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Rings in which all proper ideals are isomorphic (**)

Let R be a nonzero ring with or without an identity. A nonzero ideal different from R will be called *proper ideal* of R. A ring having no proper ideal will be called a *weakly simple ring*. Let $G = (R, +)$, the additive group of R. The 0-rank of G will be called the *0-rank* of R, which is the cardinality of a maximal independent subset of elements with infinite order in G. This terminology can be found in [2].

Def. Let R be a ring. R is called a PII-*ring*, briefly PII, if all proper ideals are isomorphic as rings.

Let $T_R = \{x \in R: x \text{ has finite order in } G\}$ and let B^r (resp. ${}^lB$) denote the right (resp. *left*) annihilator of ideal B in R. It is well-known that T_R, B^r and ${}^lB$ are ideals of R.

Lemma 1. *Suppose that R is* PII, *A a proper ideal of R. Then:*

(1) *A has infinite characteristic iff $(A, +)$ is torsion free.*
(2) *A has finite characteristic iff $pA = 0$ holds, for some prime number p.*

Proof. Let $T_A = A \cap T_R$. Then T_A is an ideal of R. If $T_A \neq 0$, then $T_A \cong A$ implies $A = T_A$. If there are two nonzero, x, y in A such that $p^m x = 0$ and $q^n y = 0$, for distinct primes p, q and positive integers m, n, then $\{x \in A: p^m x = 0\} \ncong \{y \in A: q^n y = 0\}$. If $m \geq 2$, $p^m x = 0$ and $p^{m-1} x \neq 0$, then $\{x \in A: p^2 x = 0\} \ncong \{x \in A: px = 0\}$. Thus we conclude that $pA = 0$, for some pri-

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me number p. So A has infinite characteristic iff $T_A = 0$, i.e. $(A, +)$ is torsion free.

Lemma 2. *Suppose that* R *is* PII *and* A *is a proper ideal with* $A^2 \neq 0$. *Then:*

(1) $A^n \neq 0$ *holds, for each positive integer* n.

(2) $A^n = A^{n+1}$ *iff* $A^2 = A$.

(3) *If* $(A, +)$ *is torsion free and* R *has finite* 0*-rank then* $A^2 = A$ *and* R *is Artinian.*

(4) ${}^lB = B^r$ *holds, for each proper ideal* B *of* R.

Proof. (1) Let $A^n = 0$ but $A^{n-1} \neq 0$. Then $(A^{n-1})^2 = 0$. So $A^2 = 0$, which is a contradiction.

(2) If $A^n = A^{n+1}$, then $A^n = (A^n)^2$. So $A^2 = A$.

(3) Put $A_0 = \bigcap_{n=1}^{\infty} nA$. Then $(A_0, +)$ is divisible. Since $(nA)(nA) \subseteq n(nA)$ and $nA \cong A$, for all n, $A^2 \subseteq A_0$. Thus $(A^n, +)$ is divisible since $A^n \cong A_0$. Suppose that $A \supsetneqq A^2 \supsetneqq A^3 \supsetneqq \ldots$.

Then by [2] (Theorem 4.1.3)

$$\begin{aligned} (A, +) &= (A^2, +) \oplus K_1 \qquad && K_1 \neq 0 \\ &= (A^3, +) \oplus K_2 \oplus K_1 && K_2 \neq 0 \\ &\cdots\cdots\cdots \end{aligned}$$

this contradicts the finite 0-rank of R. The same contradiction arises if $A_1 \supsetneqq A_2 \supsetneqq A_3 \supsetneqq A_4 \supsetneqq \ldots$ is a descending chain of proper ideals of R. Hence R is Artinian.

(4) Let $B^r \neq 0$. Since $(B^r \cdot B)^2 = 0$ and $B^r \cdot B \not\supseteq A$, $B^r \cdot B = 0$ and $B^r \subseteq {}^lB$. Similarly, ${}^lB \subseteq B^r$. So ${}^lB = B^r$.

Theorem 1. *Suppose that* R *is* PII, $T_R = 0$ *and* A *is a proper ideal with* $A^2 \neq 0$. *Then the following statements are equivalent:*

(1) A *is* PII *as a ring.*

(2) A *is a hereditarily idempotent ring, i.e.* $I^2 = I$ *holds for each ideal* I *of* A.

(3) $A^2 = A$.

(4) *Each ideal of* A *is an ideal of* R.

Proof. (1) $\Rightarrow$ (2). Suppose that A is PII, B is a proper ideal of A and $\overline{B}$ is the ideal generated by B in R. Then by Lemma 2, $\overline{B}^3 \neq 0$ and $\overline{B}^3 \subseteq B$. So $B \cong \overline{B}^3 \cong A$. By [1] (Lemma 3.4) $A^2 = A$ and $B^2 = B$.

(2) $\Rightarrow$ (3). Obviously.

(3) $\Rightarrow$ (4). Let B be an ideal of A. Then $\overline{B}^2 = \overline{B}$ since $\overline{B} \cong A$ and $B \subseteq \overline{B} = \overline{B}^3 \subseteq B$, i.e. $B = \overline{B}$ is an ideal of R.

(4) $\Rightarrow$ (1). Obviously.

Theorem 2. *Suppose that R is* PII, $T_R = 0$ *and A is a proper ideal of R with $A^2 \neq 0$. If R has finite 0-rank then the following statements are equivalent for all proper ideals B of R:*

(1) $B^r \neq 0$;

(2) $R = B \oplus B^r$, *where B and B^r are weakly simple rings.*

Proof. (2) $\Rightarrow$ (1). Obviously.

(1) $\Rightarrow$ (2). Since $(B \cap B^r)^2 = 0$ and $A^2 \neq 0$, $B \cap B^r = 0$. Suppose that $R \neq B \oplus B^r$. Then $B \cong B \oplus B^r$. By Lemma 2 (3) and Theorem 1, there are nonzero ideals C, D of R such that $B = C \oplus D$. Since $C \cong B \cong D$, there are nonzero ideals E, F, G, and H of R such that $C = E \oplus F$ and $D = G \oplus H$. Continuing in this way, we obtain a lot of nonzero ideals B_1, B_2, $B_3, \ldots$ such that $B = B_1 \oplus B_2 \oplus B_3 \oplus \ldots$, which contradicts the finite 0-rank of R. Similarly, if K is a proper ideal of B^r, then K is a proper ideal of R. Thus $B \cong B \oplus K$, which leads to the same contradiction.

By [1] (Theorem 3.8 and Cor. 3.9), for the ideal B described in Theorem 2, we have the following

Corollary. *R, B as above. Then:*

(1) *If R has an identity, then $B^r \neq 0$ iff $R = B \oplus B^r$ where B, B^r are simple.*

(2) *If R is commutative, then $B^r \neq 0$ iff $R = B \oplus B^r$, where B, B^r are fields.*

We now pay attention to the case that R is commutative.

Theorem 3. *Suppose that R is a commutative* PII-*ring*, $T_R = 0$ *and A is a proper ideal of R. Then:*

(1) *If $A^2 = 0$, then R is either a null on an infinite cyclic group or a local ring in which the maximal ideal consists of all elements x with $x^2 = 0$ of R.*

(2) *If* $A^2 \neq 0$, *then for each proper ideal* K *of* R,

$$(R, +) = (K, +) \oplus K_1$$

where K_1 *is a subgroup of* $(R, +)$. *In particular*, $R = R^2$.

(3) *If* $A^2 \neq 0$ *and* R *has finite* 0-*rank, then* R *is a direct sum of two fields.*

Proof. (1) Let $A^2 = 0$. Then $B^2 = 0$ holds for each proper ideal B of R.

Case 1. $R^2 = 0$. In this case, every subring of R is an ideal of R. For any $0 \neq a \in R$, let $Z[a] = \{na: n \in Z\}$, where Z denotes the set of all integers. Then $Z[a]$ is an ideal of R, in particular, a null ring. If $Z[a] \neq R$, then $Z[a] \cong A$. Assume that $R \neq nR$, for some positive integer n. Then $nR \cong Z[na]$. So $R \cong Z[a]$. Assume that $R = nR$, for all integers n. Then, by [2] (Theorem 4.1.5) $(R, +) \cong \Sigma \oplus Q$, where Q is the rational numbers additive group. Clearly, Q has a proper subgroup B that is not cyclic, e.g.

$$\frac{1}{p_1^2} \notin B = \bigcup_{i=1}^{\infty} \left\langle \frac{1}{p_1 p_2 \cdots p_i} \right\rangle$$

where $p_1, p_2, p_3 \ldots$ are all distinct prime numbers. Thus R has a proper subring that is not a null ring on an infinite cyclic group, this is a contradiction. So $R = Z[a]$, i.e. R is a null ring on an infinite cyclic group.

Case 2. $R^2 \neq 0$. We shall prove that R has an identity 1. In fact, since R is commutative, there is an element a in R such that $a^2 \neq 0$. Thus $R = Za + Ra$, where Z is the set of integers. If Ra is a proper ideal of R, then $(Ra)^2 = 0$ since $Ra \cong A$. So $2Za + Ra$ is a proper ideal of R, $a^2 = 0$ follows from $(2a)^2 = 0$. Hence $R = Ra$. Let $a = xa$. Then $(x^2 - x)a = 0$. Note that $((Ra)^r)^2 = 0$. If we put $t = x^2 - x$, then: (i) $x^2 = x$ if $t = 0$; (ii) $x - 2xt + t$ is a nonzero idempotent element if $t \neq 0$.

In either case, we obtain a nonzero idempotent element e in R. Thus $R = Re \oplus R(1-e)$, where $R(1-e) = \{y - ye: y \in R\}$. So $R(1-e) = 0$ since $R^2 \neq 0$. $e = 1$ is an identity of R.

Now, put $B = \{x \in R: x^2 = 0\}$. To show that B is an ideal, it is enough to show that B is a subgroup of $(R, +)$. Suppose that $x + y \notin B$ for some nonzero $x, y \in B$. As in the case for a, $R = R(x + y)$. So we have a z in R such that $z(x + y) = 1$. Thus $1 - zy \in (Rx)^r$ and $(1 - zy)^2 = 0$. So $2zy = 1$ and $y = 2zy^2 = 0$ contrary to $y \neq 0$.

(2) From the proof of Lemma 2, $(A, +)$ is divisible, $(K, +)$ is divisible, for each proper ideal K of R. By [2] (Theorem 4.1.3), $(R, +) = (K, +) \oplus K_1$, where K_1 is a

nonzero subgroup of $(R, +)$. In particular, take $K = R^2$. If $R \neq R^2$, then for each proper subgroup H of K_1, $R^2 \oplus H$ is a proper ideal of R. So H is divisible. Note that K_1 is also torsion free. Thus we can take a proper subgroup $H \cong Z$, which is a contradiction because Z is not divisible.

(3) From the corollary of Theorem 2, it is enough to prove that there is a proper ideal B in R such that $B^r \neq 0$. Suppose that $(Ra)^r = 0$ for all $a \neq 0 \in R$. Let $a \neq 0 \in R$. By Lemma 2, $a^n R = a^{n+1} R$ holds for some n. Thus $R = aR$, whence R is a field, which contradicts the fact that A is a proper ideal of R.

Example 1. Let F be a field. Let $R = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in F\}$. Then R is a local ring with a unique proper ideal $A = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in F\}$.

The author is unable to give an example of R described in Theorem 3 (2). But some properties of such rings are obtained.

Theorem 4. *Suppose that R is a commutative* PII*-ring. $T_R = 0$ and A is a proper ideal with $A^2 \neq 0$. Then:*

(1) *$y \in Ry$ for all $y \in R$.*
(2) *If $(Ry)^r = 0$, for some $y \neq 0 \in R$, then R has an identity.*

Proof. Note that $(A, +)$ is divisible (see the proof of Lemma 2 (3)). Firstly we shall prove that $x \in Rx$, for all proper ideals K and $x \in K$.

Let $x \neq 0 \in K$ and let $I = Zx + Rx$. Then I is a proper ideal of R, moreover $(I/Rx, +)$ is cyclic (with generator $x + Rx$). But, $(I, +)$ is divisible and so is $(Rx, +)$. Consequently, if x is not in Rx, then $(I/Rx, +) \cong \Sigma \oplus Q$ by [2] (Theorem 4.1.5), which is a contradiction. So $I = Rx$ i.e. $x \in Rx$.

Now suppose that $y \notin Ry$, for some $y \in R$. It is clear that $Ry \neq 0$. Because, if $Ry = 0$ then the ideal generated by y in R is either R or isomorphic to A, which gives $A^2 = 0$. According to the above fact, $R = nZy + Ry$, for all integers $n > 1$. Thus $y = nmy - ry$ for $m \in z$ and $r \in R$, consequently $R = (nm-1)Zy + Ry = Ry$ which is a contradiction.

If $(Ry)^r = 0$ for some $y \neq 0 \in R$, then there is a nonzero idempotent element x in R such that $y = xy$. If $R(1-x) = \{r - rx : r \in R\} \neq 0$, then $R = Rx \oplus R(1-x)$. By Theorem 1, Rx and $R(1-x)$ are PII, by [2] (Cor. 3.9), Rx and $R(1-x)$ are fields. Thus R has an identity.

Theorem 5. *Suppose that R is a commutative* PII-*ring and A is a proper ideal. If $T_R \neq 0$ then $R = T_R$. Moreover $p^2R = 0$, for some prime p.*

Proof. If $R \neq T_R$, then $pT_R = 0$ by Lemma 1, for some prime p. If $R^2 = 0$, then we can take an $a \in R$ but $a \notin T_R$. So $Z[a] \cap T_R = 0$, whence $Z[a] \cong T_R$, which to leads to $Z[a] \subseteq T_R$, a contradiction. If $R^2 \neq 0$ and $R \neq T_R$, then we can take an $a \in R$ but $a \notin T_R$, and so $na \notin T_R$ for all integers $n \neq 0$. If $R \neq aR$ then $aR \subseteq T_R$ since $aR \cong T_R$, and so $R = Za + T_R$. Thus $R \neq 2Za + T_R$ and $2Aa \subseteq T_R$, whence $a \in T_R$, this is a contradiction. Hence $R = aR$, for all $a \in R$ but $a \notin T_R$. Let $x \in R$ such that $a = ax$. So $x^2 - x \in (Ra)^r$.

(i) $A^2 = 0$: it is clear that $(Ra)^r \neq R$ since $a = ax^2$. So $((Ra)^r)^2 = 0$. Thus we obtain a nonzero idempotent element e in R, consequently e is an identity since $R^2 \neq 0$.

(ii) $A^2 \neq 0$: $((Ra)^r)^2 = ((R \cap (Ra)^r)^2 = (Ra \cap (Ra)^r)^2 = 0$. Thus $(Ra)^r = 0$, whence we still have an identity e in R.

Now let e be an identity in R. Then for all $a \in R$ but $a \notin T_R$, a is invertible. In particular $a = pe$ is invertible. Thus $peT_R = pT_R = 0$, which to leads to $T_R = 0$, a contradiction! As in the proof of Lemma 1, $p^2R = 0$ holds.

Example 2. $R = Z_{p^2} = \{\bar{1}, \bar{2}, \bar{3}, \dots, \overline{p}^2\}$, the ring of integers modulo p^2, is an example of a ring described in Theorem 5.

Example 3. $R = R_1 \oplus R_2$, where $(R_i, +) \cong (Z_p, +)$ and $R_i^2 = 0$ $(i = 1, 2)$.

Theorem 6. *Suppose that R is a commutative* PII-*ring, $R^2 \neq 0$ and $p^2R = 0$ but $pR \neq 0$, for some prime p. Then:*

(1) *R has an identity.*

(2) *R is a local ring with a unique maximal ideal B consisting of all elements x that $px = 0$ in R.*

Proof. (1) Let $x \in R$ but $px \neq 0$. Then $R = Zx + Rx$. If $x \notin Rx$ then $R^2 = (Zx + Rx)^2 = Rx \cong pR$ and $px^2 = 0$. Thus $A = \{0, px, 2px, \dots, (p-1)px\}$ is a proper ideal of R and $Rx \cong A$. If $Zx \cap Rx = 0$ then $A \oplus Rx \cong Rx$ which is impossible because Rx has just p elements. Thus, let $0 \neq nx \in Rx$. Then $n = pm$ for some integer $0 < m < p$. So $R = Zmx + Rx = Zmx + A = Zmx \cong Z_{p^2}$ and R has an identity, which contradicts $x \notin Rx$. If $x \in Rx$ then $x = ax$ thus $a^2 - a \in (Rx)^r$. Since $((Rx)^r)^2 = 0$, as in the proof of Theorem 3 (1), there is an identity in R.

(2) By (1), for all $x \in R$ but $px \neq 0$, $R = Rx$. So x is invertible, and then B is a unique maximal ideal of R.

References

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Abstract

This paper is concerned primarily with rings having the property that all proper ideals are isomorphic as rings.

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