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Strictly stable linear ordinary differential equations and similarity (**)

Alla memoria di Giorgio Sestini

1. - Let $\mathscr{M}$ denote the complex linear space of functions $M: t \to M(t)$ defined for $t \in J =]\alpha, \omega[$, $-\infty \leq \alpha < \omega \leq +\infty$, with complex $n \times n$ matrix values $M(t)$, continuous on J.

Let $\mathscr{M}'$ denote the subspace of functions $M \in \mathscr{M}$ which are continuously differentiable on J.

Let us also denote by $|M(t)|$ the euclidean norm of $M(t)$, by $\int^{\omega} |M(t)|\, dt$ the $\lim_{T \to \omega} \int^{T} |M(t)|\, dt \leq +\infty$, and, when they exist, by $M^{-1}(t)$ the inverse of $M(t)$ and by $\dot{M}(t)$ the derivative $dM(t)/dt$.

We shall consider four equivalence relations in $\mathscr{M}$ defined as follows.

Def. 1.1 (R. Conti [**1**]$_1$). $A, B \in \mathscr{M}$ are *integrally similar* if there exist $L \in \mathscr{M}'$ such that

$$|L(t)|\,|L^{-1}(t)| < l(\theta) \qquad \theta \in J \quad \theta \leq t \tag{1.1}$$

$$\int_{\theta}^{\omega} |\dot{L}(t) - A(t)L(t) + L(t)B(t)|\, dt < +\infty \qquad \theta \in J. \tag{1.2}$$

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Questo lavoro si riallaccia a quello del 1979: R. CONTI, *Equazioni differenziali lineari asintoticamente equivalenti a* $x = 0$, pubblicato su questa Rivista nel volume (4) 5** dedicato a Giorgio Sestini nel 70° compleanno. Per espresso desiderio dell'Autore, il presente lavoro viene ora pubblicato alla memoria del compianto prof. Giorgio Sestini, deceduto l'11-XII-1991. (N.d.R.)

Def. 1.2. $A, B \in \mathscr{M}$ are *strongly integrally similar* if there exist $L \in \mathscr{M}'$ such that

$$\lim L(t) = I \qquad I \text{ the unit } n \times n \text{ matrix} \tag{1.3}$$

and (1.2) are satisfied.

Def 1.3 (L. Markus [3]). $A, B \in \mathscr{M}$ are *kinematically similar if there exist* $L \in \mathscr{M}'$ *such that (1.1) and*

$$\dot{L}(t) - A(t)L(t) + L(t)B(t) = 0 \qquad t \in J \tag{1.4}$$

are satisfied.

Def. 1.4. $A, B \in \mathscr{M}$ are *strongly kinematically similar* if there exist $L \in \mathscr{M}'$ satisfying (1.3) and (1.4).

2. - Let $A \in \mathscr{M}$. Then if $X : t \to X(t)$ represents a non singular matrix solution of the linear ordinary differential equation

$$\dot{x} - A(t)x = 0 \tag{A}$$

we can associate with A its Cauchy (or evolution) operator $\mathrm{E}_A : (t, s) \to \mathrm{E}_A(t, s)$ defined by

$$\mathrm{E}_A(t, s) = X(t)X^{-1}(s) \qquad t, s \in J .$$

Def. 2.1. *A belongs to the subclass* $\mathscr{S} \subset \mathscr{M}$ if for $\theta \in J$ there exist $a(\theta) \geq 1$ such that

$$|\mathrm{E}_A(t, s)| < a(\theta) \qquad \theta \leq t, s . \tag{2.1}$$

Remark. When (2.1) holds equation (A) is called *strictly stable* («strettamente stabile») according to Guido Ascoli [**1**].

It is known that

Theorem 2.1. *$\mathscr{S}$ is the equivalence class of $0 \in \mathscr{M}$ both for kinematic and integral similarity.*

For the proof see R. Conti [2]$_2$ (pp. 74-75) where different notations and terminology are used.

3. - Def. 3.1. *A belongs to the subclass* $\mathscr{C} \subset \mathscr{M}$ if for $\theta \in J$

$$\lim_{t \to \omega} \mathrm{E}_A(t, \theta) \text{ exists and it is non singular.}$$

Clearly $\mathscr{C}$ is a subclass of $\mathscr{S}$ and it can be proved (R. Conti [2]$_3$).

Theorem 3.1. *$\mathscr{C}$ is the equivalence class of $0 \in \mathscr{M}$ both for strong kinematic and strong integral similarity.*

Since $\mathscr{C}$ is a proper subclass of $\mathscr{S}$, this means that $\mathscr{S}$ is partitioned by strong similarities into several equivalence classes. Therefore the following is an extension of Theorem 2.1.

Theorem 3.2. *The equivalence classes in $\mathscr{S}$ are the same for strong kinematic similarity and strong integral similarity.*

Proof. It is easy to verify that if $A, B \in \mathscr{M}$ and $L \in \mathscr{M}'$, then

$$L(t)\,\mathrm{E}_B(t, s) = \mathrm{E}_A(t, s)\,L(s) + \int_s^t \mathrm{E}_A(t, r)\,F(r)\,\mathrm{E}_B(r, s)\,\mathrm{d}r \qquad t, s \in J \tag{3.1}$$

holds, where

$$F(r) = \dot{L}(r) - A(r)\,L(r) + L(r)\,B(r)\,.$$

Let A, B be strongly integrally similar. By Theorem 2.1 if $A \in \mathscr{S}$ then also $B \in \mathscr{S}$ and by virtue of (1.2) we can define

$$W(s) = L(s) + \int_s^\omega \mathrm{E}_A(s, r)\,F(r)\,\mathrm{E}_B(r, s)\,\mathrm{d}r\,. \tag{3.2}$$

From (3.1) and (3.2) it follows

$$\mathrm{E}_B(t, s) - \mathrm{E}_A(t, s)\,W(s) = [L^{-1}(t) - I]\,\mathrm{E}_A(t, s)[L(s)$$

$$+ \int_s^t \mathrm{E}_A(s, r)\,F(r)\,\mathrm{E}_B(r, s)\,\mathrm{d}r] - \int_t^\omega \mathrm{E}_A(t, r)\,F(r)\,\mathrm{E}_B(r, s)\,\mathrm{d}r$$

and taking into account (1.3) we have

$$\lim_{t\to\omega}[\mathrm{E}_B(t,\ s)-\mathrm{E}_A(t,\ s)W(s)]=0\,.$$

Therefore, if we define, for a fixed $s\in J$

$$M(t)=\mathrm{E}_A(t,\ s)W(s)\,\mathrm{E}_B(s,\ t) \tag{3.3}$$

we have

$$\lim_{t\to\omega}M(t)=I\,.$$

From (3.3) we have also $M\in\mathscr{M}'$ and

$$\dot{M}(t)-A(t)M(t)+M(t)B(t)=0$$

so that A, B are strongly kinematically similar.

Remark. All what precedes remains valid if $\mathscr{M}$ denotes the space of complex $n\times n$ matrix valued functions M which are measurable and locally Lebesgue integrable on J, and by $\mathscr{M}'$ the subspace of M which are locally absolutely continuous on J.

The only difference is that the solutions of (A) are solutions in the sense of Carathéodory.

References

[1] G. Ascoli, *Osservazioni sopra alcune questioni di stabilità*, Nota I, Rend. Acc. Naz. Lincei, Cl. Sc. fis. mat. nat., (VIII) **IX** (1950), 129-134.

[2] R. Conti: $[\bullet]_1$ *Sulla t-similitudine tra matrici e la stabilità dei sistemi differenziali lineari*, Rend. Acc. Naz. Lincei, Cl. Sc. fis. mat. nat., (VIII) **XIX** (1955), 247-250; $[\bullet]_2$ *Linear differential equations and controllability*, Academic Press, 1976; $[\bullet]_3$ *Equazioni differenziali lineari asintoticamente equivalenti a* $\dot{x}=0$, Riv. Mat. Univ. Parma (4) **5** (1979), 847-853.

[3] L. Markus, *Continuous matrices and the stability of differential systems*, Math. Zeitsch., **62** (1955), 310-319.