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Principal bundles in action (**)

0 - Introduction

The notion of principal bundle represents a fundamental frame in contemporary geometry. It provides both a unitary description and a deeper comprehension of a large class of phenomena, ranging from existence of further structures on a differentiable manifold (Riemann, conformal, almost-complex etc.) to the theory of connections in the modern approach to Differential Geometry.

The aim of this paper is to gather and develop the basic features and results in the theory of principal bundles and connections on them: we hope in this way to contribute to fill a gap in the litterature, which, in spite of the increasing role played by principal bundles, seems to be quite reticent about general expositions on the subject and, therefore to provide a somehow useful tool.

The plan of the paper is the following:

Chapter 1 is devoted to the general theory of principal bundles. In Section 1 we define principal bundles, we describe some fundamental examples, properties and constructions, stressing the role of the group action. In Section 2 we take care of associated fibre bundles and, in particular, of vector bundles, presenting a principal bundles approach to canonical vector bundles constructions (dual

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(**) Received July 20, 1992. AMS classification 55 R 10. This note contains general material of the theory of principal bundles, introducting to methods and techniques, that I tried to illustrate during the *Giornate di Geometria Differenziale e Topologia*. The specific results, that I exposed in Parma, will appear soon in a joint paper with G. Tian.

bundle, subbundles, direct sum and tensor products, morphisms). Finally, in Section 3 we consider the gauge group of a principal bundle.

The subject of Chapter 2 is the theory of connections. Section 4 provides the basic features of the theory of connections, including several standard and non standard definitions, examples and local explicit descriptions (e.g. (4.10)). In Section 5 we describe pseudotensorial and tensorial forms and, by means of the results achieved in Chapter 1, we consider the induced principal bundles view point of vector bundles values forms. This includes various presentations of the exterior covariant differential operator, curvature, horizontal/vertical splitting on associated bundles and basic geometric interpretations of covariant derivative. Section 6 recalls some of the results of the holonomy theory and Section 7 is concerned with the behaviour of connections with respect to bundle morphisms and some applications (e.g. generalized Codazzi-Mainardi equation (7.1), or reduction of connections). In Section 8 we introduce a scalar product on the space of tensorial forms, we define Hodge's $*$ operator and covariant codifferential operator, pointing out some of their fundamental properties, and we describe the basic gauge-theoretic results in the theory of characteristic classes. Finally, Section 9 enlights some of the special features of linear connections and Section 10 provides a short account of the theory of moduli spaces of connections.

Manifolds and maps between them will be understood to be C^∞. $\mathcal{X}(M)$ will denote the Lie algebra of C^∞ vector fields of the manifold M. Let G be a Lie group and $\mathfrak{G}$ be its Lie algebra. Then

$$Ad(a)\colon G \to G \quad \text{is defined by} \quad Ad(a)(g) = aga^{-1}$$
$$ad\colon G \to \mathrm{Aut}(\mathfrak{G}) \quad \text{is defined by} \quad ad(a) = (Ad(a))_*[e]$$
$$\mathfrak{ad}\colon \mathfrak{G} \to \mathrm{End}(\mathfrak{G}) \quad \text{is defined by} \quad \mathfrak{ad} = (ad)_*[e].$$

Chapter 1 - Principal Bundles

1 - Preliminaries

Let M be an n-dimensional smooth manifold and let G be a Lie group. Typically G will be a matrix group i.e. a subgroup of $GL(m, \boldsymbol{R})$ as $O(m)$, $SO(m)$, S^1, $U(k)$, $SU(k)$ etc. G acts freely on the right on $M \times G$ in a trivial way

$$R_a(x, g) = (x, g)\, a = (x, ga).$$

This is the local model of a principal G-bundle over M.

Definition 1. A *principal G-bundle* $P = P(M, G)$ over M is a smooth manifold such that:

1 G acts freely on P on the right.

2 $P/G = M$ and the canonical projection $\pi: P \to M$ is smooth.

3 P is locally trivial i.e. there exists an open covering $(U_i)_{i \in I}$ of M and diffeomorphisms $\Phi_i: \pi^{-1}(U_i) \to U_i \times G$, $i \in I$, such that:

a the diagram

(1.1)
$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\Phi_i} & U_i \times G \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_1} & \\ U_i & & \end{array}$$

is commutative

b Φ_i is G-equivariant, i.e. for all $u \in P$, $a \in G$

(1.2)
$$\Phi_i(ua) = \Phi_i(u)\,a$$

or, equivalently, $\Phi_i \circ R_a = R_a \circ \Phi_i$.

Note that from **3a** it follows that $\Phi_i(\pi(u), \psi_i(u))$ for $\psi_i: \pi^{-1}(U_i) \to G$ and therefore (1.2) amounts to $\psi_i(ua) = \psi_i(u)\,a$.

$\mathfrak{U} = (U_i, \Phi_i)_{i \in I}$ is called a *bundle atlas* and every Φ_i is called a *local trivialization*. G is called the *structure group*.

Example 1 (basic example). Let

$$P = L(M) = \bigcup_{x \in M} L(x)$$

where $L(x) = \{\text{ordered basis of } T_x M\}$. Then $G = GL(n, \boldsymbol{R})$ acts freely on the right on $L(M)$ in the following way:

If $u = \{X^1, \ldots, X_n\} \in L(x) \subset L(M)$ and $a = (a_{jk}) \in GL(n, \boldsymbol{R})$, then

$$ua = \{Y_1, \ldots, Y_n\} \quad \text{with} \quad Y_j = \sum_{k=1}^{n} a_{kj} X_k \qquad 1 \leqslant j \leqslant n.$$

Equivalently, u can be interpreted as a linear isomorphism

$$u: \boldsymbol{R}^n \to T_x M$$

setting $u(e_j) = X_j$, $1 \leq j \leq n$; then ua is just the composition

$$\mathbf{R}^n \xrightarrow{a} \mathbf{R}^n \xrightarrow{u} T_x M$$

in fact
$$ua(e_j) = u\left(\sum_{k=1}^{n} a_{kj} e_k\right) = \sum_{k=1}^{n} a_{kj} X_k = Y_j .$$

It is clear that $GL(n, \mathbf{R})$ acts transitively on $L(x)$ and so $L(M)/GL(n, \mathbf{R}) = M$.

Let $\{X_1, \ldots, X_n\}$ be a local frame in the neighbourhood U of the point $x \in M$; then we can define a bijection

$$\Phi : \pi^{-1}(U) \to U \times GL(n, \mathbf{R})$$

setting
$$\Phi(u) = (\pi(u), \psi(u))$$

where $\psi(u) \in GL(n, \mathbf{R})$ is completely determined by the relation

$$u = \{X_1(\pi(u)), \ldots, X_n(\pi(u))\} \psi(u) .$$

Therefore $\psi(u)$ is the matrix representing the identity map of $T_{\pi(u)}M$ with respect to the ordered pair of basis $\{u, \{X_1(\pi(u)), \ldots, X_n(\pi(u))\}\}$. It is easy to check that Φ is $GL(n, \mathbf{R})$-equivariant and that $L(M)$ inherits through a collection of such a Φ's the structure of smooth manifold. Therefore $L(M)$ is a principal $GL(n, \mathbf{R})$-bundle, called the *bundle of linear frames* on M.

Examples 2.

a Given a connected (smooth) manifold M, its universal covering $\tilde{M}$ is a principal $\pi_1(M)$-bundle M (where $\pi_1(M)$ acts on the right on $\tilde{M}$ by deck transformations).

b Let G be a Lie group and H a closed subgroup: then H acts on the right on G by right multiplication; then G is a H-principal bundle over the analytic manifold G/H.

c Consider $S^{2m+1} = \{z \in \mathbf{C}^{m+1} \mid |z| = 1\}$ and let S^1 act on S^{2m+1} by complex multiplication. Clearly $S^{2m+1}/S^1 = \mathbf{P}^m(\mathbf{C})$. Let

$$U_k = \{[z_1, \ldots, z_{m+1}] \in \mathbf{P}^m(\mathbf{C}) \mid z_k \neq 0\} \qquad 1 \leq k \leq m+1$$

then
$$\Phi_k : \pi^{-1}(U_k) \to U_k \times S^1$$

defined by
$$\Phi_k(z) = (\pi(z), \frac{z_k}{|z_k|})$$

is a S^1-equivariant diffeomorphism and thus the fibration $S^{2m+1} \xrightarrow{\pi} \boldsymbol{P}^m(\boldsymbol{C})$ is locally trivial and so S^{2m+1} is a principal S^1-bundle over $\boldsymbol{P}^m(\boldsymbol{C})$.

d $M \times G$ is obviously a principal G-bundle over M, called the *trivial bundle.*

e Let $P(M, G)$ be a principal bundle and let $K \subset M$; then $P_{|K} = \pi^{-1}(K)$ is a principal G-bundle over K called the *restriction* to P to K.

f M is a principal $\{e\}$-bundle over itself!

Let P be a principal G-bundle over M and let $x \in M$. $P_x = \pi^{-1}(x)$ is called the *fibre* of P over x. If $u \in P_x$, then $P_x = \{ua \mid a \in G\}$ i.e. P_x is the G-orbit of u and, since G acts freely, P_x is diffeomorphic to G. $W_u = T_u P_x = \ker \pi_* [u]$ is called the *vertical subspace* of $T_u P$.

The action of G on P induces a map $\sigma\colon \mathfrak{G} \to \mathcal{X}(P)$, which is an injection. More precisely, σ is defined as follows.

For every $X \in \mathfrak{G}$, $u \in P$, $\sigma(X)(u)$ is the tangent vector at the point $t = 0$ to the curve $\gamma(t) = u \exp(tX)$. It is clear that $\sigma(X)(u) \in W_u$.

$X^* = \sigma(X)$ is called the *fundamental vertical vector field* corresponding to X. We have also that, if $t_u\colon G \to P_{\pi(u)}$ is defined by $t_u(a) = ua$, then $X^*(u) = (t_u)_*[e](X)$ and $\tau_u = (t_u)_*[e]\colon X \mapsto X^*(u)$ is an isomorphism between $\mathfrak{G}$ and W_u.

More in general, if we identify $\mathfrak{G}$ with the Lie algebra of left invariant vector fields on G, and so $X \in \mathfrak{G}$ corresponds to $\tilde{X} \in \mathcal{X}(G)$ such that $\tilde{X}(a) = (L_a)_*(X)$, then we have

$$(t_u)_*[a](\tilde{X}(a)) = (t_u \circ L_a)_*[e](X) = (t_{ua})_*[e](X) = X^*(ua)$$

and so
$$X^* = (t_u)_*(\tilde{X}).$$

Note that, for $P = L(M)$, given $X \in gl(n, \boldsymbol{R})$, we have $X^*(u) = u \circ X$.

Proposition 1. *Let P be a principal G-bundle over M; then*

a *if $X, Y \in \mathfrak{G}$, we have*

(1.3)
$$[X^*, Y^*] = [X, Y]^*$$

and so $\sigma\colon \mathfrak{G} \to \mathcal{X}(P)$ is an injective Lie algebra homomorphism

b *for every* $X \in \mathfrak{G}$ *and every* $a \in G$ *we have*

$$\sigma(ad(a^{-1})(X)) = (R_a)_*(\sigma(X)) \qquad (1.4)$$ *and so*

$$\tau_{ua} = (R_a)_* \circ \tau_u \circ ad(a). \qquad (1.5)$$

Proof.

a We have

$$[X^*, Y^*] = [(t_u)_*(\tilde{X}), (t_u)_*(\tilde{Y})] = (t_u)_*([\tilde{X}, \tilde{Y}]) = (t_u)_*([X, Y]^\sim) = [X, Y]^*.$$

b The first relation follows directly from the definition; then, just note that

$$\tau_{ua}(X) = \sigma(X)(ua) = (R_a)_*(\sigma(ad(a)(X)(u))(ua) = ((R_a)_* \circ \tau_u \circ ad(a))(X).$$

Definition 2. Let P be a principal G-bundle over M and let $U \subset M$. A *section* of P over U is a map $\sigma: U \to P$ such that $\pi \circ \sigma = id_U$.

We have the following

Proposition 2. *P admits a section over U if and only if P is trivial over U.*

Proof. Assume $\sigma: U \to P$ is a section. Define $\Phi: \pi^{-1}(U) \to U \times G$ as $\Phi(u) = (\pi(u), \psi(u))$ where $\psi: \pi^{-1}(U) \to G$ is uniquely determined by the relation $u = \sigma(\pi(u))\psi(u)$.

Since, for every $a \in G$, we have

$$ua = \sigma(\pi(u))\psi(u)\,a = \sigma(\pi(ua))\psi(ua) = \sigma(\pi(ua))\psi(u)\,a$$

it follows that $\psi(ua) = \psi(u)\,a$ and thus Φ trivializes P over U.

Viceversa, if $\Phi: \pi^{-1}(U) \to U \times G$ trivializes P over U, the $\sigma(x) = \Phi^{-1}(x, e)$ is a section of P over U.

Note also that

$$\Phi^{-1}(x, a) = \sigma(x)\,a. \qquad (1.6)$$

Corollary 1. *A principal bundle admits global sections if and only if it is trivial.*

Let P be a principal G-bundle over M and let $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$ be a bundle atlas for P with $\Phi_i = (\pi, \psi_i)$. If $U_j \cap U_k \neq \emptyset$, then for every $u \in \pi^{-1}(U_j \cap U_k)$, $a \in G$ we have

$$\psi_j(ua)\,\psi_k(ua)^{-1} = \psi_j(u)\,aa^{-1}\psi_k^{-1}(u) = \psi_j(u)\,\psi_k(u)^{-1}$$

therefore $\psi_{jk}(\pi(u)) = \psi_j(u)\,\psi_k(u)^{-1}$ is a well defined function $\psi_{ij}\colon U_j \cap U_k \to G$.

E.g. in Example 2**c**, we have $\psi_{jk}([z_1, \ldots, z_{n+1}]) = \dfrac{|z_k|\,z_j}{|z_j|\,z_k}$. The ψ_{jk}'s are called the *transition functions* of P with respect to $\mathcal{U}$.

By definition we have $\psi_j = \psi_{jk}\psi_k$ and

$$(\Phi_j \circ \Phi_k^{-1})(x, a) = (x, \psi_{jk}(x)\,a)\,. \tag{1.7}$$

If $U_i \cap U_j \cap U_k \neq \emptyset$ then the corresponding transitions functions satisfy the so called cocycle condition

$$\psi_{jk}\psi_{ki}\psi_{ij} \equiv e\,. \tag{1.8}$$

The transition functions determine completely the principal bundle. In fact, we have the following

Proposition 3. *Assume we have a smooth manifold M, a Lie group G and an open covering $(U_i)_{i \in I}$ of M such that, for every pair $(j, k) \in I \times I$ for which $U_j \cap U_k \neq \emptyset$, a map $\psi_{jk}\colon U_j \cap U_k \to G$ is given so that the cocycle conditions* (1.8) *are satisfied (This assignement is called a* 1*-cocycle with value in G on the given covering). Then there exists a unique principal G-bundle P over M admitting the ψ_{jk}'s as transition functions (with respect to the given covering).*

Proof (sketch). For every $i \in I$, let $X_i = U_i \times G$ and let $X = \bigcup_{i \in I} X_i$. Consider on X the following equivalence relation:

$$(j, x, a) \sim (k, y, b) \quad \text{if and only if} \quad x = y \in U_j \cap U_k \quad \text{and} \quad a = \psi_{jk}(x)\,b\,.$$

One can easily show that $X/\sim$ is the required principal bundle. The uniqueness is obvious.

Definition 3. A *(bundle) morphism* $f: P(M, G) \to Q(N, H)$ between principal bundles consists of a pair of maps $f = (f', f'')$ where

$f': P \to Q$ is a smooth map $\quad f'': G \to H$ is a group homomorphism

in such a way that, for every $u \in P$, $a \in G$ we have

$$f'(ua) = f'(u) f''(a). \tag{1.9}$$

Note that (1.9) implies that f' is fibre-preserving and so a morphism $f: P \to Q$ induces a map $\tilde{f}: M \to N$.

Definition 4. A morphism $f: Q(M, H) \to P(M, G)$ between principal bundles over the same manifold is called a *reduction* of P to H if $\tilde{f} = id_M$ and f'' is injective (and so H is isomorphic to a subgroup of G). A morphism $f: P(M, G) \to Q(M, G)$ between principal bundles over the same manifold with the same structure group is called an *isomorphism* if f' is a diffeomorphism, $f'' = id_G$, $\tilde{f} = id_M$.

Proposition 4. *Two principal bundles $P(M, G)$ and $Q(M, G)$ over the same manifold with the same structure groups are isomorphic if and only if for any pair of bundles atlases $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$ on P, with transition functions $\{\psi_{jk}\}$ and $\mathcal{V} = (U_i, \Sigma_i)_{i \in I}$ on Q, with transition functions $\{\xi_{jk}\}$, for any $i \in I$, there exists a function $\lambda_i: U_i \to G$ such that $\psi_{jk} = \lambda_j^{-1} \xi_{jk} \lambda_k$ on $U_j \cap U_k \neq \emptyset$.*

Proof (sketch). Given an isomorphism $f: P(M, G) \to Q(M, G)$, define $\lambda_i: U_i \to G$ from the relation

$$(\Sigma_i \circ f' \circ \Phi_i^{-1})(x, e) = (x, \lambda_i(x)).$$

If the λ_i's are given, define $f': P \to Q$ on $\pi^{-1}(U_i)$ as

$$f'(u) = \Sigma_i^{-1}((\pi(u), \lambda_i(\pi(u)) \psi_i(u))).$$

Note that, as special case of Proposition 4, we have that a principal G-bundle P is isomorphic to the trivial bundle if and only if for any bundle atlas $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$, with transition functions $\{\psi_{jk}\}$ for any $i \in I$, there exists a function $\lambda_i: U_i \to G$ such that, on $U_j \cap U_k \neq \emptyset$, $\psi_{jk} = \lambda_j^{-1} \lambda_k$.

Let $\mathcal{C} = (U_i)_{i \in I}$ be an open covering of M and let $Z^1(\mathcal{C}, G)$ be the set of 1-cocycles on $\mathcal{C}$ with values in G. Consider on $Z^1(\mathcal{C}, G)$ the following equivalence relation:

$\{\psi_{jk}\} \sim \{\xi_{jk}\}$ if and only if for any $i \in I$, there exists a function $\lambda_i: U_i \to G$ such that $\psi_{jk} = \lambda_j^{-1} \xi_{jk} \lambda_k$ on $U_j \cap U_k \neq \emptyset$.

Let $H^1(\mathcal{A}, G) = Z^1(\mathcal{A}, G)/\sim$. Then $H^1(M, G) = \varinjlim H^1(\mathcal{A}, G)$ is in 1-1 correspondence with isomorphism classes of principal G-bundles on M.

Note that, in general, $H^1(M, G)$ it simply a set with a distinguished element $\{\psi_{jk} \equiv e\}$, corresponding to the trivial bundle.

Proposition 5. *Let $P(M, G)$ be a principal bundle and let H be a Lie subgroup of G. Then P is reducible to H if and only if there exists a bundle atlas for P with H-valued transitions functions.*

Proof.

a Assume P is reducible to H; therefore there exist a principal H-bundle $Q \xrightarrow{\rho} M$ and a morphism f: $Q(M, H) \to P(M, G)$ such that $f'' = id_H$ and $\tilde{f} = id_M$. In particular, Q can be considered as a submanifold of P.

Let $\mathcal{U} = (U_i, \Sigma_i)_{i \in I}$ be a bundle atlas for Q with

$$\Sigma_i\colon \rho^{-1}(U_i) \to U_i \times H \qquad \Sigma_i(u) = (\rho(u), \xi_i(u)).$$

It is easy to extend Σ_i to $\pi^{-1}(U_i)$. Given $v \in \pi^{-1}(U_i)$, we can write $v = ua$ for $u \in \rho^{-1}(U_i)$ and $a \in G$. Therefore

$$\Phi_i(v) = (\pi(v), \psi_i(v)) = (\pi(v), \xi_i(u)\, a)$$

defines an extension of Σ_i to $\pi^{-1}(U_i)$. Moreover

$$\psi_j(v)\, \psi_k(v)^{-1} = \xi_j(u)\, \xi_k(u)^{-1}$$

and thus the corresponding transition functions are H-valued.

b Starting from H-valued transition functions, we can construct $Q(M, H)$ as in Proposition 3 and then embed Q in P as follows

$$\rho^{-1}(U_i) \to U_i \times H \to U_i \times G \to \pi^{-1}(U_i).$$

Example 3. Let (M, g) be a n-dimensional Riemannian manifold and let

$$O_g(M) = \{u \in L(M) \,|\, g(u(\zeta), u(\xi)) = \langle \zeta, \xi \rangle \text{ for all } \zeta, \xi \in \mathbf{R}^n\}.$$

It is easy to check that (the embedding into $L(M)$ of) $O_g(M)$ is a reduction of $L(M)$ to $O(n)$. $O_g(M)$ is called the *bundle of g-orthogonal frames* over M.

Viceversa, if $Q(M, O(n))$ is a reduction to $O(n)$ of $L(M)$, then we can define a Riemannian metric g on M in the following way:

If $x \in M$, take any $u \in Q_x$ and set, for $X, Y \in T_x M$,

$$g(X, Y) = \langle u^{-1}(X), u^{-1}(Y) \rangle.$$

It is clear that $Q = O_g(M)$.

Note that, if h is another Riemannian metric on M, then $O_g(M)$ and $O_h(M)$ are isomorphic. More precisely there is an automorphism of $L(M)$ taking $O_g(M)$ into $O_h(M)$. In fact, for any $x \in M$, there exists an uniquely defined $L_x \in \mathrm{End}(T_x M)$ such that

a L_x is $g(x)$-symmetric and $g(x)$-positively defined.

b For every $X, Y \in T_x M$, $h(X, Y) = g(L(X), L(Y))$.

Because of the uniqueness, we can define $f: L(M) \to L(M)$ as $f(u) = L^{-1} \circ u$. Then

c f is a bijection with inverse map $v \mapsto L \circ v$.

d For every $u \in L(M)$, $a \in GL(n, \boldsymbol{R})$ we have $f(ua) = L^{-1} \circ u \circ a = f(u)a$ and so f is an automorphism of $L(M)$.

e If $u \in O_g(M)$ then $f(u) \in O_h(M)$. In fact, for every $\zeta, \xi \in \boldsymbol{R}^n$

$$h(f(u)(\zeta), f(u)(\xi)) = h(L^{-1}(u(\zeta)), L^{-1}(u(\xi))) = g(u(\zeta), u(\xi)) = \langle \zeta, \xi \rangle.$$

If M is *orientable*, we can further reduce $O_g(M)$ to $SO(n)$ just setting

$$SO_g(M) = \{u \in O_g(M) \mid u \text{ defines the fixed orientation}\}.$$

$SO_g(M)$ is called *the bundle of oriented g-orthonormal frames* on M.

As a special case, consider (S^n, std); then $SO_{\mathrm{std}}(S^n) = SO(n+1)$. In fact

$$S^n = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} \in \boldsymbol{R}^{n+1} \,\middle|\, \|x\| = 1 \right\}$$

and $A \in SO(n+1)$, $A = (A_1, \ldots, A_n, A_{n+1})$ encodes $p = A_{n+1} \in S^n$ and an oriented orthonormal frame of $T_p S^n$, namely $\{A_1, \ldots, A_n\}$.

$SO(n)$ acts on the right on $SO(n+1)$ via the embedding $C \mapsto \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$ and it is clear that

$$S^n = SO(n+1)/SO(n) \quad \text{and} \quad \pi(A) = Ae_{n+1}.$$

Consider $n = 3$; then the map

$$\sigma\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} t & z & -y & x \\ -z & t & x & y \\ y & -x & t & z \\ -x & -y & -z & t \end{pmatrix}$$

represents a global section of $SO_{\text{std}}(S^3)$ which trivializes it via $f\colon A \mapsto (Ae_4, {}^t\sigma(Ae_4)A)$.

Examples 4.

a The reduction of $L(M)$ to $O(1, n-1)$ corresponds to the assignement of a *Lorentzian structure* on M. In contrast with the Riemannian case (cf. Corollary 2), this is not always possible. In fact, we have the following result (cf. e.g. [**1**])

i) *any non compact manifold admits a Lorentzian structure*

ii) *a compact manifold admits a Lorentzian structure if (and, in the case it is orientable, only if) its Euler-Poincaré characteristic vanishes.*

b The reduction of $L(M)$ to $C(n) = \mathbf{R} \times O(n)$ corresponds to the assignement of a *conformal structure* on M.

c Assume $n = 2k$. The reduction of $L(M)$ to $GL(k, \mathbf{C})$ corresponds to the assignement of an *almost complex structure* J on M i.e. a smooth family of endomorphisms $J_x\colon T_xM \to T_xM$ such that $J_x^2 = -id_{T_xM}$.

In fact, given an almost complex structure J on M, then

$$L_{\mathbf{C}}(M) = \{u \in L(M) \mid u \circ J_k = J \circ u\}$$

(where $J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$) is a $GL(k, \mathbf{C})$-reduction of $L(M)$.

Viceversa, given a $GL(k, \mathbf{C})$-reduction Q of $L(M)$, then $J_x = u \circ J_k \circ u^{-1}$, $u \in Q_x$, defines an almost complex structure on M.

d Again in the case $n = 2k$, the reduction of $L(M)$ to $\mathrm{Sp}(k)$ corresponds to the assignement of an *almost symplectic structure* on M.

Proposition 6. *Given a principal bundle $P = P(M, G)$ and a smooth map h of a smooth manifold N into M, there is a unique (up to isomorphism) principal bundle $Q = Q(N, G)$ with a morphism $f = (f', f'')$: $Q(N, G) \to P(M, G)$ such that $\tilde{f} = h$ and $f'' = id_G$. Q is called the induced bundle and it is denoted by $h^*(P)$.*

Proof (sketch). Let $Q = \{(y, u) \in N \times P \mid h(y) = \pi(u)\}$. Then G acts on the right on Q by $(y, u)a = (y, ua)$.

It is easy to check that this action is free and Q is a principal G-bundle over N, with projection $\rho: Q \to N$ given by $\rho(y, u) = y$. Then just set $f'(y, u) = u$.

If $\hat{Q} \xrightarrow{\mu} N$ is another principal G-bundle over N and $\hat{f} = (\hat{f}', id_G)$: $\hat{Q} \to P$ is a morphism inducing h, then the map $\hat{u} \mapsto (\mu(\hat{u}), \hat{f}'(\hat{u}))$ defines an isomorphism between $\hat{Q}$ and Q.

It is easy to check that:

given $P = P(M, G)$ and $O \xrightarrow{k} N \xrightarrow{h} M$, then $(h \circ k)^*(P) = k^*(h^*(P))$

if $h: N \to M$ is an inclusion, clearly $h^*(P) = P_{|N}$

if $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$ is a bundle atlas with transition functions $\{\psi_{jk}\}$, then $f^*(P)$ is built up from the transition functions $\{\psi_{jk} \circ f\}$ on the open covering $(f^{-1}(U_i))_{i \in I}$.

As a special example of induced bundle, we can consider the so called *square* of a principal bundle P i.e. $\pi^*(P)$ over P; note that $\sigma(u) = (u, u)$ defines a global section of $\pi^*(P)$, which is, therefore, always trivial.

We are now in position to outline fundamental homotopy properties of principal bundles.

Let $P(M, G)$ be a principal bundle, then $P \times I$ has an obvious structure of principal G-bundle over $M \times I$ and its clear that:

if $p_1: M \times I \to M$ is the first projection, then $P \times I = p_1^*(P)$

for any $t \in I$, $P_{|M \times \{t\}} \cong P$.

Definition 5. Given two principal bundles $P = P(M, G)$ and $Q = Q(N, G)$ with the same structure group, then two morphisms $f_1, f_2: P \to Q$ are said to be *homotopic* if there exists a morphism called a *homotopy* between f_1 and f_2, $F: P \times I \to Q$ such that $F(\cdot, 0) = f_1$ and $F(\cdot, 1) = f_2$.

Now we have the following

Theorem 1. *Let $P = P(M, G)$ and $Q = Q(N, G)$ be two principal bundles with the same structure group, let $f\colon P \to Q$ be a morphism and let $\tilde{g}\colon M \to N$ be a map homotopic to $\tilde{f}$, via a homotopy $K\colon M \times I \to N$. There exist a morphism $g\colon P \to Q$ and a homotopy F between f and g such that $\tilde{F} = G$. Moreover, if $f'' = id_G$, then F can be choosen in such a way that $F'' = id_G$.*

Proof. See e.g. [7].

From Theorem 1, we obtain

Proposition 7. *Let S be a principal G-bundle over $M \times I$. Then there exists a principal G-bundle P over M such that S is isomorphic to $P \times I$.*

Proof. For $t \in I$, let $r_t\colon M \to M \times I$ be defined by $r_t(x) = (x, t)$, let $P = r_0^*(S)$ and let $f\colon P \to S$ be the induced map. Then $F(x, t) = (x, t)$ is a homotopy between r_0 and r_1 and the corresponding homotopy is clearly an isomorphism.

Finally, we have

Proposition 8. *Let $P = P(M, G)$ be a principal bundle and let h_1, h_2 be two homotopic maps of N into M, then $h_1^*(P)$ and $h_2^*(P)$ are isomorphic.*

Proof. Let H be a homotopy between h_1 and h_2, then $H^*(P)$ is isomorphic to $\widehat{P} \times I$ for a principal G-bundle $\widehat{P}$ over M. Then setting, for $t \in I$, $h_t = H \circ r_t$, we obtain

$$h_t^*(P) = r_t^*(H^*(P)) \cong r_t^*(\widehat{P} \times I) \cong \widehat{P}.$$

We conclude Section 1 with the following

Definition 6. Let $P(M, G)$ and $Q(M, H)$ be two principal bundles over the same manifold; then $P \times Q$ is a principal $G \times H$-bundle over $M \times M$. We define the *sum* of P and Q as

$$P + Q = i^*(P \times Q)$$

where $i\colon M \to M \times M$ is given by $i(x) = (x, x)$.

Note that there are natural bundle morphisms $f_P: P + Q \to P$, $f_Q: P + Q \to Q$, defined respectively by $f_P(u, v) = u$ and $f_Q(u, v) = v$. In general, neither P nor Q is embedded in $P + Q$.

2 - Associated fibre bundles

Let M be an n-dimensional smooth manifold, G a Lie group, F another differentiable manifold on which G acts on the left. Therefore, G acts freely on the right on $(M \times G) \times F$ in the following way

$$(x, g, \xi)\, a = (x, ga, a^{-1}\xi)$$

and clearly
$$((M \times G) \times F)/G = M \times F .$$

The previous construction represent the local model of the *associated fibre bundle* to a principal G-bundle with standard fibre F.

In general, let $P \xrightarrow{\pi} M$ be a principal G-bundle. Then G acts freely on the right on $P \times F$ as

$$(u, \xi)\, a = (ua, a^{-1}\xi) .$$

Let $E = P \times_G F = (P \times F)/G$. From the projection $\pi: P \to M$ a projection $\pi_E: E \to M$ is induced; namely $\pi_E([u, \xi]) = \pi(u)$.

In the same manner, a local trivialization of P, $\Phi: \pi^{-1}(U) \to U \times G$, $\Phi(u) = (\pi(u), \psi(u))$, induces a local trivialization of E. In fact, we have the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(U) \times F & \xrightarrow{(\Phi,\, id_F)} & (U \times G) \times F \\ \downarrow & & \downarrow \\ \pi_E^{-1}(U) = (\pi^{-1}(U) \times F)/G & \xrightarrow{\chi} & ((U \times G) \times F)/G = U \times F \end{array}$$

where χ is given as $\chi([u, \xi]) = (\pi(u), \psi(u)\,\xi)$.

$E = E(M, F, P, G)$ is called a *bundle* with standard fibre F and structure group G, associated to the principal G-bundle P.

For any $x \in M$, $E_x = \pi_E^{-1}(x)$ is called the *fibre* of E over x and, for any $U \subset M$, a *section* of E over U is a map $\sigma: U \to E$ such that $\pi_E \circ \sigma = id_U$. $E = M \times F$ is called the *trivial bundle*.

Example 5. Let $F = \boldsymbol{R}^n$ and let $G = GL(n, \boldsymbol{R})$ act on $\boldsymbol{R}^n$ by linear transformations. Then

$$E = L(M) \times_G \boldsymbol{R}^n = TM$$

is the *tangent bundle* of M.

Remark 1. Let $P = P(M, G)$ be a principal bundle and let $E = P \times_G F$ be an associated bundle. Then, given $x \in M$, we have

1 $u \in P_x$ can be interpreted as a map u: $F \to E_x$ simply setting $u(\xi) = [u, \xi]$. It is clear that

i) u is a bijection;

ii) for any $a \in G$, we have $ua(\xi) = [ua, \xi] = [u, a\xi] = u(a\xi)$.

2 $p = [u_0, \xi_0] \in E_x$ can be interpreted as a map p: $P_x \to F$ defined by the relation $[u, p(u)] = p$. It is clear that, for any $a \in G$, we have $p(ua) = a^{-1}p(u)$, i.e. p is G-equivariant.

Viceversa, a G-equivariant map from P_x to F gives rise to an element of E_x. We will greatly generalize this construction (cf. Proposition 25).

Examples 6.

a Let $G = GL(r, \boldsymbol{R})$ act on $\boldsymbol{R}^r$ by linear transformations and let $P = P(M, G)$ be a principal bundle; then $E = P \times_G \boldsymbol{R}^r$ is called *vector bundle* of rank r associated to P.

More in general, let V be a r-dimensional real vector space, let G be a Lie group, acting on V through a representation ρ: $G \to \mathrm{Aut}(V)$, let $P = P(M, G)$ be a principal bundle and let $E = P \times_\rho V$. Then, given $x \in M$, E_x admits an intrinsic structure of vector space:

If $\pi(u) = x$, then $\lambda[u, \zeta] + \mu[u, \xi] = [u, \lambda\zeta + \mu\xi]$.

It is clear that the map u: $V \to E_x$ is linear and so, fixing once for all a basis of V, u can be interpreted as a basis of E_x. Therefore, we can proceed exactly as in Example 1, setting

$$L(E) = \bigcup_{x \in M} L_x(E)$$

with $L_x(E) = \{$ordered basis of $E_x)$ and show that $L(E)$ is a principal $GL(r, \boldsymbol{R})$-

bundle over M, reducible as P to G. It is easy to realize that

$$E = L(E) \times_{GL(r, \boldsymbol{R})} \boldsymbol{R}^r .$$

The trivial vector bundle of rank r, $M \times \boldsymbol{R}^r$, will be denoted by $\theta_r(M)$.

b Let $G = O(r+1)$ or $G = SO(r+1)$ and let $P = P(M, G)$ be a principal bundle. Then

$$S = P \times_G S^r$$

is called a *sphere bundle* of rank r.

c Let H be a closed subgroup of G, let $P = P(M, G)$ be a principal bundle and let $Q(M, H)$ be an H-reduction of P. Then

$$P = Q \times_H G .$$

d Let G_C be the complexification of G. Given a principal bundle $P = P(M, G)$, then $P_C = P \times_G G_C$ is called the *complexification* of P.

Let $E = E(M, F, P, G)$ be a bundle and let $\chi_j \colon \pi_E^{-1}(U_j) \to U_j \times F$ and $\chi_k \colon \pi_E^{-1}(U_k) \to U_k \times F$ be two local trivializations of E with $U_j \cap U_k \neq \emptyset$. Then

$$\chi_j \circ \chi_k^{-1} \colon U_k \times F \to U_j \times F$$

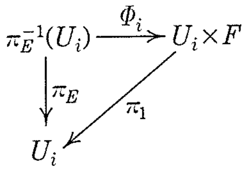
is given by

$$\chi_j \circ \chi_k^{-1}(x, \xi) = (x, \psi_{jk}(x)\, \xi) .$$

Therefore we have another possible definition of bundle.

Definition 7. Assume we have a smooth manifold E, called the total space, a smooth manifold M, called the base manifold, a smooth map $\pi_E \colon E \to M$, called the bundle projection, a smooth manifold F, called the standard fibre, and a Lie group G acting on F on the left. Then a *bundle atlas* of E consists of a system of local trivializations $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$, where $(U_i)_{i \in I}$ is an open covering of M, $\phi_i \colon \pi_E^{-1}(U_i) \to U_i \times F$ are diffeomorphisms, in such a way that:

a the diagram



is commutative

b if $U_j \cap U_k \neq \emptyset$, setting $\Phi_j \circ \Phi_k^{-1}(x, \xi) = (x, \psi_{jk}(x, \xi))$ and defining $\psi_{jk}(x)$: $F \to F$ by the relation $\psi_{jk}(x)(\xi) = \psi_{jk}(x, \xi)$, then $\psi_{jk}(x) \in G$ and the map $x \mapsto \psi_{jk}(x)$ is smooth.

Again, transition functions are enough to build up the bundle; in fact, using the same argument sketched in Proposition 3, we can prove the following

Proposition 9. *Assume we have a smooth manifold M, a Lie group G, and an open covering* $(U_i)_{i \in I}$ *of M such that for every pair* $(j, k) \in I \times I$ *for which* $U_j \cap U_k \neq \emptyset$ *a map* $\psi_{jk}: U_j \cap U_k \to G$ *is given so that the cocycle conditions* (1.8) *are satisfied.*

Then, if a smooth manifold F is given together with a left action of G, then there exists a unique bundle $E = E(F)$ *over M, with standard fibre F and structure group G, having the* ψ_{ij}*'s as transition functions (with respect to the given open covering). Moreover, for* $F = G$ *and the action given by left multiplication, we obtain a principal bundle* $P = P(M, G)$ *and* $E = P \times_G F$. *Finally, it is clear that* $E = P \times_G F$ *is trivial if and only if P is trivial.*

The notion of *induced bundle* extends in an obvious manner to *associated bundles* and clearly, if $E = P \times_G F$ and $f: N \to M$ is a smooth map, then $f^*(E) = f^*(P) \times_G F$.

Remark 2.

1 Let H be a closed subgroup of G, then G acts in a natural way on the left on G/H; therefore we have $P \times_G G/H \cong P/H$ through the map $[ua, [H]] \mapsto [ua]$. It is clear that, if H is a normal subgroup of G, then P/H is a principal G/H-bundle.

2 Given a bundle atlas $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$ in terms of local trivializations and transition functions, a section σ of E is given by a collection of maps

$\{\sigma_i\colon U_i \to F\}_{i \in I}$ such that, on $U_j \cap U_k \neq \emptyset$ we have

$$\sigma_j = \psi_{jk}\,\sigma_k \tag{2.1}$$

the relation being the following: on U_i, $\sigma(x) = \Phi_i^{-1}(x, \sigma_i(x))$. Therefore, if $v_0 \in F$ is kept fixed by G, the constant map $\sigma\colon M \to F$, $\sigma \equiv v_0$ induces a well defined global section of E; e.g. the zero section of TM.

Proposition 10. *A principal bundle $P = P(M, G)$ is reducible to the closed subgroup H of G if and only if the associated bundle $E = E(M, G/H, G, P)$ has global sections.*

Proof. Assume P is reducible to H as Q with embedding $i\colon Q \to P$. Let $\mu\colon P \to P/H = E$ be the projection; therefore $\mu \circ i\colon Q \to E$ is constant on the fibres and so it defines a section $\sigma\colon M \to E$.

Conversely if $\sigma\colon M \to E$ is a section, set $Q = \mu^{-1}(\sigma(M))$. One can easily check that Q is a H-reduction of P.

Note also that the correspondence between reductions and sections of the associated bundle is one-to-one.

Examples 7.

a Let M be a $2k$-dimensional manifold. We have seen (Example 4**c**) that a reduction of $L(M)$ to $GL(k, \mathbf{C})$ corresponds to the existence of an almost complex structure J on M. This can be identified with the section of $L(M)/GL(k, \mathbf{C})$ given by $x \mapsto [u]$ where $u \in L(M)_x$ satisfies $u \circ J_k \circ u^{-1} = J_x$; and viceversa.

b A principal bundle is trivial if and only if it is reducible to $\{e\}$.

The existence of global sections of a given bundle E is a general, natural question. For our purposes, it will be enough to recall the following

Proposition 11. *Let $E = E(M, F, P, G)$ be a bundle over a n-dimensional smooth manifold. Assume*

$$\pi_q(F) = 0 \qquad \textit{for } 1 \leqslant q \leqslant n-1. \tag{2.2}$$

Then E admits global sections.

Proof. See e.g. [2].

Corollary 2. *Any smooth manifold M admits a Riemannian structure.*

Proof. We have seen that the existence of a Riemannian structure on M is equivalent to the existence of a global section of $E = L(M)/O(n)$. Now, the standard fibre of E is diffeomorphic to a cell in $\mathbf{R}^p$ ($p = \frac{1}{2}n(n+1)$) and so, in particular, it satisfies (2.2).

More in general, the so called *Iwasawa's decomposition theorem* states, in particular, that any connected Lie group is diffeomorphic to the product of any of its massimal compact subgroups and a Euclidean space. Therefore, if G is connected, $P = P(M, G)$ can always be reduced to a maximal compact subgroup H.

Definition 8 (*operations on vector bundles*). Let E be a vector bundle over M of rank r.

1 $E^* = L(E) \times_{GL(r, \mathbf{R})} (\mathbf{R}^r)^*$ (where $GL(r, \mathbf{R})$ acts on the left on $(\mathbf{R}^r)^*$ via the action $L_a(\xi) = \xi a^{-1}$, which is *not* the dual action; call it std^{-1}) is called the *dual bundle* of E.

It is clear that, for each $x \in M$, $(E_x)^* = E_x^*$. Note also that the canonical bijection $T: L(E) \to L(E^*)$, $T: u \mapsto {}^t u^{-1}$ (i.e. if $u = \{a_1, \ldots, a_r\}$, then $T(u) = \{a_1^*, \ldots, a_r^*\}$ with $a_j^*(a_k) = \delta_{jk}$) is *not* a bundle isomorphism. In fact $T(ua) = T(u)a^{-1}$.

2 The natural action of $GL(r, \mathbf{R})$ extends to the tensor algebra $T(\mathbf{R}^r)$ and to the exterior algebra $\wedge \mathbf{R}^r$. Therefore we can define

$$E^{\otimes p} = L(E) \times_{GL(r, \mathbf{R})} (\mathbf{R}^r)^{\otimes p} \qquad \text{and} \qquad \wedge^p E = L(E) \times_{GL(r, \mathbf{R})} \wedge^p \mathbf{R}^r.$$

$\wedge^p(M)$ will denote the space of sections of $\wedge^p T^*M$, the bundle of exterior p-forms on M.

Let E_1, E_2 be two vector bundles over M of rank r_1 and r_2 respectively; then

3 $$E_1 \oplus E_2 = (L(E_1) + L(E_2)) \times_{GL(r_1, \mathbf{R}) \times GL(r_2 \mathbf{R})} \mathbf{R}^{r_1 + r_2}$$

is called the *direct sum* or *Whitney sum* of E_1 and E_2. For each $x \in M$, $(E_1 \oplus E_2)_x = E_{1x} \oplus E_{2x}$ and $L(E_1) + L(E_2)$ is a $GL(r_1, \mathbf{R}) \times GL(r_2, \mathbf{R})$-reduction of $L(E_1 \oplus E_2)$.

4 Let $\mathcal{U} = (U_i^{(\alpha)}, \Phi_i^{(\alpha)})_{i \in I}$ be a bundle atlas on $L(E_\alpha)$, with transition functions $\{\psi_{jk}^{(\alpha)}\}$, $\alpha = 1, 2$. Define $L(E_1) \otimes L(E_2)$ to be the principal $GL(r_1, \boldsymbol{R}) \otimes GL(r_2, \boldsymbol{R})$-bundle with transition functions $\{\psi_{jk}^{(1)} \otimes \psi_{jk}^{(2)}\}$; then

$$E_1 \otimes E_2 = (L(E_1) \otimes L(E_2)) \times_{GL(r_1, \boldsymbol{R}) \times GL(r_2 \boldsymbol{R})} \boldsymbol{R}^{r_1} \otimes \boldsymbol{R}^{r_2}$$

is called the *tensor product* of E_1 and E_2.

We will consider, in particular

$$\mathrm{Hom}(E_1, E_2) = E_2 \otimes E_1^*, \qquad \mathrm{End}(E) = \mathrm{Hom}(E, E) \quad \text{and} \quad \Lambda^p T^* M \otimes E,$$

the vector bundle of E-valued p-forms on M. $\Lambda^p(E)$ will denote the space of sections of $\Lambda^p T^* M \otimes E$.

Note also that $\mathrm{End}(E) = L(E) \times_{ad} gl(r, \boldsymbol{R})$ and, more in general, if E is given as $E = P \times_\varrho V$, then $\mathrm{End}(E) = P \times_{ad} \mathrm{End}(V)$.

Definition 9. Let E_1, E_2 be two vector bundles over M_1 and M_2 respectively. A *(bundle) morphism* between E_1 and E_2 consists of a pair of smooth maps $(h, \tilde{h})$ such that:

$\tilde{h}: M_1 \to M_2$ and $h: E_1 \to E_2$ is fibre-preserving, i.e. $\pi_{E_2} \circ h = \tilde{h} \circ \pi_{E_1}$ and for each $x \in M_1$, the restriction $h_x: E_{1x} \to E_{2\tilde{h}(x)}$ is linear.

It is clear that, in the case $M_1 = M_2 = M$, sections of $\mathrm{Hom}(E_1, E_2)$ correspond to bundle morphisms inducing the identity map on M.

Note that, e.g. if $\operatorname{rank} E_1 = \operatorname{rank} E_2$, then any morphism $f = (f', f'')$: $L(E_1) \to L(E_2)$ gives rise to a morphism $(h, \tilde{f})$: $E_1 \to E_2$, simply setting

$$h([u, \xi]) = [f'(u), \xi].$$

But clearly, not every morphism between E_1 and E_2 can be obtained in this way.

Definition 10. A morphism $(h, \tilde{h})$ between two vector bundles over the same manifold M is said to be *regular* if $\tilde{h} = id_M$ and $\operatorname{rank} h_x$ is constant. A regular morphism is an *isomorphism* if it is invertible. Given a vector bundle E over M, $\mathcal{E}nd(E)$ denotes the algebra of sections of $\mathrm{End}(E)$ and $\mathcal{A}ut(E)$ the group of invertible elements of $\mathcal{E}nd(E)$. Note that the subset of $\mathcal{E}nd(E)$ consisting of regular morphisms is *not* a vector space.

Definition 11. Let E, F be two vector bundles over M such that $E \subset F$. Then F is called a *vector subbundle* of E if the inclusion is a bundle morphism or, equivalently, for every $x \in M$, F_x is a vector subspace of E_x.

Definition 12. Let E be a vector bundle over M and let $F \subset E$ be a vector bundle. Assume $\operatorname{rank} F = r$ and $\operatorname{rank} E = r + p$. Consider

$$L_{r,p}(\boldsymbol{R}) = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in GL(r+p, \boldsymbol{R}) \right\}$$

and let
$$A(F) = \{u \in L(E) \mid u(\boldsymbol{R}^r) = F_{\pi(u)}\}.$$

then $A(F)$ is a $L_{r,p}(\boldsymbol{R})$-reduction of $L(E)$. Let now

$$H = \left\{ \begin{pmatrix} A & B \\ 0 & I_p \end{pmatrix} \in L_{r,p}(\boldsymbol{R}) \right\} \quad \text{and} \quad K = \left\{ \begin{pmatrix} I_r & B \\ 0 & C \end{pmatrix} \in L_{r,p}(\boldsymbol{R}) \right\}.$$

They are both normal subgroups of $L_{r,p}$ and it is easy to check that $A(F)/K = L(F)$.

Then
$$E/F = A(F)/H \times_{GL(p,\boldsymbol{R})} \boldsymbol{R}^p$$

is called the *quotient bundle* of E by F.

Note that $E \cong (E/F) \oplus F$, but *not* canonically.

We have now

Proposition 12. *Let E be a vector bundle of* $\operatorname{rank}(r+p)$ *over M. Then*

a *E admits subbundles of* $\operatorname{rank} r$ *if and only if $L(E)$ is reducible to $L_{r,p}$. More precisely, the following data are equivalent:*

1. *a subbundle $F \subset C$ of* $\operatorname{rank} r$ 2. *a $L_{r,p}$-reduction of $L(E)$*

and the correspondence is one-to-one.

b *E splits as direct sum of subbundles $E = E_1 \oplus E_2$ with* $\operatorname{rank} E_1 = r$, $\operatorname{rank} E_2 = p$, *if and only if $L(E)$ is reducible to $GL(r, \boldsymbol{R}) \times GL(p, \boldsymbol{R})$. More precisely, the following data are equivalent:*

1. *a pair of subbundles of E, (E_1, E_2) with* $\operatorname{rank} E_1 = r$, *and* $\operatorname{rank} E_2 = p$ *such that $E = E_1 \oplus E_2$*

2. *a $GL(r, \boldsymbol{R}) \times GL(p, \boldsymbol{R})$-reduction of $L(E)$*

and the correspondence is one-to-one.

c *E splits as direct sum of subbundles $E = E_1 \oplus E_2$ with* $\operatorname{rank} E_1 = r$, *and $E_2 \cong \theta_p(M)$, if and only if $L(E)$ is reducible to $GL(r, \boldsymbol{R})$. More precisely, the following data are equivalent:*

1. *a pair of subbundles of* E, (E_1, E_2) *with* $\operatorname{rank} E_1 = r$, *and* $E_2 \cong \theta_p(M)$ *such that* $E = E_1 \oplus E_2$

2. *a* $GL(r, \boldsymbol{R})$*-reduction of* $L(E)$

and the correspondence is one-to-one.

Proof.

a We have just seen that if $F \subset E$ is a subbundle of rank r, then $A(F)$ is a $L_{r,p}(\boldsymbol{R})$-reduction of $L(E)$; viceversa, given such a reduction Q, then $F = (Q/K) \times_{GL(r, \boldsymbol{R})} \boldsymbol{R}^r$ is a subbundle of rank r of E, such that $A(F) = Q$.

b If $E = E_1 \oplus E_2$, we have already pointed out that $L(E_1) + L(E_2)$ is a $GL(r, \boldsymbol{R}) \times GL(p, \boldsymbol{R})$-reduction of $L(E)$. Viceversa, let Q be such reduction; then

$$E_1 = Q \times_{GL(r, \boldsymbol{R})} \boldsymbol{R}^r \qquad E_2 = Q \times_{GL(p, \boldsymbol{R})} \boldsymbol{R}^p$$

clearly satisfy $E = E_1 \oplus E_2$ and $Q = L(E_1) + L(E_2)$.

c Assume $E = E_1 \oplus E_2$ with $E_2 \cong \theta_p(M)$; then $L(E_2)$ is reducible to $\{e\}$ and thus, both $L(E_1) + L(E_2)$ and $L(E)$ are reducible to $GL(r, \boldsymbol{R})$. Viceversa let Q be a $GL(r, \boldsymbol{R})$-reduction of $L(E)$; then, the corresponding section of $L(E)/GL(r, \boldsymbol{R})$, $x \mapsto [u(x)]$, represents a set $\{s_1, \ldots, s_p\}$ of everywhere linearly independent sections of E, defined by

$$s_j(x) = u(x)(e_{r+j}) \qquad 1 \leqslant j \leqslant p\,.$$

Therefore

$$E_2 = \operatorname{span}(s_1, \ldots, s_p)$$

is a trivial vector subbundle of E. Then just set $E_1 = Q \times_{GL(r, \boldsymbol{R})} \boldsymbol{R}^r$.

Remark 3.

1 If E admits a subbundle F of rank r, and so $L(E)$ is reducible to $L_{r,p}(\boldsymbol{R})$, then it is further reducible to $GL(r, \boldsymbol{R}) \times GL(p, \boldsymbol{R})$. In fact

$$L_{r,p}(\boldsymbol{R})/GL(r, \boldsymbol{R}) \times GL(p, \boldsymbol{R}) = M_{r,p}(\boldsymbol{R}) \cong \boldsymbol{R}^{rp}$$

any such reduction corresponding to a choice of a subbundle $F' \subset E$ of rank p such that $E = F \oplus F'$.

2 Let $G_r(\boldsymbol{R}^{r+p}) = GL(r+p, \boldsymbol{R})/L_{r,p}(\boldsymbol{R})$ be the Grassmann manifold of r-planes in $\boldsymbol{R}^{r+p}$ and let

$$G_r(E) = L(E)/L_{r,p}(\boldsymbol{R}) = L(E) \times_{GL(r+p,\boldsymbol{R})} G_r(\boldsymbol{R}^{r+p})$$

be the corresponding *Grassmann bundle*. Then $G_r(E)_x$ is the Grassmann manifold of linear r-subspaces of E_x.

Let $F \subset E$ be a subbundle of rank r and let Q be the corresponding $L_{r,p}$-reduction of $L(E)$. Then Q itself corresponds to a section of $G_r(E)$: the one assigning to every $x \in M$, $F_x \in G_r(E)_x$. In this sense, we can refer to a subbundle as the datum of $F_x \subset E_x$, varying smoothly with x.

Corollary 3. *Let E be a vector bundle of* rank $(n+p)$ *over an n-dimensional smooth manifold M; then E splits as $E = E_1 \oplus E_2$ with* $\operatorname{rank} E_1 = n$ *and $E_2 \cong \theta_p(M)$.*

Proof. Consider the homogeneous space

$$\widehat{V}_p(\boldsymbol{R}^{n+p}) = GL(n+p, \boldsymbol{R})/GL(n, \boldsymbol{R}).$$

$\widehat{V}_p(\boldsymbol{R}^{n+p})$ is called the *Stiefel manifold* of p-frames on $\boldsymbol{R}^{n+p}$.

It is known (cf. e.g. [8]) that $\widehat{V}_p(\boldsymbol{R}^{n+p})$ satisfies condition (2.2) and therefore, by Propositions 10 and 11, $L(E)$ is reducible to $GL(n, \boldsymbol{R})$ and so E splits in the desired manner.

Similarly to Proposition 12 we have

Proposition 13. *Let E be a vector bundle of* rank rp *over M; then $E = E_1 \otimes E_2$ with* $\operatorname{rank} E_1 = r$ *and* $\operatorname{rank} E_2 = p$, *if and only if $L(E)$ is reducible to $GL(r, \boldsymbol{R}) \otimes GL(p, \boldsymbol{R})$.*

The proof of the following proposition is left as an exercise

Proposition 14. *Let $h: E_1 \to E_2$ be a regular morphism between vector bundles over M; then* $\ker h$, $\operatorname{Im} h$, $\operatorname{coker} h$ *(obvious definitions) are vector bundles over M.*

Definition 13. Let G be a Lie group and let $\rho: G \to GL(r, \boldsymbol{R})$ be a representation; let E be a vector bundle of rank r over M. A *(G, ρ)-structure* (or, simply, a *G-structure*) on E is the data of a principal G-bundle P over M and of a morphism $f: P \to L(E)$ such that $f'' = \rho$ and $\tilde{f} = id_M$.

Proposition 15. *If a G-structure on E is given, then* $P \times_\rho \boldsymbol{R}^r$ *is isosmorphic to E.*

Proof. We need only to verify that $h: P \times_\rho \boldsymbol{R}^r \to E$ given by $h([u, \xi]) = [f(u), \xi]$ is a bijection.

h is injective: If $f(u) = f(v)$, then $u = vg$ for some $g \in G$ such that $\rho(g) = e$. Therefore

$$[u, \xi] = [vg, \xi] = [v, \rho(g^{-1})\xi] = [v, \xi]$$

h is surjective: Given $[w, \xi] \in E$, let $u \in P_{\pi(w)}$. Therefore, $w = f(u)\,a$ for some $a \in GL(r, \boldsymbol{R})$ and so $[w, \xi] = h([u, a\xi])$.

Examples 8.

a Reductions are obviously examples of G-structures.

b Among the G-structures which are not reductions, let us simply mention *Spin-structures* and *Spin*^{c}*-structures* (cf. e.g. [5]).

c Assume $\rho: G \to GL(r, \boldsymbol{R})$ is the trivial representation, i.e. $\rho \equiv e$. Then a G-structure gives rise to a trivialization of $L(E)$: define $\sigma: M \to L(E)$ as $\sigma(x) = f(u)$ for any $u \in P_x$. More in general, a G-structure gives rise to a $G/\ker\rho$-reduction of $L(E)$, namely $P/\ker\rho$.

Remark 4. Exactly the same argument of the proof of Proposition 15, can be used in the following situation. Let $P = P(M, G)$ be a principal bundle, let F be any manifold on which G acts on the left and let $E = P \times_G F$ be the associated bundle. Assume the given action is not effective, i.e. the underlying representation $\rho: G \to \mathrm{Diff}(F)$ is not faithful.

Then $H = G/\ker\rho$ acts effectively on F, $Q = P/\ker\rho$ is a H-principal bundle and $Q \times_H F$ is canonically isomorphic to E. Therefore any associated bundle can be presented as a bundle associated to a principal bundle through a faithful representation.

Note that, if the action we started with is trivial, then $E \cong M \times F$.

We want to conclude Section 2 with two other constructions, which will be useful in later developments.

Definition 14. We say that a principal bundle $P = P(M, G)$ admits a *flat structure* if there exists a representation $\rho: \pi_1(M) \to G$ such that $P = \tilde{M} \times_\rho G$.

We have the following characterization of principal bundles admitting flat structures:

Proposition 16. *A principal bundle $P = P(M, G)$ admits a flat structure, if and only if there exists a bundle atlas with constant transition functions.*

Proof. If $P = \tilde{M} \times_\rho G$, let $H = \pi_1(M)/\ker\rho$; then the induced map $\hat{\rho}$: $H \to G$ is a faithful representation and $\hat{M} = \tilde{M}/\ker\rho$ is a covering space for M and a H-reduction of P. Clearly the transition functions of $\hat{M}$, considered as transition functions of P, are constant. We will give a short proof of the converse implication in Proposition 38.

Finally, we have

Definition 15. We say that a principal bundle $P = P(M, G)$ admits a *projectively flat* structure if $\boldsymbol{P}(P) = P/C(G)$ ($C(G)$ being the center of G) admits a flat structure.

3 - The gauge group

Definition 16. Let $P = P(M, G)$ be a principal bundle; the group of automorphisms of P is called the *gauge group* of P and it is denoted by $\mathcal{G}(P)$.

Proposition 17. *There is a natural anti-isomorphism between $\mathcal{G}(P)$ and the group of sections of the gauge bundle of P, $\mathcal{G}_P = P \times_{Ad} G$.*

Proof. We have seen (cf. Remark 1,2) that the group of sections of $\mathcal{G}_P$ is identified with the group of G-equivariant (with respect to the adjoint action on G) maps σ: $P \to G$ i.e. those maps such that $\sigma(ua) = a^{-1}\sigma(u)\,a$.

If σ is such a map, define $f \in \mathcal{G}(P)$ as $f(u) = u\sigma(u)$. Viceversa, if $f \in \mathcal{G}(P)$, then σ: $P \to G$ determined by the equation $f(u) = u\sigma(u)$ is G-equivariant.

It is immediate to check that this correspondence is an anti-isomorphism.

Definition 17. The bundle $\mathfrak{A}_P = P \times_{ad} \mathfrak{G}$ is called the *adjoint bundle* of P.

Assume a representation ρ: $G \to \mathrm{Aut}(V)$ is given. Let $P = P(M, G)$ be a principal bundle and $E = P \times_\rho \boldsymbol{R}^r$. Since $\mathrm{End}(E) = P \times_{ad} \mathrm{End}(V)$ (cf. Definition

8,4), then both $\mathcal{G}_P$ and $\mathfrak{A}_P$ are naturally mapped in End (E), respectively by $[u, g] \mapsto [u, \rho(g)]$ and $[u, X] \mapsto [u, \rho_*(X)]$. Of course, if ρ is faithful, both these maps are embeddings.

Examples 9. Let E be a vector bundle of rank r over M.

a Let g be a Riemannian structure on E; then $L(E)$ is reducible to $G = O(r)$ as

$$P = O_g(E) = \{u \in L(E) \mid g(u(\zeta), u(\xi)) = \langle \zeta, \xi \rangle \text{ for all } \zeta, \xi \in \boldsymbol{R}^r\}.$$

Then the sections of $\mathcal{G}_P$ are the g-orthogonal elements of $\mathcal{A}ut(E)$ and the sections of $\mathfrak{A}_P$ are the g-skew-symmetric elements of $\mathcal{E}nd(E)$.

b Assume $r = 2q$ and let J be a structure of complex vector bundle on E i.e. $J \in \mathcal{A}ut(E)$ with $J^2 = -id_E$. Then $L(E)$ is reducible to $G = GL(q, \boldsymbol{C})$ as

$$P = L_{\boldsymbol{C}}(E) = \{u \in L(E) \mid u \circ J_q = J \circ u\}.$$

The sections of $\mathcal{G}_P$ (resp. of $\mathfrak{A}_P$) are the elements of $\mathcal{A}ut(E)$ (resp. $\mathcal{E}nd(E)$) commuting with J.

We have also the following

Proposition 18. *The fundamental vertical vector fields on a principal bundle $P(M, G)$ are gauge-invariant, i.e., for any $X \in \mathfrak{G}$, any $f \in \mathcal{G}(P)$, we have*

$$f_*(X^*) = X^* \tag{3.1}$$

or, more precisely, for any $u \in P$,

$$f_*[u](X^*(u)) = X^*(f(u)).$$

Proof. By definition, $f_*[u](X^*(u))$ is the tangent vector at the point $t = 0$ to the curve

$$\gamma(t) = f(u \exp(tX)) = f(u) \exp(tX).$$

This proves our assertion.

Chapter 2 - Connections on Principlal Bundles

4 - Definitions and examples

Let $P(M, G)$ be a principal bundle and let, as usual, $\pi\colon P \to M$ be the bundle projection. Consider

$$\pi^*(TM) = \{(u, \xi) \in P \times TM \mid \xi \in T_{\pi(u)}M\}$$

and let $\alpha\colon TP \to \pi^*(TM)$ be defined as follows:

if $u \in P$ and $X \in T_u P$, then $\alpha(X) = (u, \pi_*[u](X))$.

Observe that:

G acts both on TP and $\pi^*(TM)$, the former being the induced action, the latter being $R_a(\xi, u) = (ua, \xi)$.

α is G-equivariant.

Clearly, we have:

α is a regular surjective morphism.

$\ker\alpha = W = \{X \in TP \mid X \text{ is vertical}\}$.

Consequently, we have the exact sequence

$$0 \to W \to TP \xrightarrow{\alpha} \pi^*(TM) \to 0. \tag{4.1}$$

Note also that the map $\varepsilon\colon P \times \mathfrak{G} \to W$ given by

$$\varepsilon(u, X) = X^*(u) \tag{4.2}$$

is an isomorphism and so W is canonically isomorphic to a trivial bundle.

Roughly speaking, a connection on P is a G-equivariant splitting of (4.1), i.e. a G-invariant realization of $\pi^*(TM)$ in TP as direct summand of W.

We have the following

Proposition 19. *The following three assignements are equivalent.*

1 *A G-equivariant regular morphism $\Gamma\colon \pi^*(TM) \to TP$ such that* $\alpha \circ \Gamma = id_{\pi^*(TM)}$.

2 *A G-invariant subbundle H of TP, such that* $TP = H \oplus W$.

3 *A G-invariant* $GL(k, \boldsymbol{R}) \times GL(n, \boldsymbol{R})$*-reduction Q of A(W), where* $k = \dim_{\boldsymbol{R}} G$ *and A(W) is the* $L_{k,n}(\boldsymbol{R})$*-reduction of L(P) corresponding to W.*

Proof. Assume Γ is given; then just set $H = \Gamma(\pi^*(TM))$. If H is given, then define $\Gamma = (\alpha_{|H})^{-1}$; therefore **1** and **2** result to be equivalent.

Finally **2** and **3** are equivalent because of Remark 3.

We set now a more usual definition, corresponding to the datum of Proposition 19, **2**.

Definition 18. A *connection* Γ on P consists of the assignement of a G-invariant subbundle H of TP such that $TP = H \oplus W$. Therefore, for every $\in P$, a subspace H_u of $T_u P$ is assigned, in such a way that:

$H_u \oplus W_u = T_u P$.

For every $a \in G$, $H_{ua} = (R_a)_*(H_u)$.

H_u depends smoothly on u.

H_u is called the *horizontal subspace* (with respect to Γ) of $T_u P$.

Therefore, if a connection Γ is given, let $h_\Gamma : TP \to H$ be the induced projection; then for every $X \in T_u P$, we have the decomposition into *horizontal* and *vertical* components

$$X = X^{(h)} + X^{(v)} \tag{4.3}$$

with $X^{(h)} = h_\Gamma[u](X) \in H_u$ and $X^{(v)} \in W_u$.

It is clear also that $\pi_*[u]$ maps isomorphically H_u into $T_{\pi(u)} M$.

Examples 10.

1 Let $M = S^n$ and $P(M, G) = SO_{\text{std}}(S^n) = SO(n+1)$ (cf. Example 3). If $P \in SO(n+1)$, then

$$T_P SO(n+1) = \{PA \mid A \in so(n+1)\}.$$

We have the decomposition $so(n+1) = so(n) \oplus \boldsymbol{R}^n$ via the embedding

$$\boldsymbol{R}^n \to so(n+1) \qquad \xi \mapsto \begin{pmatrix} 0 & \xi \\ -{}^t\xi & 0 \end{pmatrix}$$

and so the induced decomposition $T_P SO(n+1) = H_P \oplus W_P$ where, of course,

$$H_P = \left\{ P\begin{pmatrix} 0 & \xi \\ -{}^t\xi & 0 \end{pmatrix} \middle| \xi \in \boldsymbol{R}^n \right\} \quad \text{and} \quad W_P = \{PB \mid B \in so(n)\},$$

defines a connection. In fact, for every $A \in SO(n)$, we have

$$(R_A)_*(H_P) = \left\{ P\begin{pmatrix} 0 & \xi \\ -{}^t\xi & 0 \end{pmatrix} A \middle| \xi \in \boldsymbol{R}^n \right\} = \left\{ PAA^{-1}\begin{pmatrix} 0 & \xi \\ -{}^t\xi & 0 \end{pmatrix} A \middle| \xi \in \boldsymbol{R}^n \right\}$$

$$= \left\{ PA\begin{pmatrix} 0 & A^{-1}\xi \\ -{}^t(A^{-1}\xi) & 0 \end{pmatrix} \middle| \xi \in \boldsymbol{R}^n \right\} = H_{PA}.$$

Note that $T_P SO(n+1) = \{AP \mid A \in so(n+1)\}$, but $\{BP \mid B \in so(n)\} \neq W_P$.

2 Let $P = M \times G$ be the trivial bundle; then at the point (x, a) set $H^{cf}_{(x,a)} = T_x M \times \{0\} \subset T_{(x,a)} M \times G$. This defines a connection, called the *canonical flat* connection.

Definition 19. Assume a connection Γ is given on P; then let ω be the $\mathfrak{G}$-valued 1-form on P defined as follows: for $u \in P$ and $X \in T_u P$ set

$$\omega[u](X) = \tau_u^{-1}(X^{(v)}). \tag{4.4}$$

ω is called the *connection* 1-*form* of Γ.

It is clear that $\omega[u](X) = 0$, if and only if $X \in H_u$. Moreover, we have the following

Proposition 20. *The connection* 1-*form* ω *of* Γ *satisfies the following properties*:

$$\omega(X^*) = X \quad \textit{for every } X \in \mathfrak{G} \tag{4.5}$$

ω *is G-equivariant, i.e. for every* $a \in G$, *we have*

$$(R_a)^*(\omega) = ad(a^{-1})\,\omega \tag{4.6}$$

or, more precisely, for every $X \in T_u P$,

$$((R_a)^*(\omega))[u](X) = ad(a^{-1})(\omega[u](X)).$$

Viceversa, if a $\mathfrak{G}$*-valued* 1*-form* ω *on* P *is given in such a way that* (4.5) *and* (4.6) *hold, then there exists a unique connection* Γ *on* P *having* ω *as connection* 1*-form.*

Proof. (4.5) is obvious. For $u \in P$ and $X = X^{(h)} + X^{(v)} \in T_u P$, we have

$$((R_a)_*(\omega))[u](X^{(h)}) = \omega[ua]((R_a)_*(X^{(h)})) = 0 = ad(a^{-1})\,\omega[u](X^{(h)})$$

and, recalling Proposition 1 **b**

$$((R_a)_*(\omega))[u](X^{(v)}) = \omega[ua]((R_a)_*(X^{(v)})) = \tau_{ua}^{-1}((R_a)_*(X^{(v)}))$$

$$= ad(a^{-1})(\tau_u^{-1}(X^{(v)}) = ad(a^{-1})\,\omega[u](X^{(v)}).$$

Viceversa, it is easy to check that, given ω satisfying (4.5) and (4.6), then $H_u = \ker \omega[u]$ defines a connection Γ on P, having ω as connection 1-form; the uniqueness is obvious.

Remark 5. Let ω be a connection 1-form on P; then

1 For every $u \in P$, $\omega[u] \circ \tau_u = \vartheta$, the canonical $\mathfrak{G}$-valued 1-form on G. This amounts also to the following: consider the canonical isomorphism $\varepsilon\colon P \times \mathfrak{G} \to W$ given by (4.2) and set $\varepsilon^{-1} = (\nu_1, \nu_2)$ with $\nu_1 = \pi_W$ and $\nu_2[u] = \tau_u^{-1}$. Therefore $\omega_{|W} = \nu_2$ and thus a connection 1-form is nothing but a G-equivariant extension ν_2 to TP.

2 For every $a \in G$, every $X \in T_{ua}P$, we have

$$\omega[ua](X) = ad(a^{-1})\,\omega[u]((R_{a^{-1}})_*(X)). \tag{4.7}$$

Note also that, in general, we have $X^{(h)} = X - (\omega(X))^*$.

3 In Example 10, 1

$$\omega[P](X) = ({}^tPX)_{so(n)} = so(n)\text{-component of } {}^tPX$$

i.e. if $X = P\begin{pmatrix} K & \xi \\ -{}^t\xi & 0 \end{pmatrix}$ then $\omega[P](X) = K$.

4 The connection 1-form of the canonical flat connection is simply the trivial extension of ϑ to $M \times G$ i.e. it is given by $\varphi = \pi_2^*(\vartheta)$, where $\pi_2\colon M \times G \to G$ is the natural projection.

We have the following, easy to prove result (cf. e.g. [3])

Proposition 21. *Any principal bundle admits connections.*

We denote by $\mathcal{C}(P)$ the set of connection 1-forms on the principal bundle P.

Definition 20. Let $P = P(M, G)$ be a principal bundle equipped with a connection Γ and let $X \in \mathcal{X}(M)$. Then the *horizontal lift* of X is the element $\widehat{X} \in \mathcal{X}(P)$ characterized by the following two properties

1 For every $u \in P$, $\widehat{X}(u) \in H_u$ i.e. $\widehat{X}$ is horizontal

2 $\pi_* (\widehat{X}(u)) = X(\pi(u))$.

It easy to prove the following

Proposition 22.

a *Given $X \in \mathcal{X}(M)$, there is a unique horizontal lift $\widehat{X}$ of X; moreover $\widehat{X}$ is G-invariant. Viceversa, any horizontal, G-invariant $\widehat{X} \in \mathcal{X}(P)$ is the horizontal lift of some $X \in \mathcal{X}(M)$.*

b *Let $X, Y \in \mathcal{X}(M)$; then*

i) $(X + Y)^\frown = \widehat{X} + \widehat{Y}$

ii) *If $f\colon M \to \boldsymbol{R}$ and $\widehat{f} = f \circ \pi$, then $(fX)^\frown = \widehat{f} \circ \widehat{X}$*

iii) $[X, Y]^\frown = [\widehat{X}, \widehat{Y}]^{(h)}$.

Remark 6. We want to stress the fact that, in general, $[\widehat{X}, \widehat{Y}]$ is not horizontal and so the distribution of horizontal subspaces is not involutive.

Proposition 23. *Let $X \in \mathfrak{G}$. If $Y \in \mathcal{X}(P)$ is horizontal, then $[Y, X^*]$ is horizontal. Moreover if $Z \in \mathcal{X}(M)$, then $[\widehat{Z}, X^*] = 0$.*

Proof. Let $u \in P$; then, for every $Y \in \mathcal{X}(P)$, we have

$$[Y, X^*] = \frac{\mathrm{d}}{dt} (R_{\exp(tX)})_* [(R^{-1}_{\exp(tX)})(u)](Y(R^{-1}_{\exp(tX)})(u))_{|t=0} .$$

Therefore, if Y is horizontal, then $(R_{\exp(tX)})_* [(R^{-1}_{\exp(tX)})(u)](Y(R^{-1}_{\exp(tX)})(u))$ is horizontal and so is $[Y, X^*]$. If $Y = \widehat{Z}$, then $(R_{\exp(tX)})_* [(R^{-1}_{\exp(tX)})(u)](Y(R^{-1}_{\exp(tX)})(u))$ is constant and so $[\widehat{Z}, X] = 0$.

Let $P = P(M, G)$ be a principal bundle, Γ a connection on P and ω its connection 1-form. Let $\mathfrak{U} = (U_i, \Phi_i)_{i \in I}$ be a bundle atlas with transition functions $\{\psi_{jk}\}$.

For every $i \in I$, let $\sigma_i \colon U_i \to P$ be the section corresponding to Φ_i, i.e. $\sigma_i(x) = \Phi_i^{-1}(x, e)$, and set

$$\omega_i = \sigma_i^*(\omega)$$

the ω_i's are called *local gauge potentials* of Γ with respect to $\mathfrak{U}$.

Finally, set, for $U_j \cap U_k \neq \emptyset$, $\vartheta_{jk} = \psi_{jk}^*(\vartheta)$. We have the following

Proposition 24. *On $U_j \cap U_k \neq \emptyset$, we have*

$$\omega_k = ad(\psi_{jk}^{-1})\, \omega_j + \vartheta_{jk}\,. \tag{4.8}$$

Viceversa, if for every $i \in I$, a $\mathfrak{G}$-valued 1-form ω_i is given in such a way that on $U_j \cap U_k \neq \emptyset$ (4.8) holds, then there is a unique connection Γ having the ω_i's as local gauge potentials.

Proof. See e.g. [3].

Example 11. Assume G is a matrix group; then $\vartheta[A](X) = A^{-1}X$ and therefore $\vartheta_{jk} = \psi_{jk}^*(\vartheta) = \psi_{jk}^{-1}\mathrm{d}\psi_{jk}$ and so (4.8) becomes

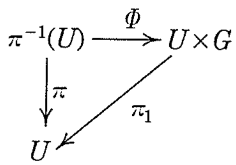
$$\omega_k = \psi_{jk}^{-1}\omega_j\psi_{jk} + \psi_{jk}^{-1}\mathrm{d}\psi_{jk} = \psi_{jk}^{-1}[\omega_j\psi_{jk} + \mathrm{d}\psi_{jk}]\,.$$

In the special case G is abelian (e.g. $G = S^1$), we have:

$$\mathrm{d}\vartheta = 0 \quad \text{and so} \quad \mathrm{d}\vartheta_{jk} = 0 \qquad \omega_k = \omega_j + \vartheta_{jk}$$

and thus $F = \mathrm{d}\omega_k = \mathrm{d}\omega_j$ is a well defined $\mathfrak{G}$-valued 2-form on M.

We want to describe ω and the horizontal lifting in terms of local trivializations. Let



be a local trivialization diagram for the principal bundle P and let σ be the induced local section. Given a connection 1-form ω on P, set $\omega_U = \sigma^*(\omega)$. Therefore, for every $x \in U$, $a \in G$, $\xi \in T_x M$, $X \in T_a G$, we have

$$((\Phi^{-1})^*(\omega))[(x, a)](\xi, X) = \omega[\sigma(x)\, a](\Phi_*^{-1}[x, a](\xi, X))$$

$$= ad(a^{-1})\, \omega[\sigma(x)]((R_{a^{-1}})_* (\Phi_*^{-1}[x, a](\xi, X)))$$

$$= ad(a^{-1})\, \omega[\sigma(x)](\Phi_*^{-1}[x, e](\xi, (R_{a^{-1}})_* (X))$$

$$= ad(a^{-1})\, \omega_U[x](\xi) + ad(a^{-1})\, \omega[\sigma(x)]((R_{a^{-1}})_* (X))^*$$

$$= (L_{a^{-1}})_*((R_a)_*(\omega_U[x](\xi)) + X) = ad(a^{-1})\, \omega_U[x](\xi) + \vartheta[a](X).$$

Therefore $\Phi_*^{-1}[x, a](\xi, X)$ is horizontal if and only if $X = -(R_a)_* \omega_U[x](\xi)$ and so in the given local trivialization, the horizontal lift of $\xi \in \mathcal{X}(M)$ at the point (x, a) is provided by

$$\xi - (R_a)_* \omega_U[x](\xi) \tag{4.9}$$

we have also

$$\hat{\xi}(\sigma(x)) = \sigma_*[x](\xi) - (\omega_U[x](\xi))^*. \tag{4.10}$$

Example 12. Let $O_g(M)$ be the principal bundle of orthonormal frames over the Riemannian manifold (M, g). Let $\Phi: \pi^{-1}(U) \to U \times O(n)$ be a local trivialization with induced local section σ; therefore $\sigma(x) = \{\eta_1(x), \ldots, \eta_n(x)\}$ is a orthonormal frame on U; let $\{\eta_1^*(x), \ldots, \eta_n^*(x)\}$ be the dual frame.

Given a connection Γ on $O_g(M)$ and its connection 1-form ω, we have

$$\omega_U = \sigma^*(\omega) = \sum_{i=1}^{n} \Gamma^j_{ik} \eta_i^* \tag{4.11}$$

where $$\Gamma_{(i)} = (\Gamma^j_{ik}) \in o(n).$$

Therefore, for $a = (a_{jk}) \in O(n)$

$$H_i(x, a) = \eta_i(x) - \sum_{h=1}^{n} \Gamma^j_{ih}(x)\, a_{hk} \tag{4.12}$$

represents the horizontal lift of η_i at the point (x, a) expressed in terms of the given local trivialization.

Remark 7. Let $\mathcal{U} = (U_i, \Phi_i)_{i \in I}$ be a bundle atlas for P with transition functions $\{\psi_{jk}\}$ and let $\omega \in \mathcal{C}(P)$. Assume that, for every $i \in I$, we have

$$\omega_{|\pi^{-1}(U_i)} = \Phi_i^* (\varphi_i) \tag{4.13}$$

where φ_i is the connection 1-form of the canonical flat connection on $U_i \times G$. (4.13) is equivalent to say that for every $j, k \in I$ such that $U_j \cap U_k \neq \emptyset$, for every $u \in \pi^{-1}(U_j \cap U_k)$, we have

$$(\Phi_j \circ \Phi_k^{-1})_* [\Phi_k(u)]: H^{cf}_{\Phi_k(u)} \to H^{cf}_{\Phi_j(u)}$$

i.e., for every $x \in U_j \cap U_k$, $\xi \in T_x M$, $a \in G$, we have

$$(\Phi_j \circ \Phi_k^{-1})_* [x, a](\xi, 0) = (\xi, 0) .$$

Now $$(\Phi_j \circ \Phi_k^{-1})_* [x, a](\xi, X) = (\xi, (\psi_{jk})_* [x](\xi)\, a + (L_{\psi_{jk}(x)})_* (X))$$

and so (4.13) is equivalent to say that the transitions functions ψ_{jk} are constant.

5 - Tensorial forms. Exterior differential operator

Let $P = P(M, G)$ be a principal fibre bundle and let V be a finite dimensional vector space equipped with a representation $\rho: G \to \mathrm{Aut}(V)$.

Definition 21. Let ϕ be a V-valued r-form on P. We say that ϕ is:

pseudotensorial (of type (V, ρ)) if for every $a \in G$ we have $(R_a)^*(\phi) = \rho(a^{-1})\,\phi$

tensorial (of type (V, ρ)) if it is pseudotensorial and *horizontal* i.e.

$$\phi(X_1, \ldots, X_r) = 0$$

whenever at least one X_j, $1 \leq j \leq r$, is vertical.

$\mathcal{PT}^r(P(M, G), V, \rho)$ (or, simpy, $\mathcal{PT}^r(P)$) will denote the space of pseudotensorial r-forms and $\mathcal{T}^r(P(M, G), V, \rho)$ (or, simpy, $\mathcal{T}^r(P)$) will denote the space of tensorial r-forms on P.

Example 13. Given $\omega_1, \omega_2 \in \mathcal{C}(P)$, we have $\alpha = \omega_1 - \omega_2 \in \mathcal{T}^1(P, \mathfrak{G}, ad)$ and so $\mathcal{C}(P)$ has a natural structure of infinite dimensional affine space having $\mathcal{T}^1(P, \mathfrak{G}, ad)$ as vector space of translations.

We have the following

Proposition 25. *Let* $E = P \times_\rho V$; *then*

(5.1) $$\mathcal{T}^r(P) \cong \bigwedge^r(E).$$

Proof (sketch). For $\phi \in \mathcal{T}^r(P)$, we define $L(\phi) \in \bigwedge^r(E)$ in the following way: if $x \in M$ and $X_1, \ldots, X_r \in T_x M$, then

(5.2) $$L(\phi)[x](X_1, \ldots, X_r) = u(\phi[u](\tilde{X}_1, \ldots, \tilde{X}_r))$$

where $u \in \pi^{-1}(x)$ and $\tilde{X}_1, \ldots, \tilde{X}_r \in T_u P$ are chosen in such a way that $\pi_*[u](\tilde{X}_j) = X_j$, $1 \leqslant j \leqslant r$.

One can easily check that $L(\phi)[x](X_1, \ldots, X_r)$ does not depend on the choice of u and of the $\tilde{X}_j$'s. Moreover, for $\psi \in \bigwedge^r(E)$, $L^{-1}(\psi) \in \mathcal{T}^r(P)$ is defined in the following way: if $u \in P$ and $\tilde{X}_1, \ldots, \tilde{X}_r \in T_u P$, then

(5.3) $$L^{-1}(\psi)[u](\tilde{X}_1, \ldots, \tilde{X}_r) = u^{-1}(\psi[\pi(u)](\pi_*(\tilde{X}_1), \ldots, \pi_*(\tilde{X}_r))).$$

It is immediate to observe that $L\colon \mathcal{T}^r(P) \to \bigwedge^r(E)$ is a linear isomorphism.

Remarks 8.

a For $r = 0$, we have $\mathcal{T}^0(P) = \mathcal{P}\mathcal{T}^0(P)$ and, as we already have seen, sections of E correspond to maps $f\colon P \to V$ such that, for every $a \in G$, $f(ua) = \rho(a^{-1}) f(u)$. In particular, if $\sigma \in \bigwedge^0(E)$, then

(5.4) $$L^{-1}(\sigma)(u) = u^{-1}(\sigma(\pi(u))).$$

Consider two special cases

1. Let $P = L(M)$ and $E = TM$. Then the previous construction identifies a vector field in M with the $\mathbf{R}^n$-valued function on $L(M)$ assigning to every frame the n-uple of coordinate of the vector field with respect to the given frame.

2. Let F be a vector bundle of rank r and let $P = L(F)$, $E = \otimes^2 F^*$. Therefore

$$E = L(F) \times_{GL(r, \mathbf{R})} (\mathbf{R}^r)^* \otimes (\mathbf{R}^r)^*.$$

Let $\iota\colon (\mathbf{R}^r)^* \otimes (\mathbf{R}^r)^* \to \mathbf{R}(r)$ be the isomorphism given by $\iota(\zeta \otimes \xi) = {}^t\zeta\xi$. Consequently, we have also

$$E = L(F) \times_{GL(r,\,\boldsymbol{R})} \boldsymbol{R}(r)$$

where $GL(r, \boldsymbol{R})$ acts on $\boldsymbol{R}(r)$ on the left as $\rho(a)X = {}^t a^{-1} X a^{-1}$.

Then any section h of E, i.e. any field of bilinear forms on F corresponds to a map $\widehat{h} \in \mathcal{T}^0(L(F), \boldsymbol{R}(r), \rho)$; therefore it satisfies $\widehat{h}(ua) = {}^t a \widehat{h}(u) a$. It is clear also that, using Remark 1,1, $\widehat{h}(u) = \iota(u^*(h))$.

b d: $\mathcal{P}\mathcal{T}^r(P) \to \mathcal{P}\mathcal{T}^{r+1}(P)$ but, in general, d *does not* map $\mathcal{T}^r(P)$ into $\mathcal{T}^{r+1}(P)$.

c If $\alpha \in \bigwedge^r(M)$ and $\phi \in \mathcal{P}\mathcal{T}^p(P)$ (resp. $\phi \in \mathcal{T}^p(P)$), then $\pi^*(\alpha) \wedge \phi$ is well defined and it belong to $\mathcal{P}\mathcal{T}^p(P)$ (resp. to $\mathcal{T}^p(P)$).

d If ρ is the trivial representation, and, therefore, $E = M \times V$, then

$$\mathcal{Q}^p(P, V) = \mathcal{T}^p(P, V, \rho) = \pi^*(\bigwedge^p(M, V))$$

i.e. $\mathcal{Q}^p(P, V)$ is the space of V-valued, horizontal, G-invariant p-forms on P.

e Again for $r = 0$, we can extend our construction to any associated bundle $E = P \times_G F$, establishing a bijection L between $\mathcal{T}^0(P, F)$ and the set of sections of E. In particular, for $F = G$ and G acting on itself through Ad, we recover the antiisomorphism of Proposition 17.

Definition 22. Let Γ be a connection with connection 1-form ω on the principal bundle $P = P(M,G)$. Then, using ω instead of Γ as index, we define

(5.5) $$\mathcal{H}_\omega \colon \mathcal{P}\mathcal{T}^r(P) \to \mathcal{T}^r(P) \quad \text{as } \mathcal{H}_\omega(\phi) = \phi \circ h_\omega$$

(5.6) $$D_\omega = \mathcal{H}_\omega \circ d \colon \mathcal{T}^r(P) \to \mathcal{T}^{r+1}(P)$$

(5.7) $$\nabla_\omega = L \circ D_\omega \circ L^{-1} \colon \bigwedge^r(E) \to \bigwedge^{r+1}(E).$$

∇_ω is usually called *covariant exterior differential operator* associated with Γ.

First of all note that, for any $\alpha \in \bigwedge^r(M)$, $\phi \in \mathcal{T}^p(P)$, we have

(5.8) $$D_\omega(\pi^*(\alpha) \wedge \phi) = \pi^*(\mathrm{d}\alpha) \wedge \phi + (-1)^r \pi^*(\alpha) \wedge D_\omega \phi.$$

Then, we have the following

Proposition 26. *For every* $\phi \in \mathcal{T}^r(P)$ *we have:*

$$D_\omega \phi = \mathrm{d}\phi + \omega \wedge \phi \tag{5.9}$$

where the wedge product is computed by means of the V-valued bilinear form on $\mathfrak{G} \times V$ *given by* $(X, v) \mapsto \rho_*(X)(v)$, *i.e.*

$$(\omega \wedge \phi)(X_1, \ldots, X_{r+1}) = \frac{1}{r!} \sum_\sigma \varepsilon(\sigma) \rho_* (\omega(X_{\sigma(1)})(\phi(X_{\sigma(2)}, \ldots, X_{\sigma(r+1)})). \tag{5.10}$$

Proof. Fix $u \in P$ and let $X_1, \ldots, X_{r+1} \in T_u P$. Therefore, we have to show that

$$\mathrm{d}\phi(X_1^{(h)}, \ldots, X_{r+1}^{(h)}) = \mathrm{d}\phi(X_1, \ldots, X_{r+1}) + \frac{1}{r!} \sum_\sigma \varepsilon(\sigma) \rho_* (\omega(X_{\sigma(1)})(\phi(X_{\sigma(2)}, \ldots, X_{\sigma(r+1)})). \tag{5.11}$$

a If all the X_j's are horizontal, then (5.11) reduces to the identity

$$\mathrm{d}\phi(X_1, \ldots, X_{r+1}) = \mathrm{d}\phi(X_1, \ldots, X_{r+1}).$$

b If two or more of the X_j's are vertical, then (5.11) clearly becomes

$$0 = \mathrm{d}\phi(X_1, \ldots, X_{r+1}).$$

Now we have

$$\mathrm{d}\phi(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \phi(X_1, \ldots, \mathring{X}_i, \ldots, X_{r+1}) + \sum_{1 \le i < j \le r+1} (-1)^{i+j} \phi([X_i, X_j], X_1, \ldots, \mathring{X}_i, \ldots, \mathring{X}_j, \ldots, X_{r+1}).$$

We can assume to extend the vertical vectors as fundamental vertical vector fields and so, recalling Proposition 23, we obtain $\mathrm{d}\phi(X_1, \ldots, X_{r+1}) = 0$.

c Finally, assuming X_1 is vertical and $X_2, \ldots, X_{r+1}$ are horizontal, we can extend X_1 as a fundamental vertical vector field X^* and $X_2, \ldots, X_{r+1}$ as horizontal lifts of elements in $\mathcal{X}(M)$; therefore, from Proposition 2.3, we obtain that (5.11) is reduced to

$$0 = X_1 \phi(X_2, \ldots, X_{r+1}) + \rho_* (\omega(X_1))(\phi(X_2, \ldots, X_{r+1})).$$

Now

$$X_1\phi(X_2, \ldots, X_{r+1}) = \frac{d}{dt}\phi((R_{\exp(tX)})_*(X_2), \ldots, (R_{\exp(tX)})_*(X_{r+1}))_{|t=0}$$

$$= \frac{d}{dt}(R_{\exp(tX)})^{-1}\phi(X_2, \ldots, X_{r+1}))$$

$$= -\rho_*(\omega(X_1))(\phi(X_2, \ldots, X_{r+1})).$$

Remark 9. If ρ is the trivial representation, then, for every $\phi \in \mathcal{T}^p(P, V, \rho)$ we have

$$D_\omega \phi = d\phi.$$

Assume now a linear differential operator $D: \mathcal{T}^p(P, V, \rho) \to \mathcal{T}^{p+1}(P, V, \rho)$ is given with the property that, for any $\alpha \in \wedge^r(M)$, $\phi \in \mathcal{T}^p(P)$, we have

$$D(\pi^*(\alpha) \wedge \phi) = \pi^*(d\alpha) \wedge \phi + (-1)^r \pi^*(\alpha) \wedge D\phi.$$

If we consider $T = D - d$, we immediately have

$$T(\pi^*(\alpha) \wedge \phi) = (-1)^r \pi^*(\alpha) \wedge T(\phi)$$

and so $T(\phi) = A\phi$, where A is an End(V)-valued 1-form on P. Moreover

1 for any $X \in \mathfrak{G}$, we have $A(X^*) = \rho_*(X)$. In fact, if $\sigma \in \mathcal{T}^0(P)$, we have

$$T(\sigma)(X^*) = (D\sigma - d\sigma)(X^*) = -d\sigma(X^*) = \sigma_*(X^*) = \rho_*(X)\sigma.$$

2 from the fact that, for every $a \in G$, $\phi \in \mathcal{T}^p(P)$, we have

$$(R_a)^* T(\phi) = \rho(a^{-1})T(\phi)$$

it follows
$$(R_a)^* A = ad(\rho(a^{-1}))A.$$

Therefore, if the representation ρ is faithful, then $\omega = \rho_*^{-1}A$ defines an element of $\mathcal{C}(P)$ and of course $D_\omega = D$. Thus, in this case, the assignement of $\omega \in \mathcal{C}(P)$ or the assignement of a linear differential operator $D: \mathcal{T}^p(P, V, \rho) \to \mathcal{T}^{p+1}(P, V, \rho)$ with property (5.8), are completely equivalent.

We need the following

Definition 23. Let $\bigwedge^k(P, \mathfrak{G})$ be the space of $\mathfrak{G}$-valued k-forms on P. Given $\phi \in \bigwedge^r(P, \mathfrak{G})$ and $\psi \in \bigwedge^s(P, \mathfrak{G})$, we define $[\phi, \psi] \in \bigwedge^{r+s}(P, \mathfrak{G})$ by

$$[\phi, \psi](X_1, \ldots, x_{r+1}) = \frac{1}{r!\, s!} \sum_\sigma \varepsilon(\sigma)[\phi(X_{\sigma(1)}, \ldots, x_{\sigma(r)}), \psi(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})].$$

A direct computation leads to the following

Proposition 27. *Given* $\phi \in \bigwedge^r(P, \mathfrak{G})$, $\psi \in \bigwedge^s(P, \mathfrak{G})$ *and* $\xi \in \bigwedge^t(P, \mathfrak{G})$, *we have*

a $[\psi, \phi] = (-1)^{rs}[\phi, \psi]$

b $(-1)^{rt}[[\phi, \psi], \xi] + (-1)^{st}[[\xi, \phi], \psi] + (-1)^{rs}[[\psi, \xi], \phi] = 0$

c $d[\phi, \psi] = [d\phi, \psi] + (-1)^r[\phi, d\psi]$.

We can now introduce a crucial definition in the theory of connections on principal bundles.

Definition 24. Let Γ be a connection with connection 1-form ω on the principal bundle $P = P(M, G)$. Then $\Omega_\omega = D_\omega \omega$ is called the *curvature form* of Γ.

We have now

Proposition 28 (structure equation).

$$\Omega_\omega = d\omega + \tfrac{1}{2}[\omega, \omega]. \tag{5.12}$$

Proof. We have to prove that, for any $X, Y \in T_u P$, we have

$$d\omega(X^{(h)}, Y^{(h)}) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

Now

a if both X and Y are horizontal, we have an identity

b assume $X = A^*$, $Y = B^*$ for $A, B \in \mathfrak{G}$; then

$$d\omega(X, Y) = -\omega([X, Y]) = -[A, B] = -[\omega(X), \omega(Y)]$$

c assume $X = \tilde{Z}$, for $Z \in \mathcal{X}(M)$ and $Y = A^*$, for $A \in \mathfrak{G}$; then by Remark 6, again we have that both sides vanish.

Corollary 4. *For every* $\phi \in \mathcal{T}^r(P)$ *we have*

$$D_\omega^2 \phi = \Omega_\omega \wedge \phi. \tag{5.13}$$

Proof. We have

$$D_\omega^2 \phi = D_\omega(\mathrm{d}\phi + \omega \wedge \phi) = \mathrm{d}(\mathrm{d}\phi + \omega \wedge \phi) + \omega \wedge (\mathrm{d}\phi + \omega \wedge \phi)$$
$$= \mathrm{d}\omega \wedge \phi - \omega \wedge \mathrm{d}\phi + \omega \wedge \mathrm{d}\phi + \omega \wedge (\omega \wedge \phi).$$

Now a direct computation shows that

$$\omega \wedge (\omega \wedge \phi) = \tfrac{1}{2}[\omega, \omega] \wedge \phi$$

and the proof is complete.

Remarks 10.

a If $\alpha \in \mathcal{T}^p(P, \mathfrak{G}, ad)$, then $[\,,]$ is nothing but the bilinear form induced by the adjoint representation and therefore (5.9) reduces to

$$D_\omega \alpha = \mathrm{d}\alpha + [\omega, \alpha]. \tag{5.14}$$

b In more explicit terms, the structure equation (5.12) is

$$\Omega_\omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)].$$

Therefore, in particular, if $X, Y \in H_u$, then

$$\Omega_\omega(X, Y) = -\omega([X, Y])$$

and so, the curvature form measures how much the distribution of horizontal subspaces fails to be integrable.

c (5.13) says that the curvature form measures how much D_ω fails to give rise to a complex.

d If $\omega, \tilde{\omega} \in \mathcal{C}(P)$ and so $\tilde{\omega} = \omega + \alpha$ with $\alpha \in \mathcal{T}^1(P, \mathfrak{G}, ad)$, then

$$\Omega_{\tilde{\omega}} = \Omega_\omega + D_\omega \alpha + \tfrac{1}{2}[\alpha, \alpha]. \tag{5.15}$$

In effect

$$\Omega_{\tilde{\omega}} = D_{\tilde{\omega}} \tilde{\omega} = \mathrm{d}\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \mathrm{d}\omega + \mathrm{d}\alpha + \tfrac{1}{2}[\omega, \omega] + [\omega, \alpha] + \tfrac{1}{2}[\alpha, \alpha]$$
$$= \Omega_\omega + D_\omega \alpha + \tfrac{1}{2}[\alpha, \alpha].$$

e The canonical flat connection on $M \times G$ has vanishing curvature. This follows from the Maurer-Cartan equation

$$\mathrm{d}\varphi = \pi_2^*(\mathrm{d}\vartheta) = -\tfrac{1}{2}\pi_2^*([\vartheta, \vartheta]) = -\tfrac{1}{2}[\varphi, \varphi].$$

Structure equation has another important consequence.

Proposition 29 (second Bianchi's identity). *We have*

$$D_\omega \Omega_\omega = 0. \tag{5.16}$$

Proof. We have

$$D_\omega \Omega_\omega = \mathrm{d}\Omega_\omega + [\omega, \Omega_\omega] = \mathrm{d}(\mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega]) + [\omega, \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega]]$$
$$= \tfrac{1}{2}[\mathrm{d}\omega, \omega] - \tfrac{1}{2}[\omega, \mathrm{d}\omega] + [\omega, \mathrm{d}\omega] = [\mathrm{d}\omega, \omega] + [\omega, \mathrm{d}\omega] = 0.$$

We consider now some properties of the covariant exterior differential operator ∇_ω.

Proposition 30. *For every* $\alpha \in \bigwedge^r(M)$, $\phi \in \bigwedge^p(E)$, *we have*

$$\nabla_\omega(\alpha \wedge \phi) = \mathrm{d}\alpha \wedge \phi + (-1)^r \alpha \wedge \nabla_\omega \phi. \tag{5.17}$$

Proof.

$$\nabla_\omega(\alpha \wedge \phi) = L \circ D_\omega \circ L^{-1}(\alpha \wedge \phi) = L(D_\omega(\pi^*(\alpha) \wedge L^{-1}(\phi))$$
$$= L(\mathrm{d}\pi^*(\alpha) \wedge L^{-1}(\phi)) + (-1)^r \pi^*(\alpha) \wedge (L \circ D_\omega \circ L^{-1})(\phi) = \mathrm{d}\alpha \wedge \phi + (-1)^r \alpha \wedge \nabla_\omega \phi.$$

Proposition 31. *Let* $\Phi\colon \pi^{-1}(U) \to U \times G$ *be a local trivialization and let* $\sigma(x) = \Phi^{-1}(x, e)$ *be the induced local section and for any* $v \in V$ *set* $\sigma_v(x) = [\sigma(x), v]$. *Let* $\omega \in \mathcal{C}(P)$ *and let* $\omega_U = \sigma^*(\omega)$ *be the induced local gauge potential. Then*

$$(\nabla_\omega \sigma_v)[x] = \sigma(x)(\rho_*(\omega_U[x])\, v). \tag{5.18}$$

Proof. If $\Phi(u) = (x, \psi(u))$, then

$$f(u) = L^{-1}(\sigma_v)(u) = L^{-1}(\sigma_v)(\sigma(x)\psi(u)) = \rho(\psi^{-1}(u))\, L^{-1}(\sigma_v)(\sigma(x))$$
$$= \rho(\psi(u)^{-1})\, \sigma^{-1}(x)(\sigma_v(x)) = \rho(\psi^{-1}(u))\, v.$$

Therefore

$$(\nabla_\omega \sigma_v)[x](\xi) = L(D_\omega f)[x](\xi) = \sigma(x)(D_\omega f[\sigma(x)](\widehat{\xi}(\sigma(x)))$$

$$= \sigma(x)(\mathrm{d}f[\sigma(x)](\widehat{\xi}(\sigma(x)))) = \sigma(f_*[\sigma(x)](\widehat{\xi}(\sigma(x)))).$$

Now (4.10) gives

$$\widehat{\xi}(\sigma(x)) = \sigma_*[x](\xi) - (\omega_U[x](\xi))^*.$$

Moreover, since $f \circ \sigma \equiv v$, we have $(f \circ \sigma)_* \equiv 0$ and so

$$f_*[\sigma(x)](\widehat{\xi}(\sigma(x))) = -f_*[\sigma(x)]((\omega_U[x](\xi))^*).$$

Finally $f_*[\sigma(x)] = -(\rho \circ \psi)_*[\sigma(x)]v$ and, for every $Z \in \mathfrak{G}$, $\psi_*[\sigma(x)](Z^*) = Z$.
Therefore we get

$$(\nabla_\omega \sigma_v)[x](\xi) = \sigma(x)(f_*[\sigma(x)](\widehat{\xi}(\sigma(x)))) = \sigma(x)\rho_*(\omega_U[x](\xi)v).$$

Definition 25. Given $x \in M$, $X \in T_x M$ and $\sigma \in \wedge^0(E)$

$$((\nabla_\omega)_X \sigma)(x) = (\nabla_\omega \sigma)[x](X)$$

is called the *covariant derivative* of σ at x in the direction X.

It is easy to check that $((\nabla_\omega)_X \sigma)(x)$ depends only on the restriction of σ to a curve through x tangent to X.

Assume to assign an exterior covariant differential operator to E i.e. a linear differential operator

$$\nabla \colon \wedge^p(E) \to \wedge^{p+1}(E)$$

such that, for every $\alpha \in \wedge^r(M)$, $\phi \in \wedge^p(E)$ we have

$$\nabla(\alpha \wedge \phi) = \mathrm{d}\alpha \wedge \phi + (-1)^r \alpha \wedge \nabla\phi.$$

Then, representing E as $E = P \times_\rho V$ $(P = P(M, G))$, we can define $D = L^{-1} \circ \nabla \circ L \colon \mathcal{T}^p(P) \to \mathcal{T}^{p+1}(P)$ and, if we choose P and G in such a way ρ is faithful (cf. Remark 4), we can reconstruct $\omega \in \mathcal{C}(P)$ such that $\nabla_\omega = \nabla$. Therefore, again, if ρ is faithful, the assignement of $\omega \in \mathcal{C}(P)$ or the assignement of an exterior covariant differential operator are completely equivalent.

Note finally that, since $\bigwedge^r(E)$ is locally generated by elements of the form $\alpha \otimes \sigma$, for $\alpha \in \bigwedge^r(M)$ and $\sigma \in \bigwedge^0(E)$, in order to define ∇ it is sufficient to give a linear differential operator

$$\nabla\colon \bigwedge^0(E) \to \bigwedge^1(E)$$

such that, for every $f \in C^\infty(M)$, $\sigma \in \bigwedge^0(E)$ we have

$$\nabla f\sigma = \mathrm{d}f \otimes \sigma + f\,\nabla\sigma .$$

We can easily perform some functorial constructions with exterior covariant differential operators; in fact we have the following

Definition 26. An exterior covariant differential operator ∇ on E induces an exterior covariant differential operator ∇^* on E^* defined on $\bigwedge^0(E^*)$ by means of the following relation

For every $\sigma^* \in \Lambda^0(E^*)$, $\tau \in \Lambda^0(E)$, we have

$$\mathrm{d}\sigma^*(\tau) = (\nabla^*\sigma^*)(\tau) + \sigma^*(\nabla\tau). \tag{5.19}$$

Exterior covariant differential operators ∇_i on E_i, $i = 1, 2$, induce exterior covariant differential operators $\nabla_1 \oplus \nabla_2$ on $E_1 \oplus E_2$ and $\nabla_1 \otimes \nabla_2$ on $E_1 \otimes E_2$, defined on $\Lambda^0(E_1 \oplus E_2)$ and $\Lambda^0(E_1 \otimes E_2)$ respectively as

$$(\nabla_1 \oplus \nabla_2)(\sigma_1 \oplus \sigma_2) = \nabla_1\sigma_1 \oplus \nabla_2\sigma_2 \tag{5.20}$$

$$(\nabla_1 \otimes \nabla_2)(\sigma_1 \otimes \sigma_2) = \nabla_1\sigma_1 \otimes \sigma_2 \oplus \sigma_1 \otimes \nabla_2\sigma_2 . \tag{5.21}$$

Definition 27. $R_\omega = L(\Omega_\omega) \in \bigwedge^2(\mathfrak{G}_P)$ is called the *curvature tensor* of ω.

One can easily prove that

$$R_\omega(X, Y) = [(\nabla_\omega)_X, (\nabla_\omega)_Y] - (\nabla_\omega)_{[X,Y]} . \tag{5.22}$$

Let $P = P(M, G)$ be a principal bundle and let $E = P \times_G F$ be a bundle associated to P and let Γ be a connection on P. We can define an horizontal distribution on TE in the following way. Let $TP = H \oplus W$ be the splitting defined by Γ; embed H in $T(P \times F)$ in the trivial way and let $p\colon P \times F \to E$ be the natural projection. Since H is G-invariant $H^E = p_*(H)$ is a well defined subbundle of TE such that, for every $a \in E$, $T_aE = H^E_a \oplus T_aE_{\pi_E(a)}$.

In other words, define

for $(u, \xi) \in P \times F$ $\qquad H_{(u, \xi)} = H_u \times \{0\}$

for $p(u, \xi) = a$ $\qquad H_a^E = p_*[(u, \xi)](H_{(u, \xi)})$.

We have now the following

Definition 28. A section σ of E is said to be *parallel* (with respect to a given connection on P) if, for every $x \in M$, $\sigma_*[x](T_x M) = H^E_{\sigma(x)}$.

Consider the special case $F = V$, G acting on V through a representation $\rho: G \to \mathrm{Aut}(V)$. Let σ be a section of E and let $f = L^{-1}(\sigma)$. A direct computation shows that, for $x \in M$ and $X \in T_x M$, if $u \in \pi^{-1}(x)$, then the horizontal/vertical decomposition of $\sigma_*[x](X)$ is given by

(5.23) $$\sigma_*[x](X) = p_*[(u, f(u))](\widehat{X}, 0) + ((\nabla_\omega)_X \sigma)(x).$$

Therefore, we have the following

Proposition 32. *$\sigma \in \bigwedge^0(E)$ is parallel if and only if for every $x \in M$ we have $((\nabla_\omega)\sigma)(x) = 0$, i.e. σ is ∇_ω-covariant constant.*

We want now to establish some basic facts on the action of the gauge group on the space of connections. First of all we have the following

Proposition 33. *Let $P = P(M, G)$ be a principal bundle; then the gauge group $\mathcal{G}(P)$ acts on the right on $\mathcal{C}(P)$ by $R_f(\omega) = f^*(\omega)$.*

Proof. We have only to check that, given $\omega \in \mathcal{C}(P)$ and $f \in \mathcal{G}(P)$, then $f^*(\omega) \in \mathcal{C}(P)$. That is the case, in fact: for every $X \in \mathfrak{G}$, we have

$$f^*(\omega)(X^*) = \omega(f_*(X^*)) = \omega(X^*) = X$$

and for every $a \in G$, we have

$$(R_a)^*(f^*(\omega)) = (f \circ R_a)^*(\omega) = (R_a \circ f)^*(\omega) = f^*((R_a)^*(\omega))$$

$$= f^*(ad(a^{-1})\omega) = ad(a^{-1}) f^*(\omega).$$

Let $\rho: G \to \mathrm{Aut}(V)$ be a representation of G into a finite dimensional vector space; let $f \in \mathcal{G}(P)$ and let $\widehat{f}$ be the corresponding element of $\mathcal{T}^0(P, G)$ (and so $f(u) = u\widehat{f}(u)$). Since f_* maps vertical subspaces into vertical subspaces, if

$\alpha \in \mathcal{T}^r(P, V, \rho)$ then $f^*(\alpha) \in \mathcal{T}^r(P, V, \rho)$. Moreover, it is easy to check that

$$f^*(\sigma) = \hat{f}^{-1}(\alpha) \tag{5.24}$$

where $\hat{f}^{-1}\colon \mathcal{T}^r(P, V, \rho) \to \mathcal{T}^r(P, V, \rho)$ is defined as

$$\hat{f}^{-1}(\alpha)(X_1, \ldots, X_r) = \rho(\hat{f}^{-1})\,\alpha(X_1, \ldots, X_r).$$

The proof of the following proposition is straightforward

Proposition 34. *Let* $\omega \in \mathcal{C}(P)$, $f \in \mathcal{G}(P)$ *and set* $\tilde{\omega} = f^*(\omega)$. *Then*

$$H_p^{\tilde{\omega}} = (f^{-1})_*(H_{f(p)}^{\omega}) \qquad h_{\tilde{\omega}} = (f^{-1})_* \circ h_\omega \circ f_* \qquad \mathcal{H}_{\tilde{\omega}} = f^* \circ \mathcal{H}_\omega \circ (f^{-1})^*.$$

Corollary 5. *We have*

$$D_{\tilde{w}} = \hat{f}^{-1} \circ D_\omega \circ \hat{f} \tag{5.25}$$

and consequently

$$\tilde{\omega} = \omega + \hat{f}^{-1} \circ D_\omega \hat{f} \qquad \nabla_{\tilde{\omega}} = L(\hat{f}^{-1}) \circ \nabla_\omega \circ L(\hat{f}). \tag{5.26}$$

Proof. The only point that deserves some comments is the definition of $\hat{f}^{-1} \circ D_\omega \hat{f}$.

Consider the adjoint representation $ad\colon G \to \mathrm{Aut}(\mathfrak{G}) \subset \mathrm{End}(\mathfrak{G})$ (note that ad is not faithful, its kernel being $C(G)$).

G acts on the left on $\mathrm{End}(\mathfrak{G})$ as

$$L_a \theta = a\theta = ad(a^{-1}) \circ \theta \circ ad(a).$$

This gives a representation $\rho\colon G \to \mathrm{Aut}(\mathrm{End}(\mathfrak{G}))$.

Let $\sigma \in \mathcal{T}^0(P, \mathrm{Aut}(\mathrm{End}(\mathfrak{G})), \rho)$ be defined by $\sigma(u) = ad(\hat{f}(u))$. Then a direct computation shows that

$$(\sigma^{-1} \circ D_\omega \sigma)[u](X) = \sigma^{-1}(u) \circ \sigma_*[u](X) + \sigma^{-1}(u) \circ \alpha\delta(\omega[u](X)) \circ \sigma(u) - \alpha\delta(\omega[u](X))$$

$$= \alpha\delta((L_{\hat{f}^{-1}(u)})_* \circ \hat{f}_*[u](X) + ad(\hat{f}^{-1}(u))\,\omega[u](X) - \omega[u](X)).$$

Therefore $\hat{f}^{-1} \circ D_\omega \hat{f} \in \mathcal{T}^1(P, \mathfrak{G}, ad)$ can be defined as

$$\hat{f}^{-1} \circ D_\omega \hat{f}(u) = (L_{\hat{f}^{-1}(u)})_* \circ \hat{f}_*[u] + ad(\hat{f}^{-1}(u))\,\omega[u] - \omega[u].$$

6 - Holonomy groups

Let $P = P(M, G)$ be a principal bundle equipped with a connection Γ.

Definition 29. A smooth curve $\mu: [a, b] \to P$ is said to be *horizontal* if, for every $t \in [a, b]$, $\mu'(t) \in H_{\mu(t)}$.

Definition 30. Let $\gamma: [a, b] \to M$ be a smooth curve. A *horizontal lift* $\hat{\gamma}$ of γ is a horizontal smooth curve $\hat{\gamma}: [a, b] \to P$ such that for every $t \in [a, b]$, $\pi(\hat{\gamma}(t)) = \gamma(t)$.

We have the following

Proposition 35. *Let $\gamma: [a, b] \to M$ be a smooth curve. For every $u \in P_{\gamma(a)}$, there exists a unique horizontal lift $\hat{\gamma}$ of γ with $\hat{\gamma}(a) = u$.*

Proof. See e.g. [3].

Definition 31. Let $\gamma: [a, b] \to M$ be a smooth curve. The *parallel displacement* along γ is a map

$$\tilde{\gamma}: P_{\gamma(a)} \to P_{\gamma(b)}$$

defined as follows:

for every $u \in P_{\gamma(a)}$, $\tilde{\gamma}(u) = \hat{\gamma}(b)$, where $\hat{\gamma}$ is the unique horizontal lift of γ starting from u.

The proof of the following proposition is left as an exercise.

Proposition 36. *Let $\gamma: [a, b] \to M$ be a smooth curve; then*

1 *The parallel displacement along γ commutes with the action of G, i.e. for every $a \in G$ we have $\tilde{\gamma} \circ R_a = R_a \circ \tilde{\gamma}$. In particular, $\tilde{\gamma}$ is a bijection.*

2 *If $\gamma^{-1}(t) = \gamma(a + b - t)$, then $(\gamma^{-1})^{\sim} = \tilde{\gamma}^{-1}$.*

3 *If $\nu: [b, c] \to M$ is another smooth curve such that $\nu(b) = \gamma(b)$ and $\nu \cdot \gamma$ denotes the composite curve, then $(\nu \cdot \gamma)^{\sim} = \tilde{\nu} \circ \tilde{\gamma}$.*

Consider a representation $\rho: G \to \mathrm{Aut}(V)$ and let $E = P \times_{\rho} V$ be the associated bundle. We have the following

Definition 32. Let: $x \in M$, γ be a curve in M starting from x, $s \in E_x$. We define the *parallel transport* σ of s along γ in the following way. Let $u \in P_x$, let

$\xi = u^{-1}(s)$ and let $\tilde{\gamma}$ be the parallel desplacement along γ starting from u. Then set $\sigma(t) = \tilde{\gamma}(t)(\xi)$.

It is clear that σ is independent of the choice of $u \in P_x$; and that σ is covariant constant along γ.

Give $x \in M$, let $C(x)$ be the *loop space* at x, i.e. the set of all closed smooth curves starting and ending at the point x and let $C_0(x)$ be the subset of $C(x)$ consisting of the loops which are homotopic to zero. Then we set

Definition 33. $\Phi(x) = \{\tilde{\gamma} \mid \gamma \in C(x)\}$ is called the *holonomy group* of Γ with reference point x.

$\Phi_0(x) = \{\tilde{\gamma} \mid \gamma \in C_0(x)\}$ is called the *restricted holonomy* of Γ with reference point x.

Both $\Phi(x)$ and $\Phi_0(x)$ can be realized as subgroups of G.

In fact, fix $u \in P_x$; therefore, for any $\gamma \in C(x)$, there exists $a(\gamma) \in G$ such that $\tilde{\gamma}(u) = ua(\gamma)$, which, in virtue of Proposition 36, 1, completely describes $\tilde{\gamma}$. It is clear that the map $\tilde{\gamma} \mapsto a(\gamma)$ maps isomorphically $\Phi(x)$ and $\Phi_0(x)$ into two subgroups of G, denoted respectively by $\Phi(u)$ and $\Phi_0(u)$ and called the *holonomy group* and the *restricted holonomy group* with reference point u.

We have now the following

Proposition 37. *Consider on P the following equivalence relation:*

$u \sim v$ if u and v can be joined by a horizontal curve.

Then:

1 $\Phi(u) = \{a \in G \mid u \sim ua\}$.

2 *If* $v = ua$, *then* $\Phi(v) = \mathcal{A}d(a^{-1})(\Phi(u))$ *and* $\Phi_0(v) = \mathcal{A}d(a^{-1})(\Phi_0(u))$. *Therefore* $\Phi(u)$ *and* $\Phi(v)$ *(resp.* $\Phi_0(u)$ *and* $\Phi_0(v)$*) are conjugate in* G.

3 *If* $u \sim v$, *then* $\Phi(u) = \Phi(v)$ *and* $\Phi_0(u) = \Phi_0(v)$.

Proof.

1 !

2 Given $b \in \Phi(u)$, from 1 it follows that $u \sim ub$ and so $v = ua \sim uba = va^{-1}ba$, i.e. $a^{-1}ba \in \Phi(v)$ and therefore $\Phi(v) = \mathcal{A}d(a^{-1})(\Phi(u))$. Strictly analogous argument for the restricted holonomy groups.

3 If $u \sim v$ then, for every $a \in G$, $ua \sim va$ and so, by transitivity, $u \sim ua \Leftrightarrow v \sim va$. Let $\hat{\gamma}$ be a horizontal curve from u to v. Given $a \in \Phi_0(u)$, then there exists a horizontal curve $\hat{\tau}$ from u to ua such that $\pi(\hat{\tau}) \in C_0(\pi(u))$. Then, clearly, $(R_a \circ \hat{\tau}) \cdot \hat{\gamma} \cdot \hat{\tau}^{-1}$ is a horizontal curve from v to va which projects to a loop in $C_0(\pi(v))$ and also so $a \in \Phi_0(v)$; therefore $\Phi_0(u) \subset \Phi_0(v)$ and thus $\Phi_0(u) = \Phi_0(v)$.

Corollary 6. *The holonomy groups $\Phi(u)$, $u \in P$, are all isomorphic.*

Proof. Just note that, given $u, v \in P$, then there exists $a \in G$ such that $u \sim va$.

We gather in the following some basic results on holonomy (see e.g. [3], for the proofs).

Theorem 2. *Let $P = P(M, G)$ be a principal bundle equipped with a connection Γ and fix $u \in P$. Then $\Phi(u)$ is a Lie subgroup of G, whose identity component is $\Phi_0(u)$.*

Theorem 3. *Let $P = P(M, G)$ be a principal bundle equipped with a connection Γ. Let u be an arbitrary point of P and let $P(u) = \{v \in P \mid v \sim u\}$. Then $P(u)$ is a $\Phi(u)$-reduction of P and Γ is reducible to a connection on $P(u)$* (cf. Definition 34).

Theorem 4. *Let $P = P(M, G)$ be a principal bundle equipped with a connection Γ; let Ω be the curvature form of Γ and let $u \in P$. Then the Lie algebra of $\Phi(u)$ is equal to the subspace of $\mathfrak{G}$ spanned by all elements of the form $\Omega[v](X, Y)$, for $v \in P(u)$ and $X, Y \in H_v$.*

Theorem 5. *Let $P = P(M, G)$ be a principal bundle; if $\dim M \geq 2$, then there exists a connection Γ on P such that, for all $u \in P$, $P(u) = P$.*

We are now in position to prove the following

Proposition 38. *Let $P = P(M, G)$ be a principal bundle; then the following facts are equivalent:*

1 *P admits a flat structure.*

2 *P admits a bundle atlas with constant transition functions.*

3 *There exists $\omega \in \mathcal{C}(P)$ such that $\Omega_\omega \equiv 0$.*

Proof.

$1 \Rightarrow 2$ Cf. Proposition 16.

$2 \Rightarrow 3$ Cf. Remark 7 and Remark 10 **e**.

$3 \Rightarrow 1$ Cf. Theorems 2, 3 and 4, we have that

$$P = P(u) \times_{\Phi(u)} G .$$

$\Phi_0(u)$ is trivial, $\Phi(u)$ is discrete and $P(u)$ is a covering space of M. Moreover, we have a surjective homomorphism $\rho\colon \pi_1(M) \to \Phi(u)$. $P(u) = \tilde{M}/\ker\rho$ and so, $P = \tilde{M} \times_\rho G$.

7 - Connections and bundle morphisms

We want to describe the behaviour of connections with respect to bundle morphisms (cf. again [3] for the proofs).

Proposition 39. *Let $f = (f', f'')\colon P_1(M_1, G_1) \to P_2(M_2, G_2)$ be a bundle morphism between principal bundles, such that the induced map $\tilde{f}\colon M_1 \to M_2$ is a diffeomorphism. Let Γ_1 be a connection on P_1 with connection 1-form ω_1. Then*

a *There exists a unique connection Γ_2 on P_2 with connection 1-form ω_2 such that*

$$f^*(\omega_2) = f''_* \circ \omega_1$$

and so the horizontal subspaces of Γ_1 are mapped by f into the horizontal subspaces of Γ_2.

b *If Ω_1 and Ω_2 are the curvature forms of Γ_1 and Γ_2 respectively, then*

$$f^*(\Omega_2) = f''_* \circ \Omega_1 .$$

c *If $u_1 \in P_1$ and $u_2 = f'(u_1)$, then f'' maps the Γ_1-holonomy group $\Phi(u_1)$ onto the Γ_2-holonomy group $\Phi(u_2)$ and the restricted Γ_1-holonomy group $\Phi_0(u_1)$ onto the restricted Γ_2-holonomy group $\Phi_0(u_2)$.*

Remark 11. As a consequence of Proposition 39, we have that, given two principal bundles $P = P(M, G)$ and $Q = Q(M, H)$, an element $\omega \in \mathcal{C}(P + Q)$ uniquely determines $\omega_P \in \mathcal{C}(P)$ an $\omega_Q \in \mathcal{C}(Q)$ such that $\omega = f_P^*(\omega_P) + f_Q^*(\omega_Q)$.

Proposition 40. *Let* $f = (f', f'')$: $P_1(M_1, G_1) \to P_2(M_2, G_2)$ *be a bundle morphism between principal bundles, such that* f'': $G_1 \to G_2$ *is a group isomorphism. Let* Γ_2 *be a connection on* P_2 *with connection* 1*-form* ω_2*. Then*

a *There exists a unique connection* Γ_1 *on* P_1 *with connection* 1*-form* ω_1 *such that*

$$f^*(\omega_2) = f''_* \circ \omega_1$$

and so the horizontal subspaces of Γ_1 *are mapped by* f *into the horizontal subspaces of* Γ_2.

b *If* Ω_1 *and* Ω_2 *are the curvature forms of* Γ_1 *and* Γ_2 *respectively, then*

$$f^*(\Omega_2) = f''_* \circ \Omega_1 .$$

c *If* $u_1 \in P_1$ *and* $u_2 = f'(u_1)$, *then* f'' *maps the* Γ_1*-holonomy group* $\Phi(u_1)$ *onto the* Γ_2*-holonomy group* $\Phi(u_2)$ *and the restricted* Γ_1*-holonomy group* $\Phi_0(u_1)$ *onto the restricted* Γ_2*-holonomy group* $\Phi_0(u_2)$.

We have also the following

Proposition 41. *Let* Γ_P *and* Γ_Q *be connections on the principal bundles* $P = P(M, G)$ *and* $Q = Q(M, H)$ *respectively and let* ω_P *and* ω_Q *be their connection* 1*-forms. Then*

a *There exists a unique connection* Γ *on* $P + Q$ *with connection* 1*-form* ω, *such that*

$$\omega = f_P^*(\omega_P) + f_Q^*(\omega_Q)$$

and so the horizontal subspaces of Γ *are mapped by* f_P *(resp.* f_Q*) into the horizontal subspaces of* Γ_P *(resp.* Γ_Q*).*

b *If* Ω_P, Ω_Q, Ω *are the curvature forms of* Γ_P, Γ_Q, Γ *respectively, then*

$$\Omega = f_P^*(\Omega_P) + f_Q^*(\Omega_Q).$$

c *Let* $(u, v) \in P + Q$. *The holonomy group* $\Phi(u, v)$ *of* Γ *is a subgroup of the product* $\Phi(u) \times \Phi(v)$ *of the holonomy groups of* Γ_P *and* Γ_Q *and the same statement holds for the restricted holonomy groups.*

The following construction is an important application of previous results.

Let (E, g) be a vector bundle of rank $r = p + q$ equipped with a Riemannian structure and let $F \subset E$ be a subbundle of rank p. Then $F^\perp$ is the subbundle of E

defined by the condition: for every $x \in M$, $F_x^\perp = (F_x)^\perp$. In order to simplify our notations, set $S = F^\perp$; it is obvious that rank $S = q$ and $E = F \oplus S$. Furthermore $O_g(F) + O_g(S)$ is a $(O(p) \times O(q))$-reduction of $O_g(E)$ with embedding

$$i\colon O_g(F) + O_g(S) \to O_g(E).$$

Let $f_F\colon O_g(F) + O_g(S) \to O_g(E)$ and $f_S\colon O_g(F) + O_g(S) \to O_g(E)$ be the natural maps. Let $\omega \in \mathcal{C}(O_g(E))$. Then $i^*(\omega)$ splits as

$$i^*(\omega) = \widehat{\omega} + \alpha.$$

According to the orthogonal decomposition

$$o(p+q) = [o(p) \oplus o(q)] \oplus \mathfrak{M}$$

where $$\mathfrak{M} = \left\{\begin{pmatrix} 0 & -{}^tA \\ A & 0 \end{pmatrix} \middle| A \in M_{q,p}(\boldsymbol{R})\right\} \cong M_{q,p}(\boldsymbol{R})$$

we have $$\widehat{\omega} \in \mathcal{C}(O_g(F) + O_g(S))$$

and $$\widehat{\omega} = f_F^*(\widehat{\omega}_F) + f_S^*(\widehat{\omega}_S) = \begin{pmatrix} f_F^*(\widehat{\omega}_F) & 0 \\ 0 & f_S^*(\widehat{\omega}_S) \end{pmatrix}$$

for $\widehat{\omega}_F \in \mathcal{C}(O_g(F))$ and $\widehat{\omega}_S \in \mathcal{C}(O_g(S))$. Moreover

$$\alpha \in \mathcal{T}^1(O_g(F) + O_g(S), \mathfrak{M}, ad)$$

and $$\alpha = \begin{pmatrix} 0 & -{}^t\sigma \\ \sigma & 0 \end{pmatrix}$$

for $$\sigma \in \mathcal{T}^1(O_g(F) + O_g(S), M_{q,p}(\boldsymbol{R}), \rho)$$

(where $\rho\colon O(p) \times O(q) \to \mathrm{Aut}(M_{q,p}(\boldsymbol{R}))$ is given by $\rho(A, B)(X) = BXA^{-1}$).

We can extend $\widehat{\omega}$ as an element of $\mathcal{C}(O_g(E))$ and α as an element of $\mathcal{T}^1(O_g(E), o(r), ad)$; then, according to Remark 10**d**

$$\Omega_\omega = \Omega_{\widehat{\omega}} + D_{\widehat{\omega}}\alpha + \tfrac{1}{2}[\alpha, \alpha].$$

Therefore, on $O_g(F) + O_g(S)$, we have

$$\Omega_\omega = \begin{pmatrix} f_F^*(\Omega_{\widehat{\omega}_F}) - \tfrac{1}{2}[{}^t\sigma, \sigma] & -D_{\widehat{\omega}}{}^t\sigma \\ D_{\widehat{\omega}}\sigma & f_S^*(\Omega_{\widehat{\omega}_S}) - \tfrac{1}{2}[\sigma, {}^t\sigma] \end{pmatrix} \tag{7.1}$$

(7.1) is called the *Codazzi-Mainardi* equation. σ is called the *second fundamental form* of F in E. Therefore $-{}^t\sigma$ is the second fundamental form of S in E.

It is immediate to check that, if $s \in \wedge^0(F)$, then $\nabla_\omega s$ decomposes according to the splitting $\wedge^1(E) = \wedge^1(F) \oplus \wedge^1(S)$ as

$$\nabla_\omega s = \nabla_{\hat{\omega}_F} s + L(\sigma)\, s\,. \tag{7.2}$$

Consider now, as special case of Proposition 39, a reduction $Q(M, H) \subset P(M, G)$.

Definition 34. A connection on $P(M, G)$ constructed from a connection on $Q(M, H)$ is said to be *reducible* to $Q(M, H)$.

The proof of the following lemma is straightforward

Lemma 1. *Let $P = P(M, G)$ be a principal bundle, let $Q = Q(M, H)$ be a H-reduction of P and let Γ be a connection on P. If for every $u \in Q$ the horizontal subspace of $T_u P$ is tangent to Q, then Γ is reducible to a connection on Q.*

We have the following

Proposition 42. *Let $P = P(M, G)$ be a principal bundle and let $Q = Q(M, H)$ be a H-reduction of P. A connection Γ on P is reducible to H, if and only if the section of $E = E(M, G/H, G, P)$ corresponding to Q is Γ-parallel.*

Examples 14. Let $E = L(E) \times_{GL(r, \mathbf{R})} \mathbf{R}^r$ be a vector bundle of rank r and let Γ be a connection on $L(E)$ with connection 1-form ω.

a Let g be a Riemannian structure on E and let $O_g(E)$ be the corresponding $O(r)$-reduction of $L(E)$. Then Γ is reducible to $O_g(E)$ if and only if $\nabla_\omega g = 0$. In this case, Γ is called a *g-Riemannian* connection.

b Assume $r = 2q$ and let J be a structure of complex vector bundle on E. Let $L_{\mathbf{C}}(E)$ be the corresponding $GL(q, \mathbf{C})$-reduction of $L(E)$. Then Γ is reducible to $L_{\mathbf{C}}(E)$ if and only if $\nabla_\omega J = 0$. In this case, Γ is called a *J-holomorphic* connection.

8 - Scalar product. Hodge's $*$ operator. Codifferential

Let E be a vector bundle of rank r over the n-dimensional manifold M. Recall that

$$E = L(E) \times_{GL(r,\boldsymbol{R})} \boldsymbol{R}^r \qquad \rho = \mathrm{std}$$

$$E^* = L(E) \times_{GL(r,\boldsymbol{R})} (\boldsymbol{R}^r)^* \qquad \rho = \mathrm{std}^{-1}$$

$$E^* \otimes E = L(E) \times_{GL(r,\boldsymbol{R})} gl(r, \boldsymbol{R}) = \mathfrak{A}_{L(E)} \qquad \rho = ad$$

$$E^* \otimes E^* = L(E) \times_{GL(r,\boldsymbol{R})} gl(r, \boldsymbol{R}) \qquad \rho = {}^t ad$$

(where ${}^t ad(a) X = {}^t a^{-1} X a^{-1}$).

Let now h be a Riemannian structure on E and let $\widehat{h} \in \mathcal{T}^0(L(E), gl(r, \boldsymbol{R}), {}^t ad)$ be the corresponding element (Remark 8**a**). Let $\langle , \rangle$ be the (pointwise) scalar product defined on $\mathcal{T}^0(L(E), \boldsymbol{R}^r, \mathrm{std})$ as

(8.1) $$\langle \sigma, \tau \rangle = {}^t\sigma(u)\, \widehat{h}(u)\, \tau(u)$$

and on $\mathcal{T}^0(L(E), (\boldsymbol{R}^r)^*, \mathrm{std}^{-1})$ as

(8. 2) $$\langle \sigma^*, \tau^* \rangle(u) = \sigma^*(u)\, \widehat{h}^{-1}(u)\, {}^t\tau^*(u).$$

Clearly, $\langle \sigma, \tau \rangle$ and $\langle \sigma^*, \tau^* \rangle$ are well defined functions on M.

Moreover b: $\mathcal{T}^0(L(E), \boldsymbol{R}^r, \mathrm{std}) \to \mathcal{T}^0(L(E), (\boldsymbol{R}^r)^*, \mathrm{std}^{-1})$, defined by $b(\sigma)(u) = {}^t\sigma(u)\, \widehat{h}(u)$, satisfies

(8.3) $$\langle b(\sigma), b(\tau) \rangle = \langle \sigma, \tau \rangle.$$

b extends as b: $\mathcal{T}^p(L(E), \boldsymbol{R}^r, \mathrm{std}) \to \mathcal{T}^p(L(E), (\boldsymbol{R}^r)^*, \mathrm{std}^{-1})$ simply setting

(8.4) $$b(\pi^*(a) \otimes \sigma) = \pi^*(a) \otimes b(\sigma).$$

note also that $b^{-1}(\sigma^*)(u) = \widehat{h}^{-1}(u)\, {}^t\sigma^*(u)$.

Then $\langle , \rangle$ extends naturally to $\mathcal{T}^p(L(E), \boldsymbol{R}^r, \mathrm{std})$ and $\mathcal{T}^p(L(E), (\boldsymbol{R}^r)^*, \mathrm{std}^{-1})$, simply setting

(8.5) $$\langle \pi^*(a) \otimes \sigma, \pi^*(b) \otimes \tau \rangle = g(a, b) \langle \sigma, \tau \rangle.$$

We can now set the following

Definition 35. Assume M is oriented with volume element $d\mu(g)$. If $\alpha, \beta \in \mathcal{T}^p(L(E), \boldsymbol{R}^r, \mathrm{std})$ (or $\mathcal{T}^p(L(E), (\boldsymbol{R}^r)^*, \mathrm{std}^{-1})$) have compact support, then set

$$(\alpha, \beta) = \int \langle \alpha, \beta \rangle \, \mathrm{d}\mu(g) . \tag{8.6}$$

Under the further assumption M is oriented with volume element $\mathrm{d}\mu(g)$, we can also set the following

Definition 36. Define

$$* : \mathcal{T}^{p}(L(E), \boldsymbol{R}^{r}, \mathrm{std}) \to \mathcal{T}^{n-p}(L(E), (\boldsymbol{R}^{r})^{*}, \mathrm{std}^{-1})$$

by means of the relation

$$\langle * \, \alpha, \beta \rangle = \langle \alpha \wedge \beta, \, \mathrm{d}\mu(g) \rangle \tag{8.7}$$

for $\alpha \in \mathcal{T}^{p}(L(E), \boldsymbol{R}^{r}, \mathrm{std})$ and $\beta \in \mathcal{T}^{n-p}(L(E), (\boldsymbol{R}^{r})^{*}, \mathrm{std}^{-1})$.

Remark 12. If $\alpha = \pi^{*}(a) \otimes \sigma$, then $* \, \alpha = \pi^{*}(* \, a) \otimes b(\sigma)$.

Definition 37. Let $\omega \in \mathcal{C}(L(E))$. Define

$$D_{\omega}^{*} : \mathcal{T}^{p}(L(E), \boldsymbol{R}^{r}, \mathrm{std}) \to \mathcal{T}^{p-1}(L(E), \boldsymbol{R}^{r}, \mathrm{std})$$

as

$$D_{\omega}^{*} = (-1)^{p} *^{-1} \circ D_{\omega} \circ * \, . \tag{8.8}$$

Proposition 43. *If M is compact, then for $\alpha \in \mathcal{T}^{p}(L(E), \boldsymbol{R}^{r}, \mathrm{std})$, $\beta \in \mathcal{T}^{p+1}(L(E), \boldsymbol{R}^{r}, \mathrm{std})$ we have*

$$(D_{\omega} \alpha, \beta) = (\alpha, D_{\omega}^{*} \beta) . \tag{8.9}$$

Proof. We have

$$\int_{M} d(\alpha \wedge * \, \beta) = 0 = \int_{M} D_{\omega} \alpha \wedge * \, \beta + (-1)^{p} \alpha \wedge D_{\omega}(* \, \beta)$$

$$= \int_{M} D_{\omega} \alpha \wedge * \, \beta + (-1)^{2p+1} \alpha \wedge * \, (D_{\omega}^{*} \beta)$$

i.e.

$$(D_{\omega} \alpha, \beta) = \int_{M} D_{\omega} \alpha \wedge * \, \beta = \int_{M} \alpha \wedge * \, (D_{\omega}^{*} \beta) = (\alpha, D_{\omega}^{*} \beta) .$$

We have also the following two results. We omit the proofs that can be obtained by direct computation.

Lemma 2. *If* $\omega \in \mathcal{C}(O_p(E))$, *then*

$$D_\omega \circ b = b \circ D_\omega . \tag{8.10}$$

Proposition 44. *If* $\omega \in \mathcal{C}(O_p(E))$, *then*

$$D_\omega^* \alpha = -\theta_h \rfloor (D_{\mu \otimes \omega})_{\theta_h} \alpha \tag{8.11}$$

where $\{\theta_1, \ldots, \theta_n\}$ *is the horizontal lifting of a local g-orthonormal frame in TM, and* $(D_{\mu \otimes \omega})_X$ *is defined as follows*

$$(D_{\mu \otimes \omega})_X \pi^*(a) \otimes \sigma = \pi^*((\nabla_\mu)_{\pi^*(X)} a) \otimes \sigma + \pi^*(a) \otimes (D_\omega \sigma)(X)$$

μ *being the Levi-Civita connection 1-form.*

Let $\sigma \in \mathcal{T}^0(L(E), gl(r, \boldsymbol{R}), ad)$. Then define $\sigma^\# \in \mathcal{T}^0(L(E), gl(r, \boldsymbol{R}), ad)$ as

$$\sigma^\#(u) = h^{-1}(u)\, {}^t\sigma(u)\, h(u) \tag{8.12}$$

and, given $\tau \in \mathcal{T}^0(L(E), gl(r, \boldsymbol{R}), ad)$

$$\langle \sigma, \tau \rangle = \operatorname{tr} \sigma(u)\, \tau^\#(u) . \tag{8.13}$$

$\langle \sigma, \tau \rangle$ is a well defined function on M and we can set definitions and obtain results in strict analogy with those established for $\mathcal{T}^0(L(R), \boldsymbol{R}^r, \text{std})$.

More in general, we can extend the previous setting to the case of a principal G-bundle P over a Riemannian manifold (M, g) equipped with a Riemannian structure on its adjoint bundle $\mathfrak{A}_P$.

We are now in position to describe the basic gauge-theoretic result in the theory of characteristic classes. We refer to [6] for a general account.

We need some algebraic preliminaries.

Definition 38. Let V be a finite dimensional $\boldsymbol{K}$-vector space; set $S^0(V) = \boldsymbol{K}$ and, for $k \geqslant 1$, let $S^k(V)$ be the $\boldsymbol{K}$-vector space of k-linear symmetric maps $V \times \ldots \times V \to \boldsymbol{K}$; equivalently, $f \in S^k(V)$ is a linear map $V \otimes \ldots \otimes V \to \boldsymbol{K}$ which is invariant under the action of the symmetric group. Set $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$.

If $f \in S^k(V)$ and $g \in S^h(V)$, then we can define $fg \in S^{k+h}(V)$ by the formula

$$fg(v_1, \dots, v_{k+h}) = \frac{1}{(k+h)!} \sum_\sigma f(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+h)}).$$

Clearly this assigns on $S(V)$ the structure of commutative algebra.

Definition 39. A map $p: V \to \boldsymbol{K}$ is said to be *polynomial* if, given a basis $\{\theta_1, \dots, \theta_m\}$ of V^*, then $p \in \boldsymbol{K}[\theta_1, \dots, \theta_m]$, i.e. p can be expressed as a polynomial in $\theta_1, \dots, \theta_m$.

E.g. if $V = \boldsymbol{K}(n)$, then $p: X \mapsto \det X$ is polynomial. With respect to the standard basis $\{E^*_{jk}\}$ of V^*, we have

$$p = \sum_\sigma \varepsilon(\sigma) E^*_{\sigma(1)1} \dots E^*_{\sigma(n)n}.$$

Let $P = \bigoplus_{k=0}^{\infty} P^k(V)$ be the commutative algebra of polynomial functions on V expressed as direct sum of spaces of homogeneous polynomials. We have the following

Lemma 3.

a *The map T: $S(V) \to P(V)$ given by $T(f)(v) = f(v, \dots, v)$ is an algebra isomorphism; the inverse map T^{-1} is usually called polarization.*

b *Let K be a group of linear transformations of V and let $S_K(V)$, $P_K(V)$ be the subalgebras of $S(V)$ and $P(V)$, respectively, consisting of K-invariant elements. Then T induces an isomorphism of $S_K(V)$ onto $P_K(V)$.*

We are mainly interested in the case $V = \mathfrak{G}$, $K = G$, acting on $\mathfrak{G}$ via the adjoint representation. We have the following fact.

Lemma 4. *Let $f \in S^k_G(\mathfrak{G})$ and let $X_1, \dots, X_k$, $Y \in \mathfrak{G}$; then*

$$\sum_{r=1}^{k} f(X_1, \dots, [Y, X_r], \dots, X_k) = 0. \tag{8.14}$$

Proof. Let $g_t = \exp tY$; then by the G-invariance of f, we have

$$f(ad(g_t)X_1, \dots, ad(g_t)X_k) = f(X_1, \dots, X_k)$$

and so, differentiating

$$0 = \frac{d}{dt} f(ad(g_t)X_1, \ldots, ad(g_t)X_k) = \sum_{r=1}^{k} f(X_1, \ldots, [Y, X_r], \ldots, X_k).$$

Let now $P = P(M, G)$ be a principal bundle. $\wedge^p(P, \mathfrak{G}^k)$ will denote the space of $\mathfrak{G}^{\otimes k}$-valued p-forms on P (cf. Definition 23 for the case $k = 1$). Therefore, computing the wedge product by means of the bilinear form $\otimes$, we have that if $\alpha \in \wedge^p(P, \mathfrak{G}^k)$ and $\beta \in \wedge^q(P, \mathfrak{G}^h)$, then $\alpha \wedge \beta \in \wedge^{p+q}(P, \mathfrak{G}^{k+h})$ is given by

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+h}) = \frac{1}{(k+h)!} \sum_{\sigma} \varepsilon(\sigma)\, \alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \otimes \beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+h)}).$$

It is clear that, if $\alpha \in \wedge^p(P, \mathfrak{G}^k)$ and $f \in S^k(\mathfrak{G})$, then $f(\alpha) = f \circ \alpha \in \wedge^p(P)$, and, in particular if $\alpha = \pi^*(\gamma)$ for $\gamma \in \wedge^p(M)$ and $s \in \wedge^0(P, \mathfrak{G}^k)$, then $f(\alpha) = \pi^*(\gamma) f(s)$.

The proof of the following proposition is straightforward.

Proposition 45.

a *Let* $\alpha \in \wedge^p(P, \mathfrak{G}^k), \beta \in \wedge^q(P, \mathfrak{G}^h), \gamma \in \wedge^r(P, \mathfrak{G}^l)$ *and let* $f \in S^{k+h+l}(P, \mathfrak{G})$. *Then*

$$f(\alpha \wedge \beta \wedge \gamma) = (-1)^{kh} f(\beta \wedge \alpha \wedge \gamma). \tag{8.15}$$

b *Let* $\alpha \in \wedge^p(P, \mathfrak{G}^k)$ *and* $f \in S^k(\mathfrak{G})$. *Then*

$$f(d\alpha) = df(\alpha). \tag{8.16}$$

Remark 13. From Lemma 4 it follows also that, if $\alpha_j \in \wedge^{p_j}(P, \mathfrak{G})$, $1 \leqslant j \leqslant k$, $\alpha \in \wedge^1(P, \mathfrak{G})$ and $f \in S^k_G(\mathfrak{G})$, then

$$\sum_{h=1}^{k} (-1)^{p_1 + \cdots + p_h} f(\alpha_1 \wedge \ldots \wedge [\alpha, \alpha_h] \wedge \ldots \wedge \alpha_k) = 0. \tag{8.17}$$

G acts on $\mathfrak{G}^{\otimes k}$ through the tensor product of the adjoint representation, i.e.

$$a(X_1 \otimes \ldots \otimes X_k) = ad(a) X_1 \otimes \ldots \otimes ad(a) X_k.$$

Let $\mathcal{T}^p(P, \mathfrak{G}^k, ad)$ be the space of $\mathfrak{G}^{\otimes k}$-valued tensorial p-forms on P. Recall that, if $\mathcal{A}^p(P)$ denotes the space of G-invariant p-forms on P, then

1 $\pi^* : \wedge^p(M) \to \mathcal{A}^p(P)$ is an isomorphism.

2 For any $\omega \in \mathcal{C}(P)$, on $\mathcal{A}^p(P)$, we have $d = D_\omega = \pi^* \circ d_M \circ (\pi^*)^{-1}$ (cf. Remark 9).

We have immediately the following

Lemma 5. *If $\alpha \in \mathcal{T}^p(P, \mathfrak{G}^k, ad)$ and $f \in S_G^k(\mathfrak{G})$, then $f(\alpha) \in \mathcal{A}^p(P)$.*

Then, we have the following remarkable

Proposition 46. *For every $\omega \in \mathcal{C}(P)$, every $\alpha \in \mathcal{T}^p(P, \mathfrak{G}^k, ad)$ and every $f \in S_G^k(\mathfrak{G})$, we have*

$$\mathrm{d}f(\alpha) = f(D_\omega \alpha). \tag{8.18}$$

Proof. It is enough to consider the case $\alpha = \pi^*(\gamma) \otimes s_1 \otimes \ldots \otimes s_k$, for $\gamma \in \wedge^p(M)$ and $s_j \in \mathcal{T}^0(P, \mathfrak{G}, ad)$, $1 \leqslant j \leqslant k$. Thus

$$D_\omega \alpha = \mathrm{d}\pi^*(\gamma) \otimes s_1 \otimes \ldots \otimes s_k + (-1)^p \pi^*(\gamma) \otimes D_\omega(s_1 \otimes \ldots \otimes s_k)$$

$$D_\omega(s_1 \otimes \ldots \otimes s_k) = d(s_1 \otimes \ldots \otimes s_k) + \omega \wedge (s_1 \otimes \ldots \otimes s_k)$$

$$= d(s_1 \otimes \ldots \otimes s_k) + \sum_{r=1}^{k} s_1 \otimes \ldots \otimes [\omega, s_r] \otimes \ldots \otimes s_k$$

and so the result follows from (8.14).

We have now the following

Proposition 47. *For every $\omega \in \mathcal{C}(P)$, every $f \in S_G^k(\mathfrak{G})$, $f(\Omega_\omega^k)$ is exact.*

Proof. Set

$$A_\omega(f) = \sum_{r=0}^{k-1} a_r\, f(\omega \wedge [\omega, \omega]^r \wedge \Omega_\omega^{k-r-1})$$

with
$$a_r = \frac{(-1)^r k!\,(k-1)!}{2^r (k+r)!\,(k-r-1)!}.$$

Then a direct computation shows that $\mathrm{d}A_\omega(f) = f(\Omega_\omega^k)$. E.g., consider the case $k = 2$; then $a_0 = 1$, $a_1 = -\dfrac{1}{6}$ and so

$$A_\omega(f) = f(\omega \wedge \Omega_\omega) - \frac{1}{6} f(\omega \wedge [\omega, \omega]).$$

Now $dA_\omega(f) = df(\omega \wedge \Omega_\omega) - \frac{1}{6} df(\omega \wedge [\omega, \omega])$

$$= f(d\omega \wedge \Omega_\omega) - f(\omega \wedge d\Omega_\omega) - \frac{1}{6} f(d\omega \wedge [\omega, \omega]) + \frac{1}{6} f(\omega \wedge d[\omega, \omega]).$$

Then $d\omega = \Omega_\omega - \frac{1}{2}[\omega, \omega] \quad d\Omega_\omega = -\frac{1}{2}[\omega, \Omega_\omega] \quad d[\omega, \omega] = 2[d\omega, \omega].$

Therefore, substituting, we obtain

$$dA_\omega(f) = f(\Omega_\omega \wedge \Omega_\omega) + \frac{2}{3} f(\omega \wedge [\omega, \Omega_\omega]) - \frac{2}{3} f([\omega, \omega] \wedge \Omega_\omega)$$

$$+ \frac{1}{12} f([\omega, \omega] \wedge [\omega, \omega]) - \frac{1}{6} f(\omega \wedge [[\omega, \omega], \omega]).$$

Finally $[[\omega, \omega], \omega] = 0$ and so, by Remark 13, $f([\omega, \omega] \wedge [\omega, \omega]) = 0$ and, again by Remark 13

$$f(\omega \wedge [\omega, \Omega_\omega]) - f([\omega, \omega] \wedge \Omega_\omega) = 0.$$

We have now

Proposition 48. *Assume Riemannian structure are assigned on M and $\mathfrak{A}_P$. Then for every $\omega \in \mathcal{C}(P)$, every $\alpha \in \mathcal{T}^p(P, \mathfrak{G}^k, ad)$ and every $f \in S_G^k(\mathfrak{G})$, we have*

$$d^* f(\alpha) = f(D_\omega^* \alpha). \tag{8.19}$$

Proof. Again, it is enough to consider the case $\alpha = \pi^*(\gamma) \otimes s$, for $\gamma \in \bigwedge^p(M)$ and $s \in \mathcal{T}^0(P, \mathfrak{G}, ad)$. Thus

$$d^* f(\alpha) = (-1)^p * d * f(\alpha) = (-1)^p * df(*\alpha) = (-1)^p * f(D_\omega * \alpha) = (-1)^p f(* D_\omega * \alpha) = f(D_\omega^* \alpha).$$

We have now the following fundamental result

Theorem 6 (Chern-Weil). *Let $P = P(M, G)$ be a principal bundle and let $\omega \in \mathcal{C}(P)$. Let $p \in S_G^k(\mathfrak{G})$ and set $p(\Omega_\omega) = p(\Omega_\omega^k)$. Then*

1 *$p(\Omega_\omega) = \pi^*(\alpha(p(\Omega_\omega)))$ for $\alpha(p(\Omega_\omega)) \in \bigwedge^{2k}(M)$ with* $d\alpha(p(\Omega_\omega)) = 0$.

2 *The map $\mathcal{W}_P : S_G(\mathfrak{G}) \to H^*(M, \mathbf{K})$ given by $\mathcal{W}_P(p) = \alpha(p(\Omega_\omega))$ does not depend on the choice of the connection and it is an algebra homomorphism (called the Chern-Weil homomorphism).*

3 *The image of $\mathcal{W}_P$ corresponds to the characteristic classes of P.*

4 *Let $H \subset G$ be a Lie subgroup and let τ: $S_G(\mathfrak{G}) \to S_H(\mathfrak{G})$ be the restriction map. Assume Q is a H-reduction of P; then*

$$\mathcal{W}_P = \mathcal{W}_Q \circ \tau \tag{8.20}$$

and so non zero elements of $\ker \tau$ *represent a universal obstruction to H-reducibility.*

Proof.

1 It follows from (8.18) and the second Bianchi's identity (5.6).

2 Let N be any smooth manifold; then a straightforward computation leads to the following homotopy formula: define Z: $\bigwedge^p(N \times [0, 1]) \to \bigwedge^{p-1}(N)$ in the following way. If $p = 0$ then $Z \equiv 0$. If $\alpha \in \bigwedge^p(N \times [0, 1])$ is written as $\alpha = dt \wedge a + b$, then $Z(\alpha) = \int_0^1 a\, dt$. Then

$$dZ(\alpha) + Z(d\alpha) = i_0^*(\alpha) - i_1^*(\alpha) \tag{8.21}$$

where i_t: $N \to N \times [0, 1]$ is given by $i_t(x) = (x, t)$.

Let now $\omega_0, \omega_1 \in \mathcal{C}(P)$ and consider $\tilde{\omega} = (1 - t)\,\omega_0 + t\omega_1 \in \mathcal{C}(P \times [0, 1])$. It is clear that $i_0^*(\tilde{\omega}) = \omega_0$ and $i_1^*(\tilde{\omega}) = \omega_1$ and thus $i_0^*(\Omega_{\tilde{\omega}}) = \Omega_{\omega_0}$ and $i_1^*(\Omega_{\tilde{\omega}}) = \Omega_{\omega_1}$. Now, for any $p \in S_G^k(\mathfrak{G})$ we have

$$d(Z(p(\Omega_{\tilde{\omega}})) = i_0^*(p(\Omega_{\tilde{\omega}})) - i_1^*(p(\Omega_{\tilde{\omega}})) = p(\Omega_{\omega_1}) - p(\Omega_{\omega_0}).$$

3 and 4 are direct consequences of definitions and basic results of the theory of characteristic classes.

9 - Linear connections

Connections on $L(M)$ share special features. Let start with the following

Definitions 40. A *linear connection* is a connection on $L(M)$. Let $\eta \in \bigwedge^1(TM)$ be defined by $\eta(X) = X$; then

$$\theta = L^{-1}(\eta) \in \mathcal{T}^1(L(M), \boldsymbol{R}^n)$$

is defined by $\theta[u](X) = u^{-1}(\pi_*[u](X))$. θ is called the *canonical form* on $L(M)$.

Let now $\omega \in \mathcal{C}(L(M))$. Consider the map

$$\chi_\omega : TL(M) \to L(M) \times \boldsymbol{R}^n \times gl(n, \boldsymbol{R})$$

defined as follows. If $X \in T_u L(M)$, then $\chi_\omega(X) = (u, \theta(X), \omega(X))$.

We have the following

Proposition 49. *χ_ω is a bundle isomorphism and so, in particular, $TL(M)$ is isomorphic to a trivial bundle and so $L(M)$ is parallelizable. Moreover χ_ω is $GL(n, \boldsymbol{R})$-equivalent and, for any $X \in gl(n, \boldsymbol{R})$, we have $\chi_\omega(X^*(u)) = (u, 0, X)$.*

Proof. It is immediate to check that χ is a bundle morphism. Moreover, fix $u \in L(M)$; then

1 Let $X \in T_u L(M)$. If $\chi_\omega(X) = (u, 0, 0)$, then both $X \in H_u^\omega$ and $\theta(X) = 0$. Thus $X = 0$ and so χ_ω is injective.

2 Given $(\xi, X) \in \boldsymbol{R}^n \times gl(n, \boldsymbol{R})$, then $\chi_\omega(u, (u(\xi))^\frown, X^*(u)) = (u, \xi, X)$, and so χ_ω is surjective.

The rest is obvious.

We have now

Definition 41. $B(\xi)(u) = \chi_\omega^{-1}(u, \xi, 0)$ is called the *fundamental ω-horizontal vector field* corresponding to $\xi \in \boldsymbol{R}^n$.

The proof of the following proposition is straightforward.

Proposition 50.

1 *For any $a \in GL(n, \boldsymbol{R})$, $\xi \in \boldsymbol{R}^n$, we have*

$$(R_a)_* (B(\xi)) = B(a^{-1}\xi). \tag{9.1}$$

2 *If $\xi \neq 0$, then $B(\xi)$ never vanishes.*

3 *For any $A \in gl(n, \boldsymbol{R})$, $\xi \in \boldsymbol{R}^n$, we have*

$$[A^*, B(\xi)] = B(A\xi). \tag{9.2}$$

Note also that, if $\{e_1, \dots, e_n\}$ and $\{E_{jk}\}_{1 \leq j, k \leq n}$ are the standard basis of $\boldsymbol{R}^n$and $gl(n, \boldsymbol{R})$ respectively, then $\{B(e_1), \dots, B(e_n), E_{11}^*, \dots, E_{nn}^*\}$ represents a global section of $L(L(M))$.

Definition 42. Given $\omega \in \mathcal{C}(L(M))$, $\Theta_\omega = D_\omega \theta$ is called the *torsion form* of ω

and $T_\omega = L(\Theta_\omega) \in \bigwedge^2(TM)$ is called the *torsion tensor* of ω. It is easy to check that

$$T_\omega(X, Y) = (\nabla_\omega)_X Y - (\nabla_\omega)_Y X - [X, Y]. \tag{9.3}$$

As a consequence of Proposition 26 and Proposition 28, we have the following

Proposition 51. *We have*

$$\Theta_\omega = \mathrm{d}\theta + \omega \wedge \theta \tag{9.4}$$

$$D_\omega \Theta_\omega = \Omega_\omega \wedge \theta \qquad (\textit{first Bianchi' s identity}). \tag{9.5}$$

A basic result in Riemannian geometry is the following

Theorem 7. *Let (M, g) be a Riemannian manifold; then on $L(M)$ there exists a unique Riemannian connection with vanishing torsion. This connection is called the Levi-Civita connection.*

The exterior covariant differential operator of the Levi-Civita connection of (M, g) is usually denoted by ∇^M.

In general, given a vector bundle E equipped with an exterior covariant differential operator ∇, we can consider the tensor product $\tilde{\nabla} = \nabla^M \otimes \nabla$: $\bigwedge^0(\bigwedge^p E) \to \bigwedge^1(\bigwedge^p E)$. We have immediately

$$(\tilde{\nabla}_X \sigma)(X_1, \ldots, X_p) = \nabla_X \sigma(X_1, \ldots, X_p) - \sum_{j=1}^{p} \sigma(X_1, \ldots \nabla_X^M X_j, \ldots, X_p) \tag{9.6}$$

and therefore

$$\begin{aligned}(\nabla\sigma)(X_0, \ldots, X_p) &= \sum_{j=0}^{p} (-1)^j \nabla_{X_j} \sigma(X_0, \ldots, \overset{\circ}{X_j}, \ldots, X_p) \\ &+ \sum_{0 \leq i < k \leq p} (-1)^{j+k} \sigma([X_j, X_k], X_0, \ldots, \overset{\circ}{X_j}, \ldots, \overset{\circ}{X_k}, \ldots, X_p) \\ &= \sum_{j=0}^{p} (-1)^j (\tilde{\nabla}_{X_j} \sigma)(X_0, \ldots, \overset{\circ}{X_j}, \ldots, X_p).\end{aligned} \tag{9.7}$$

10 - Moduli spaces

We want to give a very brief account on moduli spaces; for a more detailed description and proofs, we refer to a forthcoming paper.

Let P be a principal G-bundle over a compact Riemannian manifold (M, g) and let ρ be a Riemannian structure on its adjoint bundle $\mathfrak{A}_P = P \times_{ad} G$. Set

$$\mathcal{G}_s(P) = L_s^2\text{-completion of the gauge group } \mathcal{G}(P)$$

$$\mathcal{L}_s^p(P, G) = L_s^2\text{-completion of } \mathcal{T}^p(P, G, ad)$$

$$\mathcal{L}_s^p(P, g) = L_s^2\text{-completion of } \mathcal{T}^p(P, \mathfrak{G}, ad)$$

(with respect to $g \otimes \rho$). Fixing $\omega_0 \in \mathcal{C}(P)$, define

$$\mathcal{C}_s(P) = \omega_0 + \mathcal{L}_s^1(P, \mathfrak{G}).$$

Therefore
$$\Omega\colon \mathcal{C}_s(P) \to \mathcal{L}_{s-1}^2(P, \mathfrak{G})$$

and
$$\Omega_*[\omega]\colon \alpha \mapsto D_\omega \alpha.$$

$\mathcal{G}_{s+1}(P)$ acts smoothly one the right on $\mathcal{C}_s(P)$. This action is not effective; in fact

$$f\colon \omega \mapsto \omega \quad \text{for every} \quad \omega \in \mathcal{C}_s(P) \Leftrightarrow f \in C(G) \subset \mathcal{G}_{s+1}(P).$$

Therefore, if we set $\mathcal{G}_{s+1}^*(P) = \mathcal{G}_{s+1}(P)/C(G)$, then $\mathcal{G}_{s+1}^*(P)$ acts effectively on $\mathcal{C}_s(P)$. Set $\mathcal{L}_{s+1}^*(P, G) = \mathcal{L}_{s+1}^0(P, G)/C(G)$.

Proposition 52. *Let $\omega \in \mathcal{C}_s(P)$; then, the following facts are equivalent*

a $D_\omega\colon \mathcal{L}_{s+1}^0(P, \mathfrak{G}) \to \mathcal{L}_s^1(P, \mathfrak{G})$ *has a non trivial kernel*

b ω *is a fixed point for some* $f \in \mathcal{G}_{s+1}^*(P)$.

Definition 43. ω is said to be *simple* if **a** (or **b**) does not hold. Set

$$\widehat{\mathcal{C}}_s(P) = \{\omega \in \mathcal{C}_s(P) \mid \omega \text{ is simple}\}$$

thus $\mathcal{G}_{s+1}^*(P)$ acts freely on $\widehat{\mathcal{C}}_s(P)$.

Note that, for any $\omega \in \widehat{\mathcal{C}}_s(P)$, we have

$$T_\omega(\mathcal{G}^*_{s+1}(P)\,\omega) = D_\omega \mathcal{L}^0_{s+1}(P, \mathfrak{G}).$$

We have the following

Proposition 53. *Let $\omega_0 \in \widehat{\mathcal{C}}_s(P)$. Then, there exists a neighbourhood of* 0 *in* $\omega_0 + \ker D^*_{\omega_0}$, *such that $U = \mathcal{G}^*_{s+1}(P)\,V$ is a neighbourhood of ω_0 in $\widehat{\mathcal{C}}_s(P)$ diffeomorphic to $V \times \mathcal{G}^*_{s+1}(P)$. More precisely, there exists a smooth map $\sigma\colon U \to \mathcal{G}^*_{s+1}(P)$ such that*

a *For every $\omega \in U$, $\sigma(\omega)\,\omega \in V$*

b *$\Sigma\colon U \to V \times \mathcal{G}^*_{s+1}(P)$ given by $\Sigma(\omega) = (\sigma(\omega)\,\omega, \sigma(\omega))$ is a $\mathcal{G}^*_{s+1}(P)$-equivariant diffeomorphism (where $\mathcal{G}^*_{s+1}(P)$ acts on the right on $V \times \mathcal{G}^*_{s+1}(P)$ as $g(\omega, f) = (\omega, g^{-1}f)$).*

Note that Σ corresponds to the following map in a usual principal bundle structure:

Given a local trivialization $\Phi\colon \pi^{-1}(U) \to U \times G$, $\Phi(u) = (\pi(u), \psi(u))$, let $s(x) = \Phi^{-1}(x, e)$ be the associated section. Then $\Sigma\colon \pi^{-1}(U) \to s(U) \times G$ is given by $\Sigma(u) = (u\psi^{-1}(u), \psi^{-1}(u))$.

Theorem 8. *$\mathfrak{M}_s(P) = \widehat{\mathcal{C}}_s(P)/\mathcal{G}^*_{s+1}(P)$ is a Hilbert manifold and $\widehat{\mathcal{C}}_s(P) \to \mathfrak{M}_s(P)$ is a principal $\mathcal{G}^*_{s+1}(P)$-bundle.*

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