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Extension theorems by Helly and Riesz revisited (**)

1 - Introduction

In some instances the true origin and authorship of a well-known theorem are not accurately portrayed in the literature. The Hahn-Banach theorem is an example of this. H. Hahn proved an extension theorem for bounded linear functionals in 1927 [3] (p. 217). S. Banach proved the same result in 1929 $[1]_1$ (p. 213) and generalized it in 1932 $[1]_2$ (p. 28). Although Banach mentions Hahn's work in 1932, there is no mention of it in 1929.

A personal communication received from W. Orlicz in 1973 indicates that Banach was unaware of Hahn's result. Orlicz informed Banach of this result and subsequently Banach cited it $[1]_1$ (p. 55). Thus the Hahn-Banach theorem is appropriately named -- or is it? J. Dieudonné in his book *History of Functional Analysis* refers to a 1911 work of F. Riesz in which he establishes a «special case of what we now call the Hahn-Banach theorem» [2] (p. 130) for L^q. An article by H. Hochstadt [5] argues that E. Helly could be credited with the origination of the Hahn-Banach theorem.

The papers of Hahn and Banach in which the extension theorem for bounded linear functionals is proven give some insight into the origin of the theorem. Both give similar necessary and sufficient conditions for the existence of a bounded linear functional having a prescribed set of values.

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The following theorem is due to Banach $[\mathbf{1}]_2$ (p. 213). Hahn proved a slightly more general result [3](p. 216).

Theorem 1. *Let* $\{x_i\}$ *be a subset of the complete normed real vector space* X, $\{c_i\}$ *a set of real numbers and* M *a positive number. A necessary and sufficient condition for the existence of a linear functional satisfying*

(a) $f(x_i) = c_i \quad (i = 1, 2, 3, \dots)$ (b) $\|f\| \leq M$

is that the inequality

$$\left|\sum_{i=1}^{n} \lambda_i c_i\right| \leq M \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|$$

holds for all finite sets of real numbers λ_i.

Hahn uses this theorem to prove the extension for bounded linear functionals. Banach uses the extension theorem for bounded linear functionals to prove this theorem. Before proving their respective versions of Theorem 1, Hahn and Banach refer to several works by E. Helly and F. Riesz. Specifically Banach refers to [4], $[\mathbf{6}]_3$ and $[\mathbf{6}]_1$. Hahn refers to $[\mathbf{4}]_2$. Since Theorem 1 is actually a restricted version of the Hahn-Banach theorem, it is reasonable to expect some sort of extension theorems in these works. By analyzing $[\mathbf{4}]_{1,2}$, $[\mathbf{6}]_1$, $[\mathbf{6}]_3$ and $[\mathbf{6}]_2$, even though the latter was cited by neither, the role of Helly and Riesz in the origin of the Hahn-Banach theorem is clarified.

2 - Extension theorems by Riesz

In $[\mathbf{6}]_2$ Riesz develops conditions for which the system of integral equations

$$\int_a^b g_i(x)\,\xi(x)\,\mathrm{d}x = c_i \qquad g_i \in L^q \qquad i = 1, 2, 3, \dots$$

has a solution $\xi(x) \in L^p$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$). He arrives at a necessary condition for solvability by noting that if a solution exists in L^p, it must satisfy each equation

$$\int_a^b \left[\sum_{i=1}^{n} \lambda_i g_i(x)\right] \xi(x)\,\mathrm{d}x = \sum_{i=1}^{n} \lambda_i c_i$$

where the λ_i are arbitrary constants. Using Hölder's inequality Riesz obtains

$$\Big|\sum_{i=1}^{n} \lambda_i c_i\Big|^q \leqslant M^q \int_a^b \Big|\sum_{i=1}^{n} \lambda_i g_i(x)\Big|^q \,\mathrm{d}\,x\,.$$

He also shows that this is a sufficient condition for solvability.

Considering the usual norm in L^q, this inequality may be written as

$$\Big|\sum_{i=1}^{n} \lambda_i c_i\Big| \leqslant M \Big\|\sum_{i=1}^{n} \lambda_i g_i(x)\Big\|\,.$$

With the appropriate substitutions it is easy to see that Riesz has proven Theorem 1 for the function space L^q.

In [**6**]$_2$, a paper cited by neither Hahn nor Banach, Riesz again deals with a form of Theorem 1. He considers a systems of integral equations

$$\int_a^b f_i(x)\;\mathrm{d}\,\alpha(x) = c_i \qquad\qquad i = 1, 2, 3, \ldots$$

where the $f_i(x)$ are continuous and the desired solution $\alpha(x)$ is a function of bounded variation with a total variation not exceeding a fixed bound M. Riesz shows that a necessary and sufficient condition for the existence of a solution function $\alpha(x)$ with the required properties is

$$\Big|\sum_{i=1}^{n} \lambda_i c_i\Big| \leqslant M \times \text{maximum of } \Big|\sum_{i=1}^{n} \lambda_i f_i(x)\Big|.$$

With the usual norm on $C[a,\ b]$ and appropriate substitutions, Riesz has proven a special case of Theorem 1, i.e., for $C[a,\ b]$.

In Chapter III of [**6**]$_3$ Riesz looks at infinite systems of equations. Specifically he considers the system

$$\sum_{k=1}^{\infty} a_{ik} x_k = c_i \qquad\qquad i = 1, 2, 3, \ldots$$

where the coefficient sequences $\{a_{ik}\}_k$ are in l^q and $\{x_k\}_k$ is in l^p with $\|\{x_k\}_k\| \leqslant M$ $(p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1)$. Riesz proves that a necessary and sufficient condition for the existence of a solution $\{x_k\}_k$ with the required properties is

$$\Big|\sum_{i=1}^{n} \lambda_i c_i\Big| \leqslant M\Big(\sum_{i=1}^{\infty} \Big|\sum_{i=1}^{n} \lambda_i a_{ik}\Big|^q\Big)^{1/q}.$$

Using the standard norms on l^q and l^p and appropriate substitutions, it is easy to see that Riesz has once again proven a version of Theorem 1; this time for l^q.

As previously noted Dieudonné gives Riesz credit for a special case of the Hahn-Banach theorem in L^q. The above shows, however, that Riesz also proved special cases of the Hahn-Banach theorem for l^q and $C[a, b]$.

3 - Extension theorems by Helly

In [4]$_2$ Helly generalizes some of Riesz' results in [6]$_3$ to separable spaces. Specifically he considers a system of infinitely many equations given by

$$\sum_{k=1}^{\infty} a_k^{(i)} x_k = c_i \qquad i = 1, 2, 3, \dots$$

for which the inequality

$$\Big|\sum_{i=1}^{n} \lambda_i c_i\Big| \leqslant M\Delta\Big(\sum_{i=1}^{n} \lambda_i a^{(i)}\Big)$$

is satisfied for arbitrary values of λ_i and n. Helly puts no restrictions on the sequences $a^{(i)}$ and $x = (x_1, x_2, \dots)$ except that they be in separable spaces. The expression $\Delta(u)$ is defined by

$$\Delta(u) = \max_{\|x\|=1} \Big|\sum_{k=1}^{\infty} u_k x_k\Big|.$$

He proves that a solution x of the system of equations exists with $\|x\| \leqslant M_1$ where $M_1 > M$.

Clearly the work done in [4]$_2$ is related to Riesz' work discussed above. In fact, Helly observes that the problem he considers is a special case of the problem to find a linear functional L such that $L(a^{(i)}) = c_i$ $(i = 1, 2, \dots)$ and $|L(u)| \leqslant M_1 \Delta(u)$ where $M_1 > M$. He points out that the construction of L employs methods similar to those considered in a related problem found in [4]$_1$.

In [4]$_1$ Helly generalizes Riesz' results published in 1911. While Riesz works with a specific linear functional, i.e., the integral, Helly considers an arbitrary linear functional on $C[a, b]$. To prove this generalization Helly uses the following lemma.

Lemma 1. *If the inequality*

$$|\sum_{i=1}^{n} \lambda_i c_i| \leq M \|\sum_{i=1}^{n} \lambda_i g_i(x)\|$$

is satisfies for all values of the numbers λ_i, *then for any other (continuous) function* $g_{n+1}(x)$ *one can find a constant* c_{n+1} *so that the inequality*

$$|\sum_{i=1}^{n+1} \lambda_i c_i| \leq M \|\sum_{i=1}^{n+1} \lambda_i g_i(x)\|$$

is satisfied for all values of the numbers λ_i.

This can be restated as the following extension theorem.

Theorem 2. *If* Y *is a finite dimensional subspace of* $C[a, b]$ *and* f *is a bounded linear functional on* Y *with* $\|f\| \leq M$, *then for any* $g \notin Y$ *there is a bounded linear functional* F *defined on the space spanned by* $Y \cup \{g\}$ *which agrees with* f *on* Y *and* $\|F\| \leq M$.

Clearly this is a restricted (finite dimensional setting, $C[a, b]$) version of the extension theorem for bounded linear functionals proven by Hahn [3] in 1927 and Banach $[1]_1$ in 1929.

Helly's work in this 1912 paper is more modern in disposition and flavor than any of Riesz' papers cited above and even his own paper published in 1921. In those papers the proofs rely heavily on properties of the elements in the space and cannot be readily generalized. In fact as mentioned above, H. Hochstadt asserts that Helly «knew and proved the theorem known today as the Hahn-Banach theorem» [5] (p. 124). A few paragraphs later in the article he reduces the assertion to the fact that Lemma 1 «is evidently a version of the Hahn-Banach theorem for $C[a, b]$» [5] (p. 124). This seems closer to the context of the work.

Hochstadt's argument is that Lemma 1 is a «basic inequality» used in the proof of the Hahn Banach theorem. An analysis of the paper by Hahn [3] and Banach $[1]_1$ establishes that their approaches to proving the extension theorem are similar. Both proofs are essentially the same two part procedure. The first part is to show that the linear functional f can be extended from $G \subset X$ to a functional F defined on the span of $G \cup \{y\}$ for any $y \notin G$ so that $\|f\| = \|F\|$ and $f(x)$

$= F(x)$ for all $x \in G$. The second part is to show that f can be extended to a functional F defined on all of X with these properties. The second part is easy to prove using transfinite induction once the first part is established. It's the first part that requires a little work.

Banach's proof of the first part of the extension theorem begins with the assumption $\|f\| = 1$. Then for any $x_1,\ x_2 \in G$,

$$f(x_1 - x_2) \leqslant \|x_2 - x_1\| = \|x_2 + y - y - x_1\| \leqslant \|x_2 + y\| + \|x_1 + y\|$$

so that

$$f(x_1) - f(x_2) \leqslant \|x_2 + y\| + \|x_1 + y\|$$

and the inequality

$$-\|x_1 + y\| - f(x_1) \leqslant \|x_2 + y\| - f(x_2)$$

follows. Let m be the least upper bound of all numbers on the left and M the greatest lower bound of all numbers on the right. Then for any λ so that $m \leqslant \lambda \leqslant M$, it is possible to extend the linear functional f defined on G to a linear functional F defined on the space spanned by $G \cup \{y\}$. Let $x \in G$, then $F(x + \alpha y) = f(x) + \alpha\lambda$.

Hahn uses a similar approach. He also considers a number between the least upper bound of $f(x) - M\|x - y\|$ and the greatest lower bound of $f(x) + M\|x - y\|$ to extend the linear functional f. Hahn notes that this procedure is used so that his version of Theorem 1 (above) may be applied.

Both Hahn and Banach are interested in having the condition $\|F\| \leqslant M$ for the extension F. This means $|F(x + \alpha y)| \leqslant M\|x + \alpha y\|$ from which it follows that

$$-f(x) - M\|x + \alpha y\| \leqslant \alpha F(y) \leqslant -f(x) + M\|x + \alpha y\|.$$

The inequality here is essentially the «basic inequality» discussed above. Starting with this last inequality is a natural way to proceed to obtain the condition $\|F\| \leqslant M$. It is not something mysterious or difficult to ascertain. It involves only an elementary property of inequalities.

Hochstadt takes Hahn and Banach to task for not citing Helly's work when they used this «basic inequality». Banach, however, cites [1] (p. 213) Helly's paper before proving his version of Theorem 1 above. This citation occurs one page after he uses the inequality. It is reasonable to conjecture that Banach felt the approach was so natural and elementary that a citation for it was unwarranted. Clearly he had nothing to gain by not citing it.

Hahn does not refer directly to Helly's 1912 paper. In fact, Hochstadt states «Whether Hahn was aware of Helly's paper is an open question» [5] (p. 125). Before proving the extension theorem for bounded linear functionals, Hahn refers [3] (p. 216) to Helly's 1921 paper, specifically the material which begins on page 75. At the bottom of page 75 Helly refers to his 1912 paper. On pages 76 and 77 Helly uses a generalization of the inequality from the 1912 paper. It would appear that Hahn was aware of Helly's 1912 paper. Once again it could be conjectured, as in Banach's case, that Hahn felt there was nothing significant to warrant a citation. Obviously there could be no conspiracy between Hahn and Banach against Helly. Most probably they approached the proof of the extension theorem in the manner described above, i.e., using the «basic inequality» because it was a natural way to proceed.

4 - Conclusion

The above analysis demonstrates that Riesz proved more than one extension theorem for linear functionals on Banach spaces. Specifically the linear functionals come from linear equations and integrals on the spaces l^p, L^p, and $C[a, b]$. Although some notational adjustments are needed, they are extension theorems none the less.

Helly generalized some of Riesz' work and presented the material in a more modern format making it readily adaptable to normed linear spaces. Again with some notational adjustments, his proof of an extension theorem for $C[a, b]$ is close in spirit to the proofs of the general extension theorem given by Hahn and Banach some 15 years later.

The assertion that Helly proved the Hahn-Banach theorem, however, is an overstatement. The «basic inequality» argument is tenuous at best. Citations in the works of Hahn and Banach indicate their awareness of Helly's result. The Hahn-Banach theorem was not a famous unsolved problem or conjecture. It is a result that envolved over the years. Neither Hahn, Banach, nor the mathematics community of the time were initially aware of its importance. Since they proved the result independently and were aware of Helly's results, there would be no reason for Hahn and Banach not the note the «basic inequality» if it warranted a special citation. There is nothing mysterious or deep about the «basic inequality». It is an elementary result and an obvious and natural way to arrive at the desired conclusion that $\|F\| \leq M$.

While extension theorems which are special cases of the Hahn-Banach theorem can be ascribed to Helly and Riesz, clearly Hahn was the first to prove the extension theorem for bounded linear functionals in an arbitrary normed linear space with the norm of the functional preserved. Banach proved Hahn's result again, unaware he was doing this, and later generalized it.

References

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[6] F. RIESZ: $[\bullet]_1$ *Untersuchen über Systeme integrierbarer Funktionen*, Math. Ann. **69** (1910), 449-497; $[\bullet]_2$ *Sur certaines systèmes singuliers d'équations intégrales*, Ann. Ec. Norm. Sup. (3) 28 (1911), 33-62; $[\bullet]_3$ *Les systèmes d'équations linéaires á une infinité d'inconnues*, Gauthier-Villars, Paris, 1913.

Sommario

È già ben saputo che il lavoro di Helly e Riesz contiene dei teoremi di estensione che prefigurano il teorema di Hahn-Banach. Benché alcuni articoli sono stati pubblicati su questi teoremi di estensione, detti articoli non sono privi di alcune incoerenze ed omissioni. Dopo un'attenta analisi di certe opere di Helly e Riesz, ed altre considerazioni, diventa chiara la loro parte nel dare origine al teorema di Hahn-Banach.

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