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Application of fixed point theorems to nonlinear integrodifferential equations (**)

1 - Introduction

In this paper we are mainly concerned with the common solution of abstract Volterra integrodifferential equations (cf. Theorem 1)

$$(1.1)\qquad \begin{aligned} & u'(t)+Au(t) \\ &= f(t,\ u(t)) + \int_{t_0}^{t} g(t,\ s,\ u(s),\ \int_{t_0}^{s} K[s,\ \tau,\ u(\tau)]\,\mathrm{d}\tau)\,\mathrm{d}s \qquad t > t_0 \geqslant 0 \\ & u(t_0) = u_0 \end{aligned}$$

$$(1.2)\qquad \begin{aligned} & u'(t)+Au(t) \\ &= \underline{f}(t,\ u(t)) + \int_{t_0}^{t} \underline{g}(t,\ s,\ u(s),\ \int_{t_0}^{s} \underline{K}[s,\ \tau,\ u(\tau)]\,\mathrm{d}\tau)\,\mathrm{d}s \qquad t > t_0 \geqslant 0 \\ & u(t_0) = u_0 \end{aligned}$$

where $-A$ is the infinitesimal generator of C_0-semigroup $\{T(t)\colon\ t \geqslant 0\}$ of bounded linear operators on a Banach space B with norm $\|\cdot\|$, $f,\ \underline{f} \in C[R_+ \times B,\ B]$, $g,\ \underline{g} \in C[R_+ \times R_+ \times B \times B,\ B]$, $K,\ \underline{K} \in C[R_+ \times R_+ \times B,\ B]$ and $R_+ = [0,\ \infty)$.

Theorem 1 is also extended to studying the common solution of (1.1) and an

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infinite family of equations (see Theorem 2)

$$(1.3) \qquad u'(t) + Au(t)$$

$$= f_j(t,\ u(t)) + \int_{t_0}^{t} g_j(t,\ s,\ u(s)),\ \int_{t_0}^{t} K_j[s,\ \tau,\ u(\tau)]\, \mathrm{d}\tau)\, \mathrm{d}s \qquad t > t_0 \geqslant 0$$

$$u(t_0) = u_0$$

wherein f_j, g_j and K_j ($j = 1,\ 2, \ldots$) play the corresponding roles of $\underline{f}$, $\underline{g}$ and $\underline{K}$ of (1.2).

The theory of the existence, uniqueness and other properties of the solutions of (1.1) and its several special forms using various assumptions and different methods have been studied, among others, by Barbu [1], Fitzgibbon [3], Hussain [5], Martin, Jr. [6], Miller [7], Sinestrari [10], Travis and Webb [11] and Webb [12]. Many problems arising in various phases of physics and other areas of mathematical sciences find themselves incorporated in the abstract formulation (1.1) (see, for instance, [2], [6], [7]).

As regards the existence of a common solution of equations (1.1) and (1.2), we utilize a fixed point theorem for two contractive type operators (cf. Lemma 1). The uniqueness of the solution is established using a result of Pachpatte [8] (cf. Lemma 2). It seems that Lemma 1 is being used for the first time to guarantee a common solution of two integrodifferential equations. A general result of Hussain [5] is obtained as a particular case of our result (cf. Corollary). The proof of Theorem 2 uses another fixed point theorem (cf. Lemma 3) and Lemma 2.

2 - Preliminaries and results

Throughout this paper B stands for a Banach space with norm $\|\cdot\|$ and $-A$ for the infinitesimal generator of C_0-semigroup of operators $T(t)$, $t \geqslant 0$, on B. A family $\{T(t)\colon\ t \in R_+\}$ of bounded linear operators from B into B is a C_0-semigroup (also called *linear semigroup of class* (C_0)) if:

(i) $T(0) =$ the identity operator and $T(t+s) = T(t)\,T(s)$ for all $t,\ s \geqslant 0$;
(ii) $T(\cdot)$ is strongly continuous in $t \in R_+$;
(iii) $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ for some $M > 0$, real ω and $t \in R_+$ (see [8], Chapt. 7).

Assume for the sake of brevity and convenience that

$$H(T,\ t_0,\ f,\ g,\ K) \equiv T(t-t_0)\,u_0 + \int_{t_0}^{t} T(t-s) f(s,\ u(s))\, \mathrm{d}s$$

$$+ \int_{t_0}^{t} T(t-s) \int_{t_0}^{s} g(s,\ \tau,\ u(\tau),\ \int_{t_0}^{t} K[\tau,\ \xi,\ u(\xi)]\, \mathrm{d}\xi)]\, \mathrm{d}\tau\, \mathrm{d}s\,.$$

Then a continuous $u(t)$ is a mild solution to (1.1) if

$$u(t) = H(T,\ t_0,\ f,\ g,\ K)\,.$$

A continuous $u(t)$ is a common mild solution to (1.1) and (1.2) if

$$H(T,\ t_0,\ f,\ g,\ K) = u(t) = H(T,\ t_0,\ \underline{f},\ \underline{g},\ \underline{K})\,.$$

The following are our assumptions to be used in our first theorem.

For all t, s in $[t_0,\ \alpha]$ and x_i, y_i in B, $i = 1,\ 2$, let there exist nonnegative numbers L_i, $i = 1,\ 2,\ 3,\ 4$ such that

(A$_1$) $\quad \|K(t,\ s,\ y_1) - \underline{K}(t,\ s,\ y_2)\| \leqslant L_1 \|y_1 - y_2\|$

(A$_2$) $\quad \|g(t,\ s,\ x_1,\ y_1) - \underline{g}(t,\ s,\ x_2,\ y_2)\| \leqslant L_2 \|x_1 - x_2\| + L_3 \|y_1 - y_2\|$

(A$_3$) $\quad \|f(t,\ x_1) - \underline{f}(t,\ x_2)\| \leqslant L_4 \|x_1 - x_2\|\,.$

Theorem 1. *Assume that* (A$_1$)-(A$_3$) *are satisfied. Then, for u_0 in B, the initial value problems* (1.1) *and* (1.2) *have a unique common mild solution u in $C([t_0,\ \alpha],\ B)$ for $t \geqslant t_0$ such that $t_0 \leqslant t \leqslant \alpha$. Moreover, the* map $u_0 \to u$ *from B into $C([t_0,\ \alpha],\ B)$ is Lipchitz continuous.*

The proof is prefaced by two lemmas stated below.

The following fixed point theorem seems to have been discovered first by Yen [13] (see also [4] (Cor. 2) and [9] Th. 14).

Lemma 1. *Let P and Q be maps on a complete metric space X. If there exist a positive integer m and a positive number $h < 1$ such that $\mathrm{d}(P^m x,\ Q^m y) \leqslant h\, \mathrm{d}(x,\ y)$ for all x, y in X, then P and Q have a unique common fixed point.*

Lemma 2 [8]. *Let $x(t)$, $a(t)$, $b(t)$ and $c(t)$ be real-valued nonnegative continuous functions defined on R_+, for which the inequality*

$$x(t) \leqslant x_0 + \int_0^t a(s)\,x(s)\,\mathrm{d}\,s + \int_0^t a(s)\Big[\int_0^s b(r)\,x(r)\,\mathrm{d}\,r\Big]\,\mathrm{d}\,s$$

$$+ \int_0^t a(s)\Big[\int_0^s b(r)\Big(\int_0^r c(z)\,x(z)\,\mathrm{d}\,z\Big)\,\mathrm{d}\,r\Big]\,\mathrm{d}\,s$$

holds for all t in R_+, wherein x_0 is a nonnegative constant. Then

$$x(t) \leqslant x_0\Big[1 + \int_0^t a(s)\,\exp\Big(\int_0^s a(r)\,\mathrm{d}\,r\Big)\Big\{1 + \int_0^s b(r)\,\exp\Big(\int_0^r [b(z)+c(z)]\,\mathrm{d}\,z\Big)\,\mathrm{d}\,r\Big\}\,\mathrm{d}\,s\Big].$$

Proof of Theorem 1. Let $C := C([t_0, \alpha], B)$. Define the norm $\|\cdot\|_C$ in C as

$$\|u\|_C = \max_{t \in [t_0, \alpha]} \|u(t)\|.$$

Then C with this norm is a Banach space. Let F, $\underline{F}$: $C \to C$ be such that

$$(Fu)(t) = H(T,\ t_0,\ f,\ g,\ K) \qquad t_0 \leqslant t \leqslant \alpha \tag{2.1}$$

$$(\underline{F}u)(t) = H(T,\ t_0,\ \underline{f},\ \underline{g},\ \underline{K}) \qquad t_0 \leqslant t \leqslant \alpha. \tag{2.2}$$

Evidently a common solution to equations (1.1) and (1.2) is a common fixed point of the operators F and $\underline{F}$.

Let M be an upper bound of $\|T(t-s)\|$ on $[t_0,\ \alpha]$. Then from (2.1), (2.2) and (A_1)-(A_3),

$$\|(Fu)(t) - (\underline{F}v)(t)\| \leqslant \int_{t_0}^t ML_4\,\|u(s) - v(s)\|\,\mathrm{d}\,s$$

$$+ \int_{t_0}^t M \int_{t_0}^s \Big[L_2\,\|u(\tau) - v(\tau)\| + L_3 \int_{t_0}^{\tau} L_1\,\|u(\xi) - v(\xi)\|\,\mathrm{d}\,\xi\Big]\,\mathrm{d}\,\tau\,\mathrm{d}\,s$$

$$\leqslant ML_4\|u - v\|_C\,(t - t_0) + ML_2\,\|u - v\|_C\,(t - t_0)^2/2 + ML_3L_1\,\|u - v\|_C\,(t - t_0)^3/6$$

$$= M(t - t_0)[L_4 + L_2\,(t - t_0)/2 + L_3L_1\,(t - t_0)^2/6]\|u - v\|_C\,.$$

Repeating this process $(n-1)$ times, it can be seen that

$$\|(F^n u)(t) - (\underline{F}^n v)(t)\|$$

$$\leqslant ((t-t_0)^n/n!)\, M^n [L_4 + L_2 (t-t_0)/2 + L_3 L_1 (t-t_0)^2/6]^n \|u-v\|_C \,.$$

Therefore

$$\|F^n u - \underline{F}^n v\|_C \leqslant h \|u - v\|_C$$

wherein $h = (1/n!)(\alpha M)^n [L_4 + L_2 \alpha/2 + L_3 L_1 \alpha^2/6]^n$.

Choosing n large enough, we can make $h < 1$, and so all the hypotheses of Lemma 1 are satisfied. Consequently, there exists a unique u in C such that

$$(Fu)(t) = u(t) = (\underline{F}u)(t)\,.$$

This unique common fixed point is a common solution of (1.1) and (1.2).

To see that (1.1) and (1.2) have exactly one common mild solution, assume that v is another common solution of (1.1) and (1.2) with $v(t_0) = v_0$ on $[t_0, \alpha]$.

Suppose $L = \max\{L_1, L_2, L_3, L_4\}$. Then

$$\|u(t) - v(t)\| = \|(Fu)(t) - (Fv)(t)\|$$

$$\leqslant M\|u_0 - v_0\| + \int_{t_0}^{t} ML\|u(s) - v(s)\|\, \mathrm{d}s$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s} ML[\|u(\tau) - v(\tau)\| + \int_{t_0}^{\tau} L\|u(\xi) - v(\xi)\|\, \mathrm{d}\xi]\, \mathrm{d}\tau\, \mathrm{d}s\,.$$

Now Lemma 2 (see also [5]) yields

$$\|u(t) - v(t)\| \leqslant M\|u_0 - v_0\| R(t) \qquad \text{wherein}$$

$$R(t) = [1 + \int_{t_0}^{t} ML \exp(\int_{t_0}^{s} ML\, \mathrm{d}\tau) \times \{1 + \int_{t_0}^{s} \exp(\int_{t_0}^{\tau}[1+L]\, \mathrm{d}\xi)\, \mathrm{d}\tau\}\, \mathrm{d}s]\,.$$

Hence

$$\|u - v\|_C \leqslant M\|u_0 - v_0\| R(t)\,.$$

This yields the unicity of the common mild solution, and the Lipchitz continuity of the map $u_0 \to u$. This completes the proof.

It may be mentioned that Fitzgibbon [3] and Webb [12] using different assumptions and methods have studied certain special forms of (1.1). However, the following result is a straight forward derivation from Theorem 1.

Corollary [5]. *Assume that* (A_1) *with* $K = \underline{K}$, (A_2) *with* $g = \underline{g}$ *and* $L_2 = L_3$, *and* (A_3) *with* $f = \underline{f}$ *are satisfied. Then, for* $u_0 \in B$, *the initial value problem* (1.1) *has a unique mild solution* $u \in C([t_0, \alpha], B)$ *for* $t \geqslant t_0$ *such that* $t_0 \leqslant t \leqslant \alpha$. *Moreover, the map* $u_0 \to u$ *from* B *into* $C([t_0, \alpha], B)$ *is Lipchitz continuous.*

The following result, a special case of the fixed point theorem of Husain and Sehgal [4] (Cor. 2) and Rhoades [9] (Th. 20) is applied to establishing our next theorem.

Lemma 3. *Let* P *and* P_i, $i = 1, 2, \ldots$, *be maps on a complete metric space* X. *If there exist positive integers* m_i *and positive numbers* $h_i < 1$, $i = 1, 2, \ldots$, *such that*

$$\mathrm{d}(P^{m_i}x,\ P_i^{m_i}y) \leqslant h_i \mathrm{d}(x, y) \qquad i = 1, 2, \ldots$$

for all x, y *in* X, *then there exists a unique element* u *in* X *such that*

$$Pu = u = P_i u \qquad i = 1, 2, \ldots.$$

The following are our assumptions for Theorem 2.

For all t, s in $[t_0, \alpha]$ and x_q, y_q in B, $q = 1, 2$, let there exist nonnegative numbers L_{ij}, $i = 1, 2, 3, 4$, $j = 1, 2, 3, \ldots$, such that

(B_1) $\qquad \|K(t, s, y_1) - K_j(t, s, y_2)\| \leqslant L_{1j}\|y_1 - y_2\|$

(B_2) $\qquad \|g(t, s, x_1, y_1) - g_j(t, s, x_2, y_2)\| \leqslant L_{2j}\|x_1 - x_2\| + L_{3j}\|y_1 - y_2\|$

(B_3) $\qquad \|f(t, x_1) - f_j(t, x_2)\| \leqslant L_{4j}\|x_1 - x_2\|$.

Theorem 2. *Assume that* (B_1)-(B_3) *are satisfied for each* $j = 1, 2, \ldots$. *Then, for* u_0 *in* B, *the initial value problems* (1.1) *and* (1.3) *have a unique common mild solution* u *in* $C([t_0, \alpha], B)$ *for* $t \geqslant t_0$ *such that* $t_0 \leqslant t \leqslant \alpha$. *Moreover, the* map $u_0 \to u$ *from* B *into* $C([t_0, \alpha], B)$ *is Lipschitz continuous.*

Proof. It may be completed imitating the proof of Theorem 1. However, for the sake of completeness, we give a brief sketch of it.

Let C and F: $C \to C$ be defined as in the proof of Theorem 1 (see (2.1)). Further let F_j: $C \to C$, $j = 1, 2, \ldots$, be such that

$$(F_j u)(t) = H(T, t_0, f_j, g_j, K_j) \qquad t_0 \leqslant t \leqslant \alpha. \tag{2.3}$$

Then a common solution of (1.1) and (1.3) is a common fixed point of the operators F and F_j, $j = 1, 2, \ldots$. It can be seen that for each $j \in \{1, 2, \ldots\}$,

$$\|F^n u - F_j^n v\|_C \leqslant h_j \|u - v\|_C$$

wherein $h_j = (1/n!)(\alpha M)^n \{L_{4j} + L_{2j}\alpha/2 + L_{3j} L_{1j} \alpha^2/6\}^n$.

Since, for each $j \in \{1, 2, \ldots\}$, a suitable choice of n can make $h_j < 1$, Lemma 3 guarantees the existence of a unique u in C such that

$$Fu = u = F_j u \qquad j = 1, 2, \ldots.$$

The rest part of the proof is almost a repeat of the corresponding argument given in the proof of Theorem 1.

We remark that, since Lemma 3 is true for P and an uncountable family of maps (see [4]), Theorem 2 is true for (1.1) and an uncountable family of integrodifferential equations of type (1.3).

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Abstract

The purpose of this paper is to obtain a common mild solution of a pair and a family of nonlinear integrodifferential equations with a given initial condition. Contractive type known fixed point theorems are applied to ensure the existence of a common solution of the equations.

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