϕ.

For instance, let H be the Hilbert space $L^2(M')$ endowed with the regular action of $\Gamma: (\gamma, f) \to f \circ \gamma^{-1}$ (the Hilbert structure does not depend on the particular choice of the measure of M', provided that this measure is Γ-invariant and absolutely continuous with respect to the Lebesgue measure), we get the following examples

Example 1. For every positive real value c, let us call $\psi_c(x)$ the function from M' to $\boldsymbol{R}$ defined by

$$\psi_c(x):\ y \to \exp\left(-\frac{c}{2} d(x, y)\right)$$

where d is the distance associated to g. Integrating by parts, one obtains that $\psi_c(x)$ is an element of $L^2(M')$ iff c is greater than $h_{vol}(g)$.

Defining $\phi_c(x)$ as the radial projection of $\psi_c(x)$ onto the unit sphere S of $L^2(M')$, one obtains an equivariant immersion. From the fact that the norm of the gradient (with respect to g) of the function $x \to d(x, y)$ is equal to 1 and from

Pythagoras theorem, we deduce, for every $u \in T_x M'$ and every g-orthonormal basis $\{e_i\}$ of $T_x M$

$$\|d\phi_c(u)\|_{L^2} \leqslant \frac{c}{2} g(u, u)^{\frac{1}{2}} \sum_{1 \leqslant i \leqslant n} (\|d\phi_c(e_i)\|_{L^2})^2 \leqslant \frac{c^2}{4}. \tag{2.1}$$

Example 2. Let us define $\eta_t(x)$ as the function from M' to $\boldsymbol{R}$ defined by

$$\eta_t(x)\colon y \to k(t, x, y)^{\frac{1}{2}}$$

where $k(t, x, \cdot)$ is the *heat kernel* corresponding to a given metric g, i.e. the solution of the heat equation with initial data δ_x.

As this kernel is invariant by the diagonal action of the elements of Γ (which are isometries) and as the time-diffusion preserves its integral, η_t is an equivariant application from M' into S. As the derivatives of $k(t, x, y)$ with respect to the variable x also satisfy the heat equation with respect to y, it is easy to prove that some of these derivatives are not trivial and that η_t is an immersion (see [2] for a development of this idea).

N.B.: The assumption that the applications ϕ are immersions is not important. It has been assumed only to simplify the explanations.

The efficiency of this approach is proved by the

Lemma 1. *For any metric g on M, and for any positive ε, there exist a Γ-equivariant immersion ϕ of M' in the unit sphere of $L^2(M')$ which satisfies*

(i) $$g_\phi < (\frac{1}{2}(h_{vol}(g) + \varepsilon))^2 \cdot g$$

(ii) $$\mathrm{Trace}_g(g_\phi) < (\frac{1}{2}(h_{vol}(g) + \varepsilon))^2.$$

Proof. Take $\phi = \phi_c$, where ϕ_c is given by Example 1, and choose $c = h_{vol}(g) + \varepsilon$. The lemma then comes from (2.1).

This lemma means that, when the entropy is not trivial, the minimal value of any riemannian functional on $\mathfrak{R}_M$ (rescaled by the entropy) is bounded from below in terms of its minimal value on the subset whose elements are the metrics g_ϕ.

Unfortunately, a factor n is missing in the inequality (i) in order to get a sharp inequality. On the contrary, the inequality (ii) is sharp and the equality $\mathrm{Trace}_g(g_\phi) = \frac{c^2}{4}$ for $\phi = \phi_c$ is attained for any compact locally homogeneous manifold. This and the positivity of the above crucial examples suggest the

Definitions 1.

(i) For every Γ-equivariant immersion ϕ of M' into the unit sphere of $L^2(M')$ and any metric g on M, *the L^p-energy* $E_p(g, \phi)$ of ϕ is

$$E_p(g, \phi) = \int_M \mathrm{Trace}_g(g_\phi)^{\frac{p}{2}}\, dv_g = \int_M |d\phi|_g(x)^p\, dv_g$$

where $|d\phi|_g(x)^2 = \sum_{1 \leq i \leq n} (\|d\phi(e_i)\|_{L^2})^2$ and $\{e_i\}_{1 \leq i \leq n}$ is a g-orthonormal basis of T_xM.

(ii) The *spherical energy* $E_n(g)$ of a metric g is the infimum of $E_n(g, \phi)$ with respect to ϕ (where ϕ is any Γ-equivariant immersion of M' into the intersection of the unit sphere of $L^2(M')$ with the cone of positive functions). The spherical energy is a conformal invariant.

(iii) The *spherical volume* of a manifold M is the infimum of the volume of g_ϕ with respect to ϕ (where ϕ is any Γ-invariant immersion of M' into the intersection of the unit sphere of $L^2(M')$ with the cone of positive functions). It is also the infimum of $E_n(g) \cdot n^{-\frac{n}{2}}$ with respect to g.

From the inequality (ii) of Lemma 1, we deduce the analogous, for the spherical volume, of M. Gromov's inequality for the simplicial volume (see Section 2 a).

Proposition 3 ([6]). *Let M be a compact manifold, then any metric g satisfies*

$$\left(\frac{1}{2} h_{vol}(g)\right)^n \mathrm{Vol}(g) \geq E_n(g) \geq n^{\frac{n}{2}} \cdot \textit{spherical volume of } M.$$

The main problem is then to compute the spherical volume when M is a compact manifold which admits a hyperbolic metric g_0. This computation has been performed for surfaces (see [6]) and gives

$$\left(\frac{1}{2}h_{vol}(g_0)\right)^n \operatorname{Vol}(g_0) = E_n(g_0) = n^{\frac{n}{2}} \cdot \text{spherical volume of } M.$$

So it proves the

Proposition 4 ([6], [7]). *Let M be a compact surface (i.e. $n = 2$) with genus greater than 1, then hyperbolic metrics are the only minima of the functional $g \to h_{vol}(g)^n \operatorname{Vol}(g)$ defined on the set of Riemannian metrics on M.*

By Proposition 1, this implies a previous theorem of A. Katok which proved the same proposition for the topological entropy.

The computation of the spherical volume remains an open problem in higher dimensions. However, we proved (see also Proposition 16) that, for every locally symmetric space (M, g_0) of noncompact type and of any rank, one has

$$\left(\frac{1}{2}h_{vol}(g_0)\right)^n \operatorname{Vol}(g_0) = E_n(g_0) \tag{2.2}$$

the following proposition can be deduced from this and from Proposition 3.

Proposition 5 ([6], [7]). *Let M be a manifold which admits a locally symmetric metric g_0 of noncompact type and of any rank, then g_0 is the unique minimum of the functional $g \to h_{vol}(g)^n \operatorname{Vol}(g)$ defined on the set of Riemannian metrics on M which are conformal to g_0.*

By Proposition 1, this proposition implies a previous theorem of A. Katok, which proved the same proposition for the topological entropy and for rank one symmetric spaces. The extension to symmetric spaces admitting tangent planes with curvature zero means that our method is not any more limited by the usual limits inherent to the use of dynamical systems, which needs a negativity assumption for the curvature.

The main difficulty is then to compare the entropies of g_0 and g when g lies in a different conformal class. We solved this difficulty when g is contained in a $C^{0,\alpha}$-tubular neighbourhood of the conformal class of g_0 (a subset of $\mathfrak{R}_M$ is called tubular if, a soon as it contains a metric, it contains its conformal class). We then get the

Proposition 6 ([7]). *Let M be a compact manifold whose dimension is at least 3 and which admits a hyperbolic metric g_0. For every $\alpha \in]0, 1[$ there exists an explicit $C^{0,\alpha}$-tubular neighbourhood $\mathcal{I}$ of the conformal class of g_0 such* *that* g_0 *is the unique (up to isometries) minimum of the functional* $g \to h_{vol}(g)^n \operatorname{Vol}(g)$ *defined on* $\mathcal{S}$.

By (2.2) and the Proposition 3, this proposition is a corollary of the

Proposition 7 ([7]). *Under the same assumptions,* g_0 *is the unique (up to conformal equivalence) minimum of the functional* $g \to E_n(g)$ *defined on* $\mathcal{S}$.

We shall give later (see Section 3) a proof of this proposition, which is the main one. Let us first precisely establish its corollaries as announced in Section **1**.

c Applications to rigidity problems $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, $\mathbf{Q}_4$

Let $\mathcal{S}$ still denote the tubular neighbourhood of the conformal class of g_0 defined above. From the equality case in Proposition 6, we deduce the

Proposition 8 ([7]). *Let* M *be a compact manifold which admits a hyperbolic metric denoted* g_0. *Then, for any metric* g *which satisfies*

$$\operatorname{Vol}(g) = \operatorname{Vol}(g_0) \qquad \textit{and} \qquad h_{top}(M, g) = h_{top}(M, g_0),$$

one has

(i) *if* $\dim M = 2$, *then* g *is hyperbolic*

(ii) *if* $\dim M \geqslant 3$ *and if* $g \in \mathcal{S}$, *then* g *is isometric to* g_0.

Proof. By Propositions 6 and 1, we get

$$h_{top}(M, g) \geqslant h_{vol}(M, g) \geqslant h_{vol}(M, g_0) = h_{top}(M, g_0).$$

As the extreme left and right hand sides of this sequence of inequalities are equal, all these inequalities are in fact equalities and we deduce

$$h_{vol}(g)^n \operatorname{Vol}(g) = h_{vol}(g_0)^n \operatorname{Vol}(g_0).$$

When $\dim M \geqslant 3$, the uniqueness of the minimum of the functional $g \to h_{vol}(g)^n \operatorname{Vol}(g)$ (cf. Proposition 6) ends the proof. When $\dim M = 2$, we conclude by the Proposition 4.

Remark. Let us notice that the geometry of a hyperbolic metric is, by Proposition 8, characterized by the values of only 2 real parameters and that the set $\mathcal{G}$ is not smaller than the set of negatively curved metrics on M, for it contains metrics whose curvature takes the two signs.

From Proposition 8 and the fact that the volume and the topological entropy are invariant by C^1-conjugacy (see the end of Section **1**), we deduce the

Proposition 9 ([7]). *Let M be a compact manifold which admits a hyperbolic metric denoted g_0. Then any metric $g \in \mathcal{G}$, whose geodesic flow is C^1-conjugate to the one of g_0, is isometric to g_0.*

Let us now assume that g has strictly negative sectional curvature. Then the volume and the topological entropy only depend on the marked length spectrum (see the end of Section **1**). From this and Proposition 8, we get the

Proposition 10 ([7]). *Let M be a compact manifold which admits a hyperbolic metric denoted g_0. Then any negatively curved metric $g \in \mathcal{G}$, which has the same marked length spectrum, is isometric to g_0.*

It is otherwise a classical result, by Minakshisundaram-Pleijel's asymptotic formula (see for instance [3], p. 216), that the volume only depends on the spectrum of the Laplace operator. By Margulis' estimate (see the end of Section **1**), the topological entropy only depends on the non-marked length spectrum; so, by Colin de Verdière's result (see the beginning of Section **1**), it only depends on the spectrum of the Laplace operator. Applying Proposition 8, we get the

Proposition 11 ([7]). *Let M be a compact manifold which admits a hyperbolic metric denoted g_0. Then, for any negatively curved metric g whose Laplace operator has the same spectrum as the one of g_0, one has*

(i) *if* $\dim M = 2$, *then g is hyperbolic*

(ii) *if* $\dim M \geq 3$ *and if $g \in \mathcal{G}$, then g is isometric to g_0.*

In Propositions 8, 9, 10 and 11, the results concerning dimension 2 were already known; we mention them for the sake of completeness.

d Application to Gromov's conjecture on minimal volume

Trying to get a generalization to higher dimensions of the Gauss-Bonnet's formula for surfaces is the origin of many works in Riemannian Geometry, in particular Gromov's work **[13]**. M. Gromov noticed that a weaker version of Gauss-Bonnet's formula, and of its generalization in the even dimensional case by Chern-Weil-Avez formulas, is the following: as soon as some characteristic class of the manifold is not trivial, then the function $g \to \mathrm{Vol}(g)$ is bounded from below on the set of metrics g whose sectional curvature σ_g is bounded. So M. Gromov defined the *minimal volume* of a manifold as

$$\mathrm{Min\,Vol}(M) = \mathrm{Inf}\{\mathrm{Vol}(M, g)\colon g \text{ metric on } M \text{ such that } -1 \leqslant \sigma_g \leqslant 1\}.$$

M. Gromov then got an analogous to this weaker version of Gauss-Bonnet and Chern-Weil-Avez formulas, which is still efficient in the odd dimensional case, that is the

Proposition 12 ([**13**], p. 220). *On any compact manifold M*

$$\mathrm{Min\,Vol}(M) \geqslant C(n) \cdot \mathrm{Simpl\,Vol}(M)$$

where $C(n)$ is an explicit universal constant which only depends on the dimension.

Another stronger consequence of Gauss-Bonnet's formula is that, on any surface which admits a hyperbolic metric, the minimal volume (on the set of metrics g whose sectional curvature satisfies $-1 \leqslant \sigma_g \leqslant 1$) is attained for hyperbolic metrics. So M. Gromov conjectured that this is also true in higher dimension, that is why this is called *Gromov's conjecture on minimal volume.*

Notice that this conjecture is open in any dimension greater than 2 and cannot be deduced from Chern-Weil-Avez formulas in higher even dimensions or from Gromov's Proposition 12 in other dimensions (for these formulas don't give sharp estimates for the minimal volume). This conjecture was qualified as «optimistic» in **[13]** (p. 221, after inequality (*)) and M. Gromov asked (oral communication) if it was possible to get first a local version of it, i.e. to prove that a hyperbolic metric is a local minimum of the functional $g \to \mathrm{Vol}(g)$ on the set of metrics g whose sectional curvature satisfies $-1 \leqslant \sigma_g \leqslant 1$. To this problem, we get the following answer

Proposition 13 ([**7**]). *Let M be a compact manifold whose dimension n is at least 3 and which admits a hyperbolic metric g_0. Let $\mathcal{V}$ be the $C^{0,\alpha}$-tubular neighbourhood of the conformal class of g_0 mentioned above. Then*

$$g \in \mathcal{V} \text{ and } \mathrm{Ricci}_g \geqslant -(n-1)g \Rightarrow \mathrm{Vol}(M, g) \geqslant \mathrm{Vol}(M, g_0).$$

The equality characterizes g_0 up to isometries.

Let us recall that the *Ricci curvature tensor* Ricci_g is the 2-tensor defined in each point x from the curvature 4-tensor R_g by tracing one time, i.e.

$$\mathrm{Ricci}_g(X, Y) = \sum_{1 \leqslant i \leqslant n} R_g(X, e_i, Y, e_i)$$

where $\{e_i\}_{1 \leqslant i \leqslant n}$ is any orthonormal basis of $T_x M$ and where X and Y are vectors of $T_x M$. The assumption $\mathrm{Ricci}_g \geqslant -(n-1)g$ means that every eigenvalue of Ricci_g is greater than $-(n-1)$. Notice that every hyperbolic metric satisfies $\mathrm{Ricci}_g = -(n-1)g$.

Remark to Proposition 13. M. Ville ([23]) previously proved the following result by means of the Chern-Weil-Avez formula.

Let M be a compact manifold whose dimension n is even and which admits a hyperbolic metric g_0. Let ϑ_ε be the C^2-neighbourhood of g_0 defined by $-1-\varepsilon \leqslant \sigma_g \leqslant -1+\varepsilon$. Then there exists some positive ε such that

$$g \in \vartheta_\varepsilon \text{ and } \sigma_g \geqslant -1 \Rightarrow \mathrm{Vol}(M, g) \geqslant \mathrm{Vol}(M, g_0).$$

Proof of Proposition 13. As $g \in \vartheta$ we get, by Proposition 6

$$h_{vol}(g)^n \, \mathrm{Vol}(g) \geqslant h_{vol}(g_0)^n \, \mathrm{Vol}(g_0).$$

By R. L. Bishop's comparison theorem, the assumption $\mathrm{Ricci}_g \geqslant -(n-1)g$ implies that the volume of the balls of radius R in the Riemannian universal covering of (M, g) is smaller than the volume of the ball of radius R in the hyperbolic space. By definition of the volume entropy (see Section **1 b**), this implies

$$h_{vol}(M, g) \leqslant h_{vol}(M, g_0).$$

So we get
$$\mathrm{Vol}(g) \geqslant \mathrm{Vol}(g_0).$$

Let us notice that the assumption $\mathrm{Ricci}_g \geqslant -(n-1)g$ is much weaker than the assumption $-1 \leqslant \sigma_g \leqslant 1$ of the initial question. It is a natural question to try to improve Proposition 13 by replacing the assumption $\mathrm{Ricci}_g \geqslant -(n-1)g$ by $\mathrm{Scal}_g \geqslant -n(n-1) = \mathrm{Scal}_{g_0}$, where the *scalar curvature* Scal_g is the scalar function obtained by tracing the Ricci curvature tensor Ricci_g. In this direction, we get the

Proposition 14 ([**6**]). *Let M be a compact manifold whose dimension is at least 3 and which admits a metric g_0 which satisfies at least one of the following assumptions*

(i) *g_0 is Einstein with strictly negative sectional curvature*

(ii) *(M, g_0) is a quotient of a symmetric non compact irreducible space of any rank (i.e. its sectional curvature is non positive but may vanish).*

Then there exists a C^2-tubular neighbourhood $\mathcal{I}'$ of the conformal class of g_0 such that

$$g \in \mathcal{I}' \text{ and } \mathrm{Scal}_g \geq \mathrm{Scal}_{g_0} \Rightarrow \mathrm{Vol}(M, g) \geq \mathrm{Vol}(M, g_0).$$

The equality characterizes g_0 up to isometries.

Trying to get an integral version of this result, which would be more similar to Gauss-Bonnet formula, let us consider the *Einstein-like functional* $S_n(g)$, where $S_p(g)$ is defined by

$$S_p(g) = \int_M |\mathrm{Scal}_g^-|^{\frac{p}{2}} dv_g,$$

and where $\mathrm{Scal}_g^-(x) = \mathrm{Inf}(\mathrm{Scal}_g(x), 0)$. We get the following

Proposition 15 ([**6**]). *Let M be a compact manifold whose dimension is at least 3 and which admits a metric g_0, which satisfies at least one of the following assumptions*

(i) *g_0 is Einstein with strictly negative sectional curvature*

(ii) *(M, g_0) is a quotient of a non compact irreducible space of any rank.*

Then there exists a C^2-tubular neighbourhood $\mathcal{I}'$ of the conformal class of g_0 such that g_0 is the unique (up to isometries) minimum of the functional $g \to S_n(g)$ defined on $\mathcal{I}'$.

Remarks.

1 - Proposition 15 is stronger than Proposition 14 because the inequality $\mathrm{Scal}_g \geq \mathrm{Scal}_{g_0}$ implies that

$$S_n(g) \leq \mathrm{Vol}(g)(-\mathrm{Scal}_{g_0})^{\frac{n}{2}}$$

and because, in this last inequality, the equality is attained when $g = g_0$.

2 - This proposition is related to another conjecture of M. Gromov ([**15**], p. 117): is $S_n(g)$ bounded from below in terms of the simplicial volume of M?

3 - The fact that the minimum is strict in Proposition 15 implies the following rigidity theorem which characterizes the hyperbolic metric by prescribing only one real parameter:

$$g \in \mathcal{S}' \text{ and } S_n(g) = S_n(g_0) \Rightarrow g \text{ isometric to } g_0 .$$

In fact, a stronger version of Proposition 15 is the following: if d denotes the H^1-distance between classes of homothetically equivalent metrics, we have

$$S_n(g) - S_n(g_0) \geqslant C \cdot d(g, g_0)^2 ,$$

where C is a constant which only depends on g_0 (cf. [**6**]).

About the proofs. The proof of Propositions 14 and 15 are different from the proof of Proposition 13. That is the reason why we shall say very little about them here (see [**6**] for complete proofs).

We first reduce the problem to the proof of the fact that g_0 is a strict minimum of S_1, when this functional is restricted to an infinite-dimensional submanifold Σ of the space $\mathfrak{R}_M$ of all metrics on M, which is a slice transverse to the classes of conformally equivalent metrics and on which $S_1 = S_n$ (see [**5**]).

We first have to prove (by algebraic considerations on the associated operator) that the quotient

$$Q(h) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(S_1(g_0 + t \cdot h))\right)_{t=0} \cdot \|h\|^{-2}$$

is bounded from below by a positive constant, when h lies in the Hilbert space $T_{g_0}(\Sigma)$ tangent to Σ at g_0 and that the negative values of $Q(h)$ are bounded when h is orthogonal to $T_{g_0}(\Sigma)$.

We then have to control the two perturbations of $Q(h)$ induced first by the perturbation of the operator associated to the quadratic form $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(S_1(g_0 + t \cdot h))$, which happens when we replace the second derivative at g_0 by the second derivative at g, and second by the simultaneous perturbation of the tangent space $T_{g_0}(\Sigma)$ in $T_g(\Sigma)$. From its positivity for $t = 0$, we then deduce that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} S_1((1-t)g_0 + t \cdot g)$ is positive for every $t \in [0,1]$, provided that g lies in Σ.

One of the main problems comes from the fact that we are in a critical situation for the choice of the Hilbert norm: it must be both weaker than the H^1-norm in order that $Q(h)$ is bounded from below by a positive constant and stronger than the H^1-norm in order that $Q(h)$ is continuous with respect to the above perturbations.

3 - A rough sketch of the proof

The last proposition to prove is Proposition 7. Using Definitions 1, let us notice that a Cauchy-Schwarz's argument implies the convexity of $\phi \to E_n(g_0, \phi)$ with respect to the variations $t \to ((1-t)\phi_0^2 + t\phi_1^2)^{\frac{1}{2}}$. A consequence is that any critical point ϕ_0 is a minimum and realizes the spherical energy, i.e.

$$E_n(g_0) = E_n(g_0, \phi_0).$$

We shall first identify what is this critical point and then show that, for every metric g and any Γ-equivariant immersion ϕ of M' into the unit sphere of $L^2(M')$, we have

$$E_n(g, \phi) \geqslant E_n(g_0, \phi_0).$$

There are some similitudes between this problem and the more classical study of harmonic mappings. Let us however notice that, in the classical problem, the metric g is fixed and one minimizes with respect to ϕ. Here we minimize with respect to the couple of variables (g, ϕ).

a Identification of the critical point

In this section, the Riemannian universal covering of (M, g_0) is denoted by (M', g_0'), viewed as the Euclidean unit ball B^n endowed with the hyperbolic metric. Its ideal boundary $\partial M'$ is then identified with the euclidean unit sphere S^n endowed with the usual euclidean probability measure db. Let us call O a fixed point of M', which we fix as the center of B^n. All over this section, any application ϕ from M' to $L^2(\partial M')$ will be identified with the function on $M' \times \partial M'$ defined by $\phi(x, b) = \phi(x)(b)$.

The *Poisson kernel* P is a function on $M' \times \partial M'$ defined as follows: for any $b \in \partial M'$, $x \to P(x, b)$ is the unique harmonic function whose data on $\partial M'$ is the Dirac measure δ_b.

Let us define the application ϕ_0 from M' into $L^2(\partial M')$ by

$$\phi_0(x)(b) = \phi_0(x, b) = P(x, b)^{\frac{1}{2}}. \tag{3.1}$$

As $P(x, b)\,db$ is the measure which is the limit (when t goes to infinity) of the measures $k_0(t, x, \cdot)\,dv_0$ [where $k_0(t, x, \cdot)$ is the *heat kernel* corresponding to the metric g_0 (i.e. the solution of the heat equation with initial data δ_x) and dv_0 the canonical Riemannian measure associated to g_0], ϕ_0 may be viewed as the limit of the immersions η_t of the Example 2, where the metric g is replaced by g_0.

More generally, for any Γ-equivariant immersion of M' into the unit sphere of $L^2(M')$, the limit in this sense of the heat-diffusion process with initial data $\phi(x, b)^2$ when time goes to infinity is the application ϕ_∞ from M' to the unit sphere of $L^2(\partial M')$ defined by

$$\phi_\infty(x, b)^2 = \int_{M'} \phi(x, z)^2 P(z, b)\,dv_0(z). \tag{3.2}$$

If we define the action of Γ on $L^2(\partial M')$ by the formula

$$(\gamma f)(b) = f\circ\gamma^{-1}(b)\cdot P(\gamma(O), b)^{\frac{1}{2}}$$

the action on $L^2(\partial M')$ is then the limit of the action on $L^2(M')$ and ϕ_∞ is Γ-equivariant (see [7] Proposition 1.1).

By (3.2) and the Cauchy-Schwarz's inequality, the L^p-energy decreases when replacing ϕ by ϕ_∞. This is the reason why, instead of looking for the infimum of $E_n(g, \phi)$ with respect to ϕ (where ϕ is any Γ-equivariant immersion of M' into the unit sphere of $L^2(M')$), we shall look for the infimum of $E_n(g, \phi)$ with respect to ϕ (where ϕ is any Γ-equivariant immersion of M' into the unit sphere of $L^2(\partial M')$).

It would have been more rigourous and more simple from a technical point of view (but less natural from an intuitive point of view) to start from the beginning defining the spherical volume and energy from Γ-equivariant immersions ϕ of M' into the unit sphere of $L^2(\partial M')$, as it is done in [7]. We wanted here to explain how this idea comes naturally in the theory.

The immersion ϕ_0 has the following properties

Proposition 16.

(i) ϕ_0 *is homothetic and* $g_{\phi_0} = \dfrac{1}{4n} h_{vol}(g_0)^2 g_0$

(ii) ϕ_0 *is minimal, so it is a critical point (and moreover a minimum) of the function* $\phi \to E_n(g, \phi)$.

Remark. These properties can easily be established (by the same proof) on rank-one locally symmetric spaces, so they prove the formula (2.2) and then the Proposition 5 in these cases.

Proof of (i). We refer to [1] for the basic properties of the Busemann function which are used here. In what follows, all the derivatives are considered with respect to the first variable x.

On a rank-one locally symmetric space, the Poisson kernel may be written

$$P(x, b) = \exp(-h_{vol}(g_0) B(x, b)) \tag{3.3}$$

where B is the *Busemann function* defined by

$$B(x, b) = \lim_{y \to b} (d(x, y) - d(O, y)) \tag{3.4}$$

where d is the distance associated to g_0.

If one identifies the hyperbolic unit tangent bundle UM' with $B^n \times S^{n-1}$, the unit vector $-\nabla B$ is identified with (x, b) and the Lebesgue measure $\mathrm{d}u$ of the fiber $UM'(x)$ at the point x coincides with $P(x, b)\,\mathrm{d}b = \phi_0(x, b)^2\,\mathrm{d}b$. So a direct computation gives, from formulas (3.1) and (3.3),

$$g_{\phi_0}(X, X) = \frac{1}{4} h_{vol}(g_0)^2 \int_{\partial M'} d_X B(x, b)^2 \phi_0(x, b)^2\,\mathrm{d}b$$

$$= \frac{1}{4} h_{vol}(g_0)^2 \int_{UM'(x)} g_0(X, u)^2\,\mathrm{d}u = \frac{1}{4n} h_{vol}(g_0)^2 g_0(X, X).$$

Proof of (ii). Let us call Δ the *Laplace operator* associated to the metric g_{ϕ_0}. By (i), the g_0-harmonicity of $\phi_0(\cdot, b)^2$ implies its g_{ϕ_0}-harmonicity. Then, applying the Laplace operator and the differential only to the first variable x, we get $\phi_0 \Delta\phi_0 = |\mathrm{d}\phi_0|^2$, where the norm $|\cdot|$ is the norm on $(T_x M)^*$ defined by g_{ϕ_0}. From (3.4), the g_0-norm of $\mathrm{d}B$ is equal to 1, then, by (i), the square of its g_{ϕ_0}-norm is equal to $\dfrac{4n}{h_{vol}(g_0)^2}$. As (3.3) implies that $\mathrm{d}\phi_0 = -\dfrac{1}{2} h_{vol}(g_0)\ \phi_0 \cdot \mathrm{d}B$, we have

$$\Delta\phi_0 = n\phi_0.$$

By Takahashi's theorem (or more simply, by considering that the vector *mean curvature of the imbedding in* $L^2(\partial M')$, equal to $\Delta\phi_0$, is orthogonal to the sphere), we prove that ϕ_0 is minimal (i.e. is a critical point of $\phi \to \mathrm{Vol}(g_\phi)$). As it is also homothetic, we deduce that it is a critical point of $\phi \to E_p(g_0, \phi)$ for any p. We have already seen in the beginning of Section 3 that such a critical point is automatically a minimum.

b The minimum property

Every metric can be written as $g_H(X,Y)=g_0(e^H(X),Y)$, where $x\to H_x$ is a field of symmetric endomorphisms of the tangent fiber. Defining Σ as the set of all g_H such that the trace and the divergence of H are both trivial, we first prove that Σ is a local slice, transversal to the classes of conformally equivalent metrics (see [7]).

By Hölder's inequality and the conformal invariance of E_n, it is sufficient to prove that (g_0, ϕ_0) is the minimum of $(g, \phi) \to E_2(g, \phi)$ when g lies in Σ. Let us make the change of variables

$$\phi(x, b)^2 = (1 + f(x, b))\, \phi_0(x, b)^2 .$$

As $\phi(x)$ lies in the unit sphere of $L^2(\partial M')$, we have

$$\int_{\partial M'} f(x, b)\, P(x, b)\, \mathrm{d}b = 0$$

at every point x.

For $g = g_H$, a direct computation (using (3.1), (3.3) and (1.1)) gives the following formula, where all derivatives are considered with respect to the first variable x, where all integrals are considered with respect to the second variable b and the measure $\mathrm{d}b$ and where $\langle X, Y\rangle = g_0(X, Y)$

$$4(\mathrm{Trace}_g\, g_{\phi} - \mathrm{Trace}_{g_0} g_{\phi_0}) = \int_{\partial M'} \langle e^{-H}(\nabla f), \nabla f\rangle \cdot (1+f)^{-1} \cdot P$$

$$- 2(n-1) \int_{\partial M'} \langle e^{-H}(\nabla B), \nabla f\rangle \cdot P + (n-1)^2 \int_{\partial M'} \langle (e^{-H} - I)(\nabla B), \nabla B\rangle \cdot (1+f) \cdot P .$$

We then use the triviality of the trace and of the divergence of H and the fact that the horocycles of (M', g_0') are totally ombilical (which gives the expression of the Hessian of the function B) in an integration by parts (this computation is long and technical and won't be done here, see [7]). This leads to

$$\begin{aligned} 4(E_2(g_H, \phi) - E_2(g_0, \phi_0)) &= \int_M \int_{\partial M'} \langle e^{-H}(\nabla f), \nabla f\rangle \cdot (1+f)^{-1} \cdot P \,\mathrm{d}b\, \mathrm{d}v_0 \\ &- 2(n-1) \int_M \int_{\partial M'} \langle [e^{-H} - I + (2n)^{-1}(n-1)\cdot H](\nabla B), \nabla f\rangle \cdot P \,\mathrm{d}b\, \mathrm{d}v_0 \qquad (3.5) \\ &+ (n-1)^2 \int_M \int_{\partial M'} \langle [e^{-H} - I + H](\nabla B), \nabla B\rangle \cdot (1+f) \cdot P \,\mathrm{d}b\, \mathrm{d}v_0 . \end{aligned}$$

When H is small enough with respect to the C^0-norm, the properties of the exponential function imply

$$2\langle [e^{-H} - I + H](\nabla B), \nabla B\rangle = \langle [I - \varepsilon(H)]H(\nabla B), [I - \varepsilon(H)]H(\nabla B)\rangle.$$

As $e^{-H} - I + (2n)^{-1}(n-1)\cdot H$ may be written

$$(n+1)(2n)^{-1}[I + O(H)][I - \varepsilon(H)]H,$$

the right hand side of the formula (3.5) is, by the Cauchy-Schwarz's inequality, bounded from below (up to a factor $(1 - 2^{\frac{1}{2}}(n+1)(2n)^{-1})(1-\varepsilon')$) by the sum of the two following positive quantities:

$$Q(\phi) = \int_M \int_{\partial M'} \langle \nabla f, \nabla f\rangle \cdot (1+f)^{-1} P \, db \, dv_0 \geq \Big(\int_M \int_{\partial M'} \langle \nabla f, \nabla f\rangle^{\frac{1}{2}} P \, db \, dv_0\Big)^2 \operatorname{Vol}(g_0)^{-1}$$

$$T(g_H, \phi) = \frac{1}{2}(n-1)^2 \int_M \int_{\partial M'} \langle H(\nabla B), H(\nabla B)\rangle \phi^2 \, db \, dv_0 .$$

We then get $E_2(g_H, \phi) \geq E_2(g_0, \phi_0)$ which, by Hölder's inequality, implies that $E_n(g, \phi) \geq E_n(g_0, \phi_0)$ for every g conformally equivalent to g_H and ends the proof.

c The minimum is strict

Let us suppose that, for some $g_H \in \Sigma$, we have $E_2(g_H) = E_2(g_0)$. It means that there exists a sequence ϕ_k such that

$$\lim_{k\to\infty} E_2(g_H, \phi_k) = E_2(g_0, \phi_0).$$

By the above inequalities, it implies that $Q(\phi_k)$ and $T(g_H, \phi_k)$ both go to zero. The first fact implies that the measure $\phi_k(x,b)^2 db$ converges to a harmonic measure (see [7]). By the uniqueness of the harmonic measure, we conclude that this limit is

$$\phi_0(x,b)^2 \, db = P(x,b)\, db = \text{Lebesgue measure of the fiber } UM'(x).$$

This implies that

$$2(n-1)^{-2} \lim_{k\to\infty} T(g_H, \phi_k) = \int_M \int_{\partial M'} \langle H(\nabla B), H(\nabla B)\rangle P \, db \, dv_0(x)$$

$$= \int_M \int_{UM'(x)} \langle H(u), H(u)\rangle du \, dv_0(x) = \frac{1}{n} \int_M \|H_x\|^2 dv_0(x).$$

As this limit is trivial, we have $H = 0$ and then $g_H = g_0$. Applying Hölder's inequality and the fact that every metric g which lies in a neighbourhood of g_0 is conformally equivalent to some $g_H \in \Sigma$, we end the proof as in [7].

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