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On continuous sublinear functionals in reflexive Banach spaces and applications (**)

1 - Introduction

Let $(X, \|\cdot\|)$ be a real normed space and consider the semi-inner products $\langle\, ,\rangle_i$, $\langle\, ,\rangle_s$, given by

$$\langle x, y\rangle_i = \lim_{t\to 0-} (\|y+tx\|^2 - \|y\|^2)/2t$$

$$\langle x, y\rangle_s = \lim_{t\to 0+} (\|y+tx\|^2 - \|y\|^2)/2t$$

for all $x, y \in X$.

For the sake of completeness we list some usual properties of these semi-inner products that will be used in the sequel (see for example [1]$_1$):

(i) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$

(ii) $\langle -x, y\rangle_s = \langle x, -y\rangle_s = -\langle x, y\rangle_i$ if $x, y \in X$

(iii) $\langle \alpha x, \beta y\rangle_p = \alpha\beta\langle x, y\rangle_p$ for all $x, y \in X$ and $\alpha\beta \geqslant 0$.

(iv) $\langle \alpha x + y, z\rangle_p = \alpha\langle x, x\rangle_p + \langle y, x\rangle_p$ if $x, y \in X$, $\alpha \in \mathbb{R}$

(v) $\langle x+y, z\rangle_p \leqslant \|x\|\,\|z\| + \langle y, z\rangle_p$ for $x, y, z \in X$

(vi) the element $x \in X$ is Birkhoff orthogonal over $y \in X$, i.e., $\|x+ty\| \geqslant \|x\|$ for all $t \in \mathbb{R}$ iff $\langle y, x\rangle_i \leqslant 0 \leqslant \langle y, x\rangle_s$

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(vii) the space $(X, \|\cdot\|)$ is smooth iff $\langle x, y\rangle_i = \langle x, y\rangle_s$ for all $x, y \in X$ or iff $\langle\, ,\rangle_p$ is linear in the first variable where $p = s$ or $p = i$.

For other properties of $\langle\, ,\rangle_p$ in connection to best approximation element and continuous linear functionals see [**1**]$_1$ where further references are given.

In paper [**1**]$_1$ (p. 504) we proved the following «interpolation» theorem for the continuous linear functionals.

Theorem 1. *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and f be a continuous linear functional on it. Then there exists an element $u \in X$ such that*

$$\langle x, u\rangle_i \leq f(x) \leq \langle x, u\rangle_s \quad \textit{for all} \quad x \in X \quad \textit{and} \quad \|f\| = \|u\|.$$

Note that the next decomposition theorem is also valid.

Theorem 2. *Let $(X, \|\cdot\|)$ be as above and G be its closed linear subspace. If $G^\perp$ denotes the orthogonal complement of G in the sense of Birkhoff, then*

$$X = G + G^\perp .$$

For the proof of this fact see for example [**1**]$_1$ (p. 505), where further consequences and applications are given.

The main aim of this paper is to extend the above results for continuous sublinear functionals and closed clins in real reflexive Banach spaces. Applications for inequations as in [**1**]$_2$ are also given.

2 - Semi-orthogonality in reflexive Banach spaces

A nonempty subset K of a real linear space X is called *clin in X* if the following conditions are satisfied:

(i) $x, y \in K$ imply $x + y \in K$

(ii) $x \in K$, $\alpha \geq 0$ imply $\alpha x \in K$.

A real functional p defined on a clin K is said to be *sublinear on* K if

(s) $p(x+y) \leqslant p(x)+p(y)$ for all $x, \; y \in K$

(ss) $p(\alpha x) = \alpha p(x)$ for all $x \in K$ and $\alpha \geqslant 0$.

Def. 1. The element x in real normed space $(X, \; \|\cdot\|)$ will be called *semi-orthogonal in the sense of Birkhoff over* $y \in X$ if $\langle y, \; x\rangle_i \leqslant 0$. We denote $x \perp_S y$.

It is clear that $0 \perp_S y$; $x \perp_S 0$; $x \perp_S x$ implies $x=0$ and $x \perp_S y$ implies $\alpha x \perp_S \beta y$ if $\alpha\beta \geqslant 0$. For a nonempty subset A of X we put

$$A^{\perp_S} := \{y \in X | y \perp_S x \quad \text{for all} \quad x \in A\}.$$

We also remark that $0 \in A^{\perp_S}$, $A \cap A^{\perp_S} \subseteq \{0\}$ and $x \in A^{\perp_S}$, $\alpha \geqslant 0$ imply $\alpha x \in A^{\perp_S}$.

The following theorem is a natural generalization of Theorem 2.

Theorem 3. *Let* $(X, \; \|\cdot\|)$ *be a real reflexive Banach space and* K *be a closed clin in* X. *Then the following decomposition holds*

$$X = K + K^{\perp_S}. \tag{1}$$

Proof. Let $x \in X$. If $x \in X$ then $x = x + 0$ with $x \in K$ and $0 \in K^{\perp_S}$. If $x \notin K$, since K is a closed convex set in reflexive Banach space X, then there exists a best approximation element in K refering to x, i.e., there exists $x' \in K$ so that $d(x, \; K) = \|x - x'\|$.

Let put $x'' := x - x'$ and consider $\alpha \geqslant 0$ and $y \in K$. Then we have

$$\|x'' - \alpha y\| = \|x - x' - \alpha y\| = \|x - (x' + \alpha y)\| \geqslant \|x''\|$$

because $x', \alpha y \in K$ and K is a clin in X. Hence

$$\|x'' - \alpha y\|^2 \geqslant \|x''\|^2 \qquad \text{for all} \quad \alpha \geqslant 0$$

what implies

$$(\|x'' - \alpha y\|^2 - \|x''\|^2)/2\alpha \geqslant 0 \qquad \text{for all} \quad \alpha > 0.$$

Passing at limit for $s \to 0$ $(s > 0)$ we obtain $\langle -y, \; x\rangle_s \geqslant 0$, i.e., $\langle y, \; x\rangle_i \leqslant 0$ for all $y \in K$ what means that $x'' \in K^{\perp_S}$ and the theorem is proven.

The following result holds.

Corollary. *If K is a closed linear subspace in X, then $K^{\perp s} = K^{\perp}$ where $K^{\perp}$ denotes the orthogonal complement of K in the sense of Birkhoff.*

Proof. It is clear that $K^{\perp} \subseteq K^{\perp s}$.

Now, let $x \in K^{\perp s}$. Then $\langle y, \ x\rangle_i \leqslant 0$ for all $y \in K$ and since K is a linear subspace, then it follows $\langle -y, \ x\rangle_i \leqslant 0$, i.e., $\langle y, \ x\rangle_s \geqslant 0$ what implies that $x \in K^{\perp}$ and the statement is proven.

Remark 1. If X is a Hilbert space, we recapture Theorem 2.1 from [**1**]$_2$.

The following lemma will be used in the sequel.

Lemma 1. *Let $(X, \ \|\cdot\|)$ be a Banach space and $p: X \to \mathbb{R}$ be a continuous sublinear functional on it. Then the set $K(p) := \{x \in X, \, p(x) \leqslant 0\}$ is a closed clin in X. In addition, if we assume that there exists $x_0 \in X$ such that $p(x_0) < 0$ then $K(p)$ is proper in X, i.e., $K(p)$ is not a linear subspace.*

The argument is similar to that in the proof of the Lemma 3.1 from [**1**]$_2$ and we omit the details.

Theorem 4. *Let $(X, \ \|\cdot\|)$ be a real reflexive Banach space and $p: X \to \mathbb{R}$ be a continuous sublinear functional on it such that $K(p) \neq X$. Then there exists $u \in X$, $\|u\| = 1$ such that*

$$p(x) \geqslant p(u)\langle x, \ u\rangle_i \quad \text{for all} \quad x \in K(p). \tag{2}$$

Proof. Since $K(p)$ is closed and $K(p) \neq X$ then there exists an element $w \in K^{\perp s}(p)$ such that $w \neq 0$. Since $w \notin K(p)$ we have $p(w) > 0$. On the other hand, for all $x \in K(p)$ we have

$$p(p(w)\,x - p(x)\,w) \leqslant p(p(w)\,x) + p(-p(x)\,w) = p(w)\,p(x) - p(x)\,p(w) = 0$$

and then

$$p(w)\,x - p(x)\,w \in K(p) \qquad \text{for all} \quad x \in K(p).$$

Since $w \in K^{\perp s}(p)$ we get

$$\langle p(w)\,x - p(x)\,w,\ w\rangle_i \leqslant 0 \qquad \text{for all} \quad x \in K(p).$$

Using the properties of semi-inner product $\langle\ ,\ \rangle_i$ we deduce: $p(w)\langle x,\ w\rangle_i - p(x)\|w\|^2 \leqslant 0$ for all $x \in K(p)$ what implies

$$p(x) \geqslant \frac{p(w)}{\|w\|}\langle x,\ \frac{w}{\|w\|}\rangle_i \quad \text{for all} \quad x \in K(p)$$

from where results (2).

Remark 2. If X is a Hilbert space we obtain the first part of Theorem 3.2 from [**1**]$_2$.

The following two corollaries also hold.

Corollary 1. *Let $p: X \to \mathrm{R}$ be a continuous sublinear functional on reflexive Banach space X such that $K(p) \neq X$. Then there exists an element $u \in X$, $\|u\| = 1$ with the property that*

$$\inf_{x \neq 0} \frac{p(x)}{\|x\|} \geqslant -p(u).$$

Proof. It is clear that

$$\inf_{x \neq 0} \frac{p(x)}{\|x\|} = \inf\{\, \frac{p(x)}{\|x\|} | x \in K(p) \setminus \{0\}\}.$$

By the above theorem there exists an element $u \in X$, $\|u\| = 1$ such that: $p(x) \geqslant p(u)\langle x,\ u\rangle_i$ for all $x \in K(p)$. But $\langle x,\ u\rangle_i \geqslant -\|x\|\,\|u\| = -\|x\|$ what implies that $p(x) \geqslant -p(u)\|x\|$ for all $x \in K(p)$, from where results the desired inequality.

Remark 3. The above corollary contains Theorem 3.10 from [**1**]$_2$ what works in the case of Hilbert spaces.

Corollary 2. *Let p be as above. Then there exists an element $u \in X$, $\|u\| = 1$ such that the mapping $p_u: X \to \mathrm{R}$, $p_u(x) = p(x) + p(u)\|x\|$ is a positive continuous sublinear functional on X.*

3 - Clins with the (*H*)-property in reflexive Banach spaces

We start to the following

Def. 2. Let $(X, \|\cdot\|)$ be a real normed linear space and K be a clin in it. Then K will be called with the *H-property* if the set $H(K) := K^{\perp s} \cap (-K)$ also contains nonzero elements.

Remark 4. If the clin K has the (H)-property, then K is proper in X, i.e., K is not a linear subspace in X.

Indeed, if we suppose, by absurd, that K is linear subspace and $w \in K^{\perp s} \cap(-K) \setminus \{0\}$ then $w \in -K = K$ and since $K^{\perp s} \cap K = \{0\}$ we obtain a contradiction.

The following lemma of characterization holds.

Lemma 2. *The clin K has the* (H)*-property if and only if there exists a nonzero element* $w \in K$ *such that* $\langle x, w \rangle_s \geqslant 0$ *for all* $x \in K$.

Proof. Let $-w \in K^{\perp s} \cap (-K)$ then $w \in K$ and since $-w \in K^{\perp s}$, we have $\langle x, -w \rangle_i \leqslant 0$ for all $x \in K$, i.e., $\langle x, w \rangle_s \geqslant 0$.

Conversely, if $\langle x, w \rangle_s \geqslant 0$ for all $x \in K$ then $\langle x, -w \rangle_i \leqslant 0$, i.e., $-w \in K^{\perp s}$ and since $-w \in -K$ we deduce that K has the (H)-property.

Examples. Let $f: X \to \mathbb{R}$ be a nonzero continuous linear functional on reflexive Banach space X and put $K_+(f) := \{x \in X | f(x) \geqslant 0\}$, $K_-(f) := \{x \in X | f(x) \leqslant 0\}$. Then $K_+(f)$ and $K_-(f)$ are clins with the (H)-property.

Indeed, by Theorem 1, there exists a nonzero element $u \in X$ such that: $\langle x, u \rangle_i \leqslant f(x) \leqslant \langle x, u \rangle_s$ for all $x \in X$.

Let $x \in K_+(f)$, then $\langle x, u \rangle_s \geqslant 0$ and since $f(u) = \|u\|^2 > 0$ we obtain that $u \in K_+(f)$, $u \neq 0$ and $\langle x, u \rangle_s \geqslant 0$, i.e., $K_+(f)$ has the (H)-property.

The proof of the fact that $K_-(f)$ is also a clin with the (H)-property is similar and we omit the details.

Note that the following theorem is valid.

Theorem 5. *Let* $(X, \|\cdot\|)$ *be a reflexive and strict convex Banach space and K be a closed clin in X such that* $K^{\perp s}$ *is also a clin. Then the following state-*

ments are equivalent:

(i) *K, $K^{\perp_s}$ are linear subspaces.*

(ii) *The following decomposition holds*

$$X = K \oplus K^{\perp_s}.$$

Proof. (i) $\Rightarrow$ (ii). If K is a liner subspace, then $K^{\perp_s} = K^{\perp}$ (see Corollary of Theorem 3). Since $(X, \|\cdot\|)$ is reflexive and strict convex it is known that $X = K \oplus K^{\perp}$.

(ii) $\Rightarrow$ (i). Let $u \in K$, $v \in K^{\perp_s}$ and put $x = u + v$. Then by Theorem 3 there exists $m \in K$, $n \in K^{\perp_s}$ such that $-x = m + n$. Hence $0 = (u + m) + (v + n)$ with $u + m \in K$, $v + n \in K^{\perp_s}$ and since the null element has a unique decomposition we obtain $-u = m \in K$, $-v = n \in K^{\perp_s}$, i.e., K and $K^{\perp_s}$ are linear subspaces.

Remark 5. The above theorem contains Theorem 2.4 from $[\mathbf{1}]_2$ which is valid in Hilbert spaces.

Theorem 6. *Let $(X, \|\cdot\|)$ be a reflexive and strict convex Banach space and K be a proper closed clin in X such that $K^{\perp_s}$ is also a clin. Then K has the (H)-property.*

Proof. Since K is a proper closed clin in X then by the above theorem there exists at least one element x such that

$$x = x' + x'' \qquad x' \in K \qquad x'' \in K^{\perp_s}$$

$$x = x_1 + x_2 \qquad x_1 \in K \qquad x_2 \in K^{\perp_s}$$

and $\qquad x' \neq x_1 \qquad x'' \neq x_2\,.$

By Theorem 3, there exists $y' \in K$ and $y'' \in K^{\perp_s}$ such that $-x = y' + y''$ and then

$$0 = (x' + y') + (x'' + y'') \qquad x' + y' \in K \qquad x'' + y'' \in K^{\perp_s}$$

$$0 = (x_1 + y') + (x_2 + y'') \qquad x_1 + y' \in K \qquad x_2 + y'' \in K^{\perp_s}$$

with $x' + y' \neq x_1 + y'$ and $x'' + y'' \neq x_2 + y''$.

Consequently, there exists $m \in K$, $n \in K^{\perp_s}$ with $m \neq 0$ and $n \neq 0$ such that $0 = m + n$ what implies that $n = -m$ and then the set $K^{\perp s} \cap (-K)$ also contains nonzero elements.

Corollary. *Let $(X;\ (\,,))$ be a Hilbert space. Then every proper closed clin in X has the (H)-property.*

Proof. Follows from the above theorem and to the fact that for all clin K in X, $K^{\perp s}$ is also a clin in X.

Now, we can improve Theorem 4.

Theorem 7. *Let $(X,\ \|\cdot\|)$ be a real reflexive Banach space and $p: X \to \mathbb{R}$ be a continuous sublinear functional on it such that $K(p)$ has the (H)-property. Then there exists an element $u \in X$, $\|u\| = 1$ such that*

$$p(x) \geqslant \begin{matrix} p(u)\langle x,\ u\rangle_i & \text{for all} & x \in K(p) \\ -p(-u)\langle x,\ u\rangle_i & \text{for all} & x \in X \setminus K(p)\,. \end{matrix} \tag{4}$$

Proof. Because $K(p)$ has the (H)-property, there exists $w \neq 0$, $w \in K^{\perp s}(p) \cap (-K(p))$. Since $w \in K^{\perp s}(p)$, we have $p(w) > 0$. Then by a similar argument to that in the proof of Theorem 4 we have

$$p(x) \geqslant \frac{p(w)}{\|w\|}\langle x,\ \frac{w}{\|w\|}\rangle_i \quad \text{for all} \quad x \in K(p)$$

and putting $u := w/\|w\|$ we obtain the first part of (4).

Now, let $x \in X \setminus K(p)$, then $p(x) > 0$ and since $-w \in K(p)$ it follows that $-p(-w) \geqslant 0$. We obtain: $p(p(x)(-w) - p(-w)\,x) \leqslant p(x)\,p(-w) + (-p(-w))\,p(x) = 0$ what implies that $-p(x)\,w - p(-w)\,x \in K(p)$. Since $w \in K^{\perp s}(p)$ we derive: $\langle -p(x)\,w - p(-w)\,x,\ w\rangle_i \leqslant 0$ for all $x \in X \setminus K(p)$, what implies $-p(x)\|w\|^2 - p(-w)\langle x,\ w\rangle_i \leqslant 0$ for all $x \in X \setminus K(p)$, from where results

$$p(x) \geqslant \frac{-p(-w)}{\|w\|}\langle x,\ \frac{w}{\|w\|}\rangle_i \quad \text{for all} \quad x \in X \setminus K(p)$$

and the second part of relation (4) is also valid.

Remark 6. If X is a Hilbert space we obtain the main result from $[\mathbf{1}]_2$ (see Theorem 3.2).

Remark 7. If f is a continuous linear functional on X and since $K(f) = K_-(f)$, then by (4) we have: $f(x) \geqslant f(u)\langle x, u\rangle_i$ for all $x \in X$.

On the other hand, substituting x by $-x$ we derive that $-f(x) \geqslant f(u)\langle -x, u\rangle_i = -f(u)\langle x, u\rangle_s$ for all $x \in X$, what implies

$$f(x) \leqslant f(u)\langle x, u\rangle_s \quad \text{for all} \quad x \in X.$$

Consequently, Theorem 7 gives a natural generalization of Theorem 1 for the case of sublinear and continuous functionals which has the (H)-property.

Now, let consider the set

$$L(p) := \{x \in X | p(x) + p(-x) = 0\}$$

where p is a continuous sublinear functional on Banach space X. Then $L(p)$ is a closed linear subspace in X. The proof is similar to that of Lemma 3.4 from $[\mathbf{1}]_2$ and we shall omit the details.

Def. 3. A continuous sublinear functional p is said to be of *(C)-type* (see also $[\mathbf{1}]_2$) if the set $N(p) := H(p) \cap L(p)$ also contains nonzero elements.

It is easy to see that if p is a continuous linear functional then p is of (C)-type.

The following result extend Theorem 3.4 from $[\mathbf{1}]_2$ which works in Hilbert spaces.

Theorem 8. *Let p be a continuous sublinear functional of (C)-type on reflexive Banach space X. Then there exists an element $v \in X$ such that*

$$p(x) \geqslant \langle x, v\rangle_i \quad \textit{for all} \quad x \in X.$$

Proof. Let $w \in N(p)$, $w \neq 0$. Then, as in Theorem 7, we have

$$p(x) \geqslant \frac{p(w)}{\|w\|^2}\langle x, w\rangle_i \quad \text{for all} \quad x \in K(p)$$

$$p(x) \geqslant \frac{-p(-w)}{\|w\|^2}\langle x, w\rangle_i \quad \text{for all} \quad x \in X \setminus K(p).$$

Since $p(-w) = -p(w)$, we obtain

$$p(x) \geqslant \frac{p(w)}{\|w\|^2} \langle x, \ w \rangle_i \quad \text{for all} \quad x \in X$$

and putting $v := (p(w)/\|w\|^2) w$ we obtain the desired inequality.

Remark 8. If p is linear, then the above theorems gives also Theorem 1.

4 - Applications

Let $(X, \ \|\cdot\|)$ be a real reflexive Banach space and $(e_i)_{i=\overline{1,n}}$ be a linearly independente family of vectors in X. Consider the following system of inequations $(x \in X)$

$$\text{(S)} \qquad \langle e_1, \ x \rangle_s \geqslant 0 \qquad \langle e_2, \ x \rangle_s \geqslant 0 \quad \dots \quad \langle e_n, \ x \rangle_s \geqslant 0$$

and put $K(e_1, \dots, e_n) := \{x | x = \sum_{i=1}^{n} \alpha^i e_i, \ \alpha^i \geqslant 0\}$ which is a proper closed clin in X generated by $(e_i)_{i=\overline{1,n}}$. The next result holds.

Proposition 1. *The following statements are equivalent*

(i) *$K(e_1, \dots, e_n)$ has the (H)-property in X.*

(ii) *The system (S) has a nonzero solution in $K(e_1, \dots, e_n)$.*

Proof. If $K(e_1, \dots, e_n)$ has the (H)-property, then there exists $x_0 \in K(e_1, \dots, e_n) \setminus \{0\}$ (see Lemma 2) such that $\langle x, \ x_0 \rangle_s \geqslant 0$ for all $x \in K(e_1, \dots, e_n)$ what implies that (S) has a nonzero solution in $K(e_1, \dots, e_n)$.

Conversely, if we suppose that (S) has a nonzero solution x_0 in $K(e_1, \dots, e_n)$ then for all $x := \sum_{i=1}^{n} \alpha^i e_i, \ \alpha^i \geqslant 0 \ (i = \overline{1, \ n})$ we get

$$\langle x, \ x_0 \rangle_s = \langle \sum_{i=1}^{n} \alpha^i e_i, x_0 \rangle_s = \sum_{i=1}^{n} \alpha^i \langle e_i, x_0 \rangle_s \geqslant 0$$

and by Lemma 2 it follows that $K(e_1, \dots, e_n)$ has the (H)-property.

Remark 9. If $(X; \ (\ ,\ ))$ is a Hilbert space, then for all $(e_i)_{i=\overline{1,n}}$ a linearly independente family of vectors, the system (S) has a nonzero solution in $K(e_1, \dots, e_n)$ (see [**1**]$_2$).

The following results are also valid in Hilbert spaces (see $[1]_2$).

Proposition 2. *Let $(e_i)_{i=\overline{1,n}}$ be a linearly independente family of vectors in X and $G(e_1, \ldots, e_n)$ be the Gram's matrix associated to it. Then the system of linear inequations*

$$G(e_1, \ldots, e_n)\,\overline{x}^t \geq 0 \qquad \overline{x} \in \mathbb{R}^n_+$$

has nonzero solutions.

Proposition 3. *If $(e_i)_{i=\overline{1,n}}$ is as above and $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F(\overline{x}, \overline{y}) := \overline{x}G(e_1, \ldots, e_n)\,\overline{y}^t$, then there exists $\overline{y}_0 \geq 0$ in $\mathbb{R}^n$ and $\overline{y}_0 \neq 0$ such that*

$$F(\overline{x}, \overline{y}_0) \geq 0 \qquad \text{for all} \quad \overline{x} \geq 0\,.$$

The following result is in connection to well-known theorems of J. von Neuman which are important in Games' theory (see [2] or [3] p. 107).

Proposition 4. *Let $A = (a^i_j)^{i=\overline{1,m}}_{j=\overline{1,n}}$ be a matrix with real elements and rang $(A) = m \leq n$. Then there exists $\overline{x}_0 \in \mathbb{R}^n_+$, such that: $A\overline{x}^t_0 \geq 0$ in $\mathbb{R}^n$.*

Finally, we shall give another result in connection to Ville's theorem (see [4] or [3], p. 130) which is also important in Games' theory.

Proposition 5. *Let A be a symmetric positive definite matrix and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g(\overline{x}, \overline{y}) := \overline{x}A\overline{y}^t$. Then there exists $\overline{y}_0 \in \mathbb{R}^n_+$, $\overline{y}_0 \neq 0$ such that: $g(\overline{x}, \overline{y}_0) \geq 0$ for all $\overline{x} \geq 0$.*

For the proof of these results see $[1]_2$ where further details and consequences are given.

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Résumé

Dans cet article on introduit la notionne de semi-orthogonalité en sens de Birkhoff dans un espace normé et on donne quelques théorèmes d'éstimation pour les fonctionnelles sublinéaires définites sur cet espace. On donne aussi quelques applications pour les inéquations linéaires.

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