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Analysis of a «quasi molecular» model relative to an elastic bar vibrating against a rigid wall (**)

1 - Introduction

The problem of the motion of a dynamical elastic system subject to a unilateral constraint is of considerable interest and has been studied by various methods, both from a theoretical and from numerical point of view, especially in the case of a string vibrating against a rigid wall (see, for instance, [1], [2], [3], [4], [5]).

A method of approach is to introduce a discrete model which physically approsimates the problem and to substitute such a model to the original system; this has been done, for instance, in $[5]_2$ for the problem of a string vibrating against a rigid wall.

In the present paper we shall consider the motion of an elastic bar, clamped at one end, in the presence of a rigid obstacle, represented by the half plane $\eta \geqslant 0$, against which the bar can vibrate, assuming that the corresponding shocks are perfectly elastic.

This problem is studied adopting a «discrete» model of the bar, equal to the one introduced by Greenspan $[5]_2$, in which the bar is considered as an aggregate of «quasi molecules» which exercise upon each other attraction and repulsion forces, given by empirical lows. The study of the motion of the bar is therefore reduced to that of the single «quasi molecules».

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It is interesting to note that such a «quasi molecular» approach has been applied by Greenspan also to other problems related to the bar, such as heat conduction and melting $[5]_2$, obtaining some interesting results which cannot be reproduced utilizing the classical continuum model.

Following the scheme given by Greenspan, we shall assume that the bar is costituted by n «quasi molecules» (whose coordinates are denoted by ξ_i, η_i $i = 1, \ldots, n$) which, when at rest, are placed at the vertices of equilateral triangles; the first five «quasi molecules» P_1, P_2, P_3, P_4, P_5, are fixed, while the others are subjected to the external force f_i and the obstacle is represented by the surface $\eta = 0$.

The «intermolecular» forces are, as mentioned above, of two types: attraction force F_a and repulsion force F_r; the first must prevail at «large» distances, while the second must be stronger at «small» distances. These forces are, according to Greenspan, of the form

$$F_r = A/r^\alpha \qquad F_a = B/r^\beta \tag{1.1}$$

where r is the distance of two «quasi molecules» and A, B, α, β are positive costants, determined empirically on the basis of computer experiments; it must, however, necessarly be, $\alpha > \beta$. In what follows, we shall therefore always assume that this condition is satisfied.

While, in Greenspan's works, time is also discretized, thus reducing the problem to that of an algebraic system, in the present paper the discretization is limited to the space variables and, consequently, the equations of motion costitute a system of strongly non linear ordinary differential equations.

In the framework set out above, it appears natural to simulate the action of the rigid wall by assuming that the wall exercises on the «quasi molecules» a repulsion force given by

$$F_{i\eta} = \frac{C}{(\eta_i)^\gamma} \qquad C > 0 \qquad \gamma > 1 \tag{1.2}$$

where η_1 obviously represents the distance of the «quasi molecules» P_i from the wall.

Our aim is to give *a global existence and uniqueness theorem for the solution of the equations corresponding to the model described above and for a similar one, in which, however, the attraction forces between the «quasi molecules» are «retarded»*. The introduction of such a retarded model is justified by the fact that, while the repulsion forces are «short range» forces, the attraction forces have a «long range» and therefore can be assumed to act subsequently to the repulsion forces. In this way, it is also possible to take into account the fact that the velocity for the propagation of perturbations inside the bar is finite and not infinite, as would be the case for «instantaneous» forces.

Obviously, a more sophisticated model should take into account the fact that the retarded action of the forces depends on the distance of the «quasi molecules».

2 - Mathematical formulation of the model

Let us introduce some notations which will be used in the sequel.

Let (η_i, ξ_i) be the coordinates of the «quasi molecule» P_i in the (ξ, η) reference plane and

$$\lambda_{ij} = \sqrt{(\eta_i - \eta_j)^2 + (\xi_i - \xi_j)^2} \tag{2.1}$$

the distance between the two «quasi molecules» P_i and P_j; we shall denote by p_{ij} and q_{ij} respectively the attraction and repulsion forces exercised by P_j on P_i; the components of these forces are given by

$$p_{ij,\xi} = -\frac{B}{(\lambda_{ij})^\beta}(\xi_i - \xi_j) \qquad p_{ij,\eta} = -\frac{B}{(\lambda_{ij})^\beta}(\eta_i - \eta_j)$$

$$q_{ij,\xi} = \frac{A}{(\lambda_{ij})^\alpha}(\xi_i - \xi_j) \qquad q_{ij,\eta} = \frac{A}{(\lambda_{ij})^\alpha}(\eta_i - \eta_j)$$

according to what already illustrated in **1**.

Finally, $r_{i\eta}$ will denote the force of repulsion (obviously parallel to the η-axis) exercised by the obstacle on P_i, given by

$$r_{i,\eta} = \frac{C}{(\eta_i)^\gamma} \qquad C > 0 \qquad \gamma > 1 . \tag{2.2}$$

The mathematical model corresponding to the problem outlined in **1** is therefore represented (for the non-retarded case) by the system of non linear ordinary differential equations

$$(2.3)\qquad \begin{aligned} \eta_i''(t) &= f_{i\eta}(t) + r_{i\eta}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\eta}(t) + p_{ij,\eta}(t)\} \\ \xi_i''(t) &= f_{i\xi}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\xi}(t) + p_{ij,\xi}(t)\}\,. \end{aligned}$$

Assuming that the bar is clamped at one end, we shall suppose that the «quasi molecules» $P_1, P_2, \ldots, P_5$ are fixed ($\xi_j(t) = \alpha_j\ \eta_j(t) = \beta_j\ j = 1\ldots 5$) and the index i in (2.3) varies therefore from 6 to n, moreover we shall assign the initial conditions

$$(2.4)\qquad \begin{aligned} &\eta_i(0) = \bar{\eta}_i \geqslant 0 \qquad && \xi_i(0) = \bar{\xi}_i \\ &\eta_i'(0) = \bar{\eta}_i' \geqslant 0 \qquad && \xi_i'(0) = \bar{\xi}_i'\,. \end{aligned}$$

For the retarded model, equations (2.3) are substituted by

$$(2.5)\qquad \begin{aligned} \eta_i''(t) &= f_{i\eta}(t) + r_{i\eta}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\eta}(t) + p_{ij,\eta}(t-\tau)\} \\ \xi_i''(t) &= f_{i\xi}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\xi}(t) + p_{ij,\xi}(t-\tau)\} \end{aligned}$$

with the same initial and boundary conditions; we shall however assume that (2.4) hold also for $-\tau \leqslant t \leqslant 0$, where τ is the time shift corresponding to the attraction forces.

In the following 3 and 4 we shall prove global existence and uniqueness theorems for systems (2.3), (2.5) with initial conditions (2.4).

3 - The retarded model. An existence and uniqueness theorem

Let us, first of all, prove the following

Lemma. *Consider a material point on the x axis, subject to an external* *force* $f \in L^2(0,T)$ *and to a repulsion force from the origin given by* $F = \dfrac{A}{\xi^\alpha}$ $(A>0,\ B>0,\ \alpha>\beta>1)$.

Assume that $\xi(0) = \bar{\xi} > 0$, $\xi'(0) = -k$ $(k>0)$; *then* $\xi(t) \geq E > 0$ *(E depending on* f, α, A, k, $\bar{\xi}$*).*

The equation of the motion is, in fact

$$\xi''(t) = \frac{A}{\xi^\alpha(t)} - \frac{B}{\xi^\beta(t)} + f(t). \tag{3.1}$$

Integrating on $[0, t]$ we obtain

$$\xi'(t) - \xi'(0) = \int_0^t \left\{\frac{A}{\xi^\alpha(t)} - \frac{B}{\xi^\beta(t)}\right\} \mathrm{d}t + \int_0^t f(t)\, \mathrm{d}t.$$

By the assumptions made, there exists obviously ξ^* (depending on α, β, A, B) such that $\forall \xi \leq \xi^*$, $\dfrac{A}{\xi^\alpha(t)} \geq \dfrac{2B}{\xi^\beta(t)}$. Taking $\bar{\xi} = \xi^*$ and observing that, since $\xi'(0) < 0$, there exists an interval $[0, t^*]$ in which $\xi'(t) \leq 0$, $\xi(t) \leq \bar{\xi}$ and consequently,

$$\xi'(t) \geq \xi'(0) + \frac{A}{\xi^\alpha(0)} t - C\sqrt{t} \geq -k - C\sqrt{T}.$$

Hence

$$|\xi'(t)| \leq k + C\sqrt{T}. \tag{3.2}$$

Multiplying (3.1) by $\xi'(t)$, we have, on the other hand

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\xi'^2(t) = A\frac{\mathrm{d}}{\mathrm{d}t}\frac{\xi^{-\alpha+1}(t)}{-\alpha+1} - B\frac{\mathrm{d}}{\mathrm{d}t}\frac{\xi^{-\beta+1}(t)}{-\beta+1} + f(t)\,\xi'(t)$$

and consequently, integrating on $[0, t]$

$$\frac{1}{2}\xi'^2(t)$$

$$= A\frac{\xi^{-\alpha+1}(t)}{-\alpha+1} - A\frac{\xi^{-\alpha+1}(0)}{-\alpha+1} - B\frac{\xi^{-\beta+1}(t)}{-\beta+1} + B\frac{\xi^{-\beta+1}(0)}{-\beta+1} + \frac{1}{2}k^2 + \int_0^t f(\eta)\,\xi'(\eta)\,\mathrm{d}\eta.$$

It follows, bearing in mind (3.2), and by the assumptions made,

$$|\xi^{-\alpha+1}(t)| \leqslant C_1 \qquad \text{in} \qquad [0,\ t^*] \qquad \text{i.e.} \qquad |\xi(t)| \geqslant E > 0\,.$$

We now pass to the proof of the existence and uniqueness theorem.

Theorem 1. *Assume that* $f_i(t) \in L^2(0,\ T)$ *and that* $\alpha > \beta > 2$; *there exists then a solution of* (2.5) (2.4) *with* $\eta_i(t),\ \xi_i(t) \in H^2(0,\ T)$.

Observe that, by well known results (2.5) (2.4) admit a local solution on an interval $[0,\ \bar{t}]$, with $\bar{t}$ sufficiently small.

In order to prove the global existence in $[0,\ T]$ it is then suficient to establish some a priori bounds on the solutions.

Let us divide $[0,\ T]$ in subintervals $[0,\ \tau]$, $[\tau,\ 2\tau]$ ... and consider the first of these; multiplying the first of (2.5) by $\eta_i'(t)$ and the second by $\xi_i'(t)$, we obtain

$$\eta_i'(t)\,\eta_i''(t) = \eta_i'(t) f_{i\eta}(t) + \eta'(t)\, r_{i\eta}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\eta}(t) + p_{ij,\eta}(t-\tau)\}\, \eta_i'(t)$$

(3.3)

$$\xi_i'(t)\,\xi_i''(t) = \xi_i'(t) f_{i\xi}(t) + \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{q_{ij,\xi}(t) + p_{ij,\xi}(t-\tau)\}\, \xi_i'(t)$$

with $i = 6, \ldots, n$.

Hence, bearing in mind the expression of p_{ij}, q_{ij}, $r_{i\eta}$ and adding

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_i (n_i'^2(t) + \xi_i'^2(t)) \tag{3.4}$$

$$= \sum_{6}^{n}{}_i (f_{i\eta}(t)\,\eta_i'(t) + f_{i\xi}(t)\,\xi_i'(t)) + \sum_{6}^{n}{}_i \frac{C}{(\eta_i(t))^{\gamma}}\,\eta_i'(t) + \sum_{6}^{n}{}_i \sum_{\substack{1\\ j\neq i}}^{n}{}_j \frac{A}{(\lambda_{ij}(t))^{\alpha}}$$

$$\cdot\{(\eta_i(t) - \eta_j(t))(\eta_i'(t) - \eta_j'(t)) + (\xi_i(t) - \xi_j(t))(\xi_i'(t) - \xi_j'(t))\}$$

$$+ \sum_{6}^{n}{}_i \sum_{\substack{1\\ j\neq i}}^{n}{}_j \{p_{ij,\eta}(t-\tau)\,\eta_i'(t) + p_{ij,\xi}(t-\tau)\,\xi_i'(t)\}$$

which can also be written

$$(3.5)\quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_i(\eta_i'^2(t)+\xi_i'^2(t))+\frac{C}{\gamma-1}\sum_{6}^{n}{}_i\frac{\mathrm{d}}{\mathrm{d}t}(\eta_i(t))^{1-\gamma}+\frac{A}{\alpha-2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_i\sum_{\substack{1\\ j\neq i}}^{n}{}_j(\lambda_{ij}(t))^{2-\alpha}$$

$$=\sum_{6}^{n}{}_i(f_{i\eta}(t)\,\eta_i'(t)+f_{i\xi}(t)\,\xi_i'(t))+\sum_{6}^{n}{}_i\sum_{\substack{1\\ j\neq i}}^{n}{}_j\{p_{ij,\eta}(t-\tau)\,\eta_i'(t)+p_{ij,\xi}(t-\tau)\,\xi_i'(t)\}\,.$$

Integrating (3.5) between 0 and $t\in[0,\ \tau]$, we have

$$(3.6)\quad \frac{1}{2}\sum_{6}^{n}{}_i(\eta_i'^2(t)+\xi_i'^2(t)-\eta_i'^2(0)-\xi_i'^2(0))+\frac{C}{\gamma-1}\sum_{6}^{n}{}_i(\eta_i(t))^{1-\gamma}$$

$$-(\eta_i(0))^{1-\gamma})+\frac{A}{\alpha-2}\sum_{6}^{n}{}_i\sum_{\substack{1\\ j\neq i}}^{n}{}_j((\lambda_{ij}(t))^{2-\alpha}-(\lambda_{ij}(0))^{2-\alpha})$$

$$=\int_0^t\{\sum_{6}^{n}{}_i(f_{i\eta}(t)\,\eta_i'(t)+f_{i\xi}(t)\,\xi_i'(t))+\sum_{6}^{n}{}_i\sum_{\substack{1\\ j\neq i}}^{n}{}_j\{p_{ij,\eta}(t-\tau)\,\eta_i'(t)+p_{ij,\xi}(t-\tau)\,\xi_i'(t))\}\,\mathrm{d}t\,.$$

Bearing in mind that the terms p_{ij} are known, since they are calculated on the interval $[-\tau,\ 0]$, and that some of the terms in (3.6) are positive, it follows from (3.6) that

$$(3.7)\quad \sum_{6}^{n}{}_i(\eta_i'^2(t)+\xi_i'^2(t)-\eta_i'^2(0)-\xi_i'^2(0))\leqslant\int_0^t\{\sum_{6}^{n}{}_i(f_{i\eta}(t)\,\eta_i'(t)+f_{i\xi}(t)\,\xi_i'(t)\}\,\mathrm{d}t\leqslant M_1\,.$$

Hence the total kinetic energy of the «quasi-molecules» is bounded and, by the Lemma proved at the beginning of this section, $\lambda_{ij}\geqslant\rho_1>0$. Consequently, the repulsion forces p_{ij} are bounded on the interval $[0,\ \tau]$

$$\sup_{0\leqslant t\leqslant\tau}|p_{ij}(t)|\leqslant\sigma_1<+\infty\,.$$

In exactly the same way it can be shown that

$$\sum_{6}^{n}{}_i(\eta_i'^2(t)+\xi_i'^2(t)-\eta_i'^2(0)-\xi_i'^2(0))\leqslant M_2$$

when $t\in[\tau,\ 2\tau]$. In fact, by what has been proved before, the terms p_{ij} calculated on $[0,\ \tau]$ are bounded. We can then repeat the procedure for the subsequent intervals $[2\tau, 3\tau]$ ... and so prove that

$$\sum_{6}^{n} {}_i (\eta_i'^2(t) + \xi_i'^2(t) - \eta_i'^2(0) - \xi_i'^2(0)) \leq M$$

on the whole of $[0, T]$.

By the Lemma already recalled, it is therefore also $\lambda_{ij}(t) \geq \rho > 0$, $\eta_i(t) \geq \bar{\delta} > 0$ with $\rho = \max(\rho_1 \rho_2 \ldots)$. Consequently the functions $q_{ij}(t)$, $p_{ij}(t-\tau)$, $r_{i\eta}(t)$ are bounded on $[0, T]$. From (2.5) it follows then, by the assumptions made on $f_i(t)$, that η_i'', $\xi_i'' \in L^2(0, T)$; this proves a global a priori estimate and, consequently, the global existence theorem.

It can be observed that (3.6) corresponds to an «energy conservation equation».

Theorem 2. *The solution* $\{\eta_i(t), \xi_i(t)\}$ *given in* Theorem 1 *is unique.*

Assume, in fact, that there exist two solutions $\{\eta_i^{(1)}(t), \xi_i^{(1)}(t)\}$, $\{\eta_i^{(2)}(t), \xi_i^{(2)}(t)\}$; denoting by $p^{(1)}(t)$, $q^{(1)}(t)$, $r^{(1)}(t)$, $p^{(2)}(t)$, $q^{(2)}(t)$, $r^{(2)}(t)$ the attraction and repulsion forces relative to the two solutions and setting

$$w_i(t) = \eta_i^{(1)}(t) - \eta_i^{(2)}(t) \qquad z_i = \xi_i^{(1)}(t) - \xi_i^{(2)}(t)$$

we obtain

$$\text{(3.9)} \qquad \{\eta_i^{(1)\prime\prime}(t) - \eta_i^{(2)\prime\prime}(t)\}\, w_i'(t)$$

$$= C w_i'(t)\left\{ \frac{1}{(\eta_i^{(1)}(t))^\gamma} - \frac{1}{(\eta_i^{(2)}(t))^\gamma} \right\} + A \sum_{\substack{1 \\ j \neq i}}^{n} {}_j w_i'(t) \left\{ \frac{\eta_i^{(1)}(t) - \eta_j^{(1)}(t)}{(\lambda_{ij}^{(1)}(t))^\alpha} - \frac{\eta_i^{(2)}(t) - \eta_j^{(2)}(t)}{(\lambda_{ij}^{(2)}(t))^\alpha} \right\}$$

$$+ \sum_{\substack{1 \\ j \neq i}}^{n} {}_j w_i'(t) \{ p_{ij,\eta}^{(1)}(t-\tau) - p_{ij,\eta}^{(2)}(t-\tau) \}$$

$$\text{(3.10)} \qquad \{\xi_i^{(1)\prime\prime}(t) - \xi_i^{(2)\prime\prime}(t)\}\, z_i'(t)$$

$$= \sum_{\substack{1 \\ j \neq i}}^{n} {}_j z_i'(t) \{ p_{ij,\xi}^{(1)}(t-\tau) - p_{ij,\xi}^{(2)}(t-\tau) \} + A \sum_{\substack{1 \\ j \neq i}}^{n} {}_j z_i'(t) \left\{ \frac{\xi_i^{(1)}(t) - \xi_j^{(1)}(t)}{(\lambda_{ij}^{(1)}(t))^\alpha} - \frac{\xi_i^{(2)}(t) - \xi_j^{(2)}(t)}{(\lambda_{ij}^{(2)}(t))^\alpha} \right\}.$$

Let us now study some of the terms appearing in (3.9), (3.10)

$$(3.11)\qquad w_i'(t)\{\frac{1}{(\eta_i^{(1)}(t))^\gamma}-\frac{1}{(\eta_i^{(2)}(t))^\gamma}\}=\frac{(\eta_i^{(2)}(t))^\gamma-(\eta_i^{(1)}(t))^\gamma}{(\eta_i^{(1)}(t)\,\eta_i^{(2)}(t))^\gamma}w_i'(t)$$

$$=\frac{(\eta_i^{(2)}(t)-\eta_i^{(1)}(t))}{(\eta_i^{(1)}(t)\,\eta_i^{(2)}(t))^\gamma}g(t)\,w_i'(t)=\frac{g(t)}{(\eta_i^{(1)}(t)\,\eta_i^{(2)}(t))^\gamma}w_i(t)\,w_i'(t)$$

where $g(t)$ is a continuous function such that

$$(\eta_i^{(2)}(t))^\gamma-(\eta_i^{(1)}(t))^\gamma=g(t)(\eta_i^{(2)}(t)-\eta_i^{(1)}(t)).$$

Moreover

$$(3.12)\qquad l_i=\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\{\frac{\eta_i^{(1)}-\eta_j^{(1)}(t)}{(\lambda_{ij}^{(1)}(t))^\alpha}-\frac{\eta_i^{(2)}(t)-\eta_j^{(2)}(t)}{(\lambda_{ij}^{(2)}(t))^\alpha}\}$$

$$=\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\{\eta_i^{(1)}(t)-\eta_j^{(1)}(t)\}\{\frac{(\lambda_{ij}^{(2)}(t))^\alpha-(\lambda_{ij}^{(1)}(t))^\alpha}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}$$

$$+\frac{1}{(\lambda_{ij}^{(2)}(t))^\alpha}(\eta_i^{(1)}(t)-\eta_i^{(2)}(t)-\eta_j^{(1)}(t)+\eta_j^{(2)}(t))\}$$

$$=\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\{\frac{(\eta_i^{(1)}(t)-\eta_j^{(1)}(t))^2+(\xi_i^{(1)}(t)-\xi_j^{(1)}(t))^2}{(\lambda_{ij}^{(1)}(t))^\alpha(\lambda_{ij}^{(2)}(t))^\alpha}$$

$$-\frac{(\eta_i^{(2)}(t)-\eta_j^{(2)}(t))^2+(\xi_i^{(2)}(t)-\xi_j^{(2)}(t))^2}{(\lambda_{ij}^{(1)}(t))^\alpha(\lambda_{ij}^{(2)}(t))^\alpha}\}\,h(t)\{\eta_i^{(1)}(t)-\eta_j^{(1)}(t)\}$$

$$+\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\frac{1}{(\lambda_{ij}^{(2)}(t))^\alpha}\{w_i(t)-w_j(t)\}$$

$$=\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\{\eta_i^{(1)}(t)-\eta_j^{(1)}(t)\}\{\frac{(\eta_i^{(1)}(t)-\eta_j^{(1)}(t)-\eta_i^{(2)}(t)+\eta_j^{(2)}(t))}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}a(t)$$

$$+\frac{(\xi_i^{(1)}(t)-\xi_j^{(1)}(t)-\xi_i^{(2)}(t)+\xi_j^{(2)}(t))}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}b(t)\}+\sum_{\substack{1\\ j\neq i}}^{n}{}_j\, w_i'(t)\frac{1}{(\lambda_{ij}^{(2)}(t))^\alpha}\{w_i(t)-w_j(t)\}$$

$$= \sum_{\substack{1\\j\neq i}}^{n} {}_j w_i'(t)\{\eta_i^{(1)}(t)-\eta_j^{(1)}(t)\}\{\frac{(w_i(t)-w_j(t))\,a(t)}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}+\frac{(z_i(t)-z_j(t))\,b(t)}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}\}$$

$$+\sum_{\substack{1\\j\neq i}}^{n} {}_j w_i'(t)\frac{1}{(\lambda_{ij}^{(2)}(t))^\alpha}\{w_i(t)-w_j(t)\}$$

where $h(t)$, $a(t)$, $b(t)$ are appropriate continuous functions on $[0T]$.

Consequently

$$(3.13)\qquad l_i=\sum_1^n {}_j w_i'(t)\{(w_i(t)-w_j(t))\,u_{ij}(t)+(z_i(t)-z_j(t))\,v_{ij}(t)\}\qquad \text{where}$$

$$u_{ij}(t)=\frac{(\eta_i^{(1)}(t)-\eta_j^{(1)}(t))}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}a(t)+\frac{1}{(\lambda_{ij}^{(2)}(t))^\alpha}\qquad v_{ij}(t)=\frac{b(t)}{(\lambda_{ij}^{(1)}(t)\,\lambda_{ij}^{(2)}(t))^\alpha}$$

are bounded on $[0, T]$, bearing in mind that $\lambda(t)\geqslant\rho$, as proved in the Lemma.

Let us now add (3.13) with respect to i; we obtain, setting $l(t)=\sum_6^n {}_i l_i(t)$,

$$l(t)=\sum_6^n {}_i\sum_{\substack{1\\j\neq i}}^{n} {}_j w_i'(t)\{(w_i(t)-w_j(t))\,u_{ij}(t)+(z_i(t)-z_j(t))\,v_{ij}(t)\}\qquad \text{hence}$$

$$|l(t)|\leqslant\sum_6^n {}_i\sum_{\substack{1\\j\neq i}}^{n} {}_j\{|w_i'(t)||w_i(t)|\,|u_{ij}(t)|+|w_i'(t)||w_j(t)|\,|u_{ij}(t)|\}$$

$$+\sum_6^n {}_i\sum_{\substack{1\\j\neq i}}^{n} {}_j\{|z_i(t)||w_i'(t)|\,|v_{ij}(t)|+|w_i'(t)||z_j(t)|\,|v_{ij}(t)|\}$$

$$\leqslant\sum_6^n {}_i\{|w_i'(t)|^2K_1(t)+|w_i(t)|^2K_2(t)+|z_i(t)|^2K_3(t)\}$$

where K_m $(m=1,\ 2,\ 3)$ are continuous functions on $[0T]$. Bearing in mind that, by the boundary conditions, $\xi_i'(t)=\eta_i'(t)=0$ when $(i=1,2,\dots,5)$, we obtain then from (3.9), (3.11), (3.13)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_1^n{}_i |w_i'^2(t)|$$

$$\leqslant \sum_6^n{}_i |w_i'(t)||w_i(t)| F(t) + A\sum_1^n{}_i \{|w_i'(t)|^2 K_1(t) + |w_i(t)|^2 K_2(t) + |z_i(t)|^2 K_3(t)\}$$

$$+ \sum_6^n{}_i \sum_{\substack{1\\ j\neq i}}^n{}_j |w_i'(t)\{p_{ij,\eta}^{(1)}(t-\tau) - p_{ij,\eta}^{(2)}(t-\tau)\}|$$

$$\leqslant \sum_1^n{}_i \{|w_i'(t)|^2 C_1(t) + |w_i(t)|^2 D_1(t) + |z_i(t)|^2 K_3(t)\}$$

$$+ \sum_6^n{}_i \sum_{\substack{1\\ j\neq i}}^n{}_j |w_i'(t)\{p_{ij,\eta}^{(1)}(t-\tau) - p_{ij,\eta}^{(2)}(t-\tau)\}|$$

where $F(t) = \dfrac{Cg(t)}{(\eta_i^{(1)}(t)\,\eta_i^{(2)}(t))^\gamma}$ and $C_1(t)$, $D_1(t)$, are continuous functions on $[0T]$. In exactly the same way it can be proved that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_1^n{}_i |z_i'^2(t)| \leqslant \sum_1^n{}_i \{|z_i'(t)|^2 C_2(t) + |z_i(t)|^2 D_2(t) + |w_i(t)|^2 E_2(t)\}$$

$$+ \sum_6^n{}_i \sum_{\substack{1\\ j\neq i}}^n{}_j |z_i'(t)\{p_{ij,\xi}^{(1)}(t-\tau) - p_{ij,\xi}^{(2)}(t-\tau)\}|$$

where $C_2(t)$, $D_2(t)$, $E_2(t)$ are continuous functions on $[0T]$. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_1^n{}_i |z_i'^2(t) + w_i'^2(t)|$$

$$\leqslant \sum_1^n{}_i \{|z_i'(t)|^2 C_2(t) + |z_i(t)|^2 D_3(t) + |w_i(t)|^2 E_3(t) + |w_i'(t)|^2 C_1(t)\}$$

$$+ \sum_6^n{}_i \sum_{\substack{1\\ j\neq i}}^n{}_j |w_i'(t)\{p_{ij,\eta}^{(1)}(t-\tau) - p_{ij,\eta}^{(2)}(t-\tau)\}| + \sum_6^n{}_i \sum_{\substack{1\\ j\neq i}}^n{}_j |z_i'(t)\{p_{ij,\xi}^{(1)}(t-\tau) - p_{ij,\xi}^{(2)}(t-\tau)\}|$$

where $D_3(t)$, $E_3(t)$ are continuous functions on $[0T]$. Let us divide the interval $[0, T]$ in subintervals $[0, \tau]$ $[\tau, 2\tau]$ ... and consider the interval $[0, \tau]$. On this interval the retarded terms corresponding to the two solutions coincide and we obtain therefore

$$\tag{3.14} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_1^n{}_i|z_i'^2(t)+w_i'^2(t)|$$

$$\leqslant \sum_1^n{}_i\{|z_i'(t)|^2 C_2(t)+|z_i(t)|^2 D_3(t)+|w_i(t)|^2 E_3(t)+|w_i'(t)|^2 C_1(t)\}$$

with the initial condition $z_i'(0)=w_i'(0)=0$.

Relationship (3.14) is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi'^2(t)\leqslant \alpha\phi'^2(t)+\beta\phi^2(t) \qquad \text{with } \phi'(0)=0=\phi(0)$$

from which follows, bearing in mind that $\phi(t)=\int_0^t \phi'(\eta)\,\mathrm{d}\eta$, by Gronwall's lemma, $\phi'(t)=0$ and, consequently $z_i(t)=w_i(t)=0$ in $[0,\ \tau]$. In exactly same way, uniqueness can be proved in the intervals $[\tau,\ 2\tau]\ldots$.

4 - The non-retarded model. An existence and uniqueness theorem

Theorem 3. *Assume that* $f_i(t)\in L^2(0,\ T)$ *and* $\alpha>\beta>2$, *there exists then a solution of* (2.3), (2.4) *with* $\eta_i(t),\ \xi_i(t)\in H^2(0,\ T)$.

The existence theorem can be proved following the same procedure of Theorem 1, substituting to the retarded terms $p_{i\eta}(t-\tau)$ and $p_{i\xi}(t-\tau)$ the non-retarded terms $p_{i\eta}(t)$, $p_{i\eta}(t)$.

Multiplying the first equation of (2.3) by $\eta_i'(t)$ and the second by $\xi_i'(t)$, we obtain

$$\tag{4.1} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\eta_i'^2(t)$$

$$=f_{i\eta}(t)\,\eta_i'(t)+\frac{C}{(\eta_i(t))^r}\eta_i'(t)+\sum_{\substack{1\\ j\neq i}}^n{}_j\frac{A(\eta_i(t)-\eta_j(t))}{(\lambda_{ij}(t))^\alpha}\eta_i'(t)-\sum_{\substack{1\\ j\neq i}}^n{}_j\frac{B(\eta_i(t)-\eta_j(t))}{(\lambda_{ij}(t))^\beta}\eta_i'(t)$$

$$\tag{4.2} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\xi_i'^2(t)$$

$$=f_{i\xi}(t)\,\xi_i'(t)+\sum_{\substack{1\\ j\neq i}}^n{}_j\frac{A(\xi_i(t)-\xi_j(t))}{(\lambda_{ij}(t))^\alpha}\xi_i'(t)-\sum_{\substack{1\\ j\neq i}}^n{}_j\frac{B(\xi_i(t)-\xi_j(t))}{(\lambda_{ij}(t))^\beta}\xi_i'(t)\,.$$

Adding the two equations (4.1), (4.2) we have

$$(4.3)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_{i}(\eta_i'^2(t)+\xi_i'^2(t))+\frac{C}{\gamma-1}\sum_{6}^{n}{}_{i}\frac{\mathrm{d}}{\mathrm{d}t}(\eta_i(t))^{1-\gamma}$$

$$+\frac{A}{\alpha-2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_{i}\sum_{\substack{1\\ j\neq i}}^{n}{}_{j}(\lambda_{ij}(t))^{2-\alpha}-\frac{B}{\beta-2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{6}^{n}{}_{i}\sum_{\substack{1\\ j\neq i}}^{n}{}_{j}(\lambda_{ij}(t))^{2-\beta}$$

$$=\sum_{6}^{n}{}_{i}(f_{i\eta}(t)\,\eta_i'(t)+f_{ij}(t)\,\xi_{ij}'(t)).$$

Hence integrating (4.3) between 0 and $t\in[0,\ T]$

$$(4.4)\quad \frac{1}{2}\sum_{6}^{n}{}_{i}\{(\eta_i'^2(t)+\xi_i'^2(t))-(\eta_i'^2(0)+\xi_i'^2(0))\}+\frac{C}{\gamma-1}\sum_{6}^{n}{}_{i}\{(\eta_i(t))^{1-\gamma}-(\eta_i(0))^{1-\gamma}\}$$

$$+\sum_{6}^{n}{}_{i}\sum_{\substack{1\\ j\neq i}}^{n}{}_{j}\{H(\lambda_{ij}(t))^{2-\alpha}-H(\lambda_{ij}(0))^{2-\alpha}-K(\lambda_{ij}(t))^{2-\beta}+K(\lambda_{ij}(0))^{2-\beta}\}$$

$$=\int_0^t\{\sum_{6}^{n}{}_{i}(f_{i\eta}(t)\,\eta_i'(t)+f_{i\xi}(t)\,\xi_i'(t))\}\,\mathrm{d}t$$

where $H=\dfrac{A}{\alpha-2}$, $K=\dfrac{B}{\beta-2}$. Let us study the function

$$(4.5)\qquad \phi(t)=H(\lambda_{ij}(t))^{2-\alpha}-K(\lambda_{ij}(t))^{2-\beta}$$

considering the following two cases and bearing in mind that $\alpha>\beta>2$

(a) $\qquad \lambda_{ij}(t)<\sqrt[\alpha-\beta]{H/K}$

we then have $\phi(t)>0$

(b) $\qquad \lambda_{ij}(t)\geqslant\sqrt[\alpha-\beta]{H/K}.$

In this case

$$|\phi(t)|\leqslant H\Big(\frac{K}{H}\Big)^{(\alpha-2)/(\alpha-\beta)}+K\Big(\frac{K}{H}\Big)^{(\beta-2)/(\alpha-\beta)}=N.$$

We have therefore, in both cases, since $(\eta_i(t))^{1-\gamma} > 0$,

$$(4.6) \qquad \frac{1}{2}\sum_{6}^{n}{}_i(\eta_i'^2(t) + \xi_i'^2(t)) - (\eta_i'^2(0) + \xi_i'^2(0))$$

$$\leqslant \sum_{6}^{n}{}_i \sum_{1}^{n}{}_j \{H(\lambda_{ij}(0))^{2-\alpha} - K(\lambda_{ij}(0))^{2-\beta}\} + N + \int_0^t \{\sum_{6}^{n}{}_i (f_{i\eta}(t)\,\eta_i'(t) + f_{i\xi}(t)\,\xi_i'(t))\}\,\mathrm{d}t$$

from which a priori estimates analogous to the ones of Theorem 1 hold.
The global existence is therefore proved.

Theorem 4. *The solution $\{\eta_i(t),\ \xi_i(t)\}$ given in Theorem 3 is unique.*

Assume in fact, that there exist two solutions $\{\eta_i^{(1)}(t),\ \xi_i^{(1)}(t)\}$, $\{\eta_i^{(2)}(t),\ \xi_i^{(2)}(t)\}$ of (4.1), (4.2).

Setting $\quad w_i(t) = \eta_i^{(1)}(t) - \eta_i^{(2)}(t) \qquad z_i(t) = \xi_i^{(1)}(t) - \xi_i^{(2)}(t) \quad$ we obtain

$$(4.7) \qquad \{\eta_i^{(1)\prime\prime}(t) - \eta_i^{(2)\prime\prime}(t)\}\, w_i'(t)$$

$$= Cw_i'(t)\{\frac{1}{(\eta_i^{(1)}(t))^{\gamma}} - \frac{1}{(\eta_i^{(2)}(t))^{\gamma}}\} + A\sum_{\substack{1\\ j\neq i}}^{n}{}_j w_i'(t)\{\frac{\eta_i^{(1)}(t) - \eta_j^{(1)}(t)}{(\lambda_{ij}^{(1)}(t))^{\alpha}} - \frac{\eta_i^{(2)}(t) - \eta_j^{(2)}(t)}{(\lambda_{ij}^{(2)}(t))^{\alpha}}\}$$

$$+ \sum_{\substack{1\\ j\neq i}}^{n}{}_j w_i'(t)\{p_{ij,\eta}^{(1)}(t) - p_{ij,\eta}^{(2)}(t)\}\,.$$

$$(4.8) \qquad \{\xi_i^{(1)\prime\prime}(t) - \xi_i^{(2)\prime\prime}(t)\}\, z_i'(t)$$

$$= A\sum_{\substack{1\\ j\neq i}}^{n}{}_j z_i'(t)\{\frac{\xi_i^{(1)}(t) - \xi_j^{(1)}(t)}{(\lambda_{ij}^{(1)}(t))^{\alpha}} - \frac{\xi_i^{(2)}(t) - \xi_j^{(2)}(t)}{(\lambda_{ij}^{(2)}(t))^{\alpha}}\} + \sum_{\substack{1\\ j\neq i}}^{n}{}_j z_i'(t)\{p_{ij,\xi}^{(1)}(t) - p_{ij,\xi}^{(2)}(t)\}\,.$$

We can observe that the terms

$$\sum_{\substack{1\\ j\neq i}}^{n}{}_j z_i'(t)\{p_{ij,\eta}^{(1)}(t) - p_{ij,\eta}^{(2)}(t)\} \qquad \sum_{\substack{1\\ j\neq i}}^{n}{}_j w_i'(t)\{p_{ij,\xi}^{(1)}(t) - p_{ij,\xi}^{(2)}(t)\}$$

can be studied in the same way as (3.13) obtaining inequalities analogous to (3.14). Hence, proceeding as in Theorem 2, the uniqueness of the solution follows.

Remark. The results obtained can be extended to the case $\beta = 2$ by modifying stightly the proof of Theorems 1 and 3. It should be noted however that, as shown by Greenspan, the values for α and β utilized in this model are both considerably larger than 2.

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Abstract

In the present paper we consider the motion of an elastic bar, clamped at one end, in the presence of a rigid obstacle against wich the bar can vibrate, assuming that the corresponding shocks are perfectly elastic.

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