Riv. Mat. Univ. Parma (4) **16** (1990), 183-194

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Reciprocity and variational theorems in a generalized thermoelastic theory for non-simple materials with voids (**)

1 - Introduction

In a previous paper $[3]_1$ we were concerned with a *generalized* thermodynamic theory for elastic solids with voids; more precisely, we started from a modified entropy production inequality, originated by Green & Laws [7] (cf. also [15]), in order to get a temperature-rate dependent formulation of thermoelasticity for porous materials.

As is known [8], [19], such a formulation — one of the so-called *generalized* (see also [13], [12]) is mainly acknowledged since it makes possible a finite velocity of propagation for thermal perturbations (the *second sound* effect), thereby avoiding a well-known flaw of the classical theory.

On the other hand, the introduction of an additional degree of freedom, connected for each material particle with the fraction of (elementary) volume which is void of matter, proves to be useful to describe the mechanical behaviour of elastic solids having small pores, or *voids*, in their constituent structure [16], [4].

In the present paper, we aim to carry on the study of these topics in the more general framework of *non-simple* materials. In short, these (elastic) materials are defined by including the gradient of the strain among the independent kinematical variables; the resulting theory thus accounts for a wider range of spatial dependence in the response functionals [14], $[20]_{1,2}$.

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(**) This work has been performed under the auspices of G.N.F.M. of the Italian Research Council and with a 60 per cent grant from M.P.I. (Italy). – Ricevuto: 22-XI-1989.

Starting from the field equations established for this context in [2], we shall deal with *reciprocity* (of Graffi type), *variational* (of Gurtin type) and *minumum* (of Reiss type) principles (1). Classical results will be recovered from our's as particular cases.

We recall that Graffi [6] obtained reciprocal results in elastodynamics by using the *convolution* (in time) of the relevant fields, thus avoiding recourse to Laplace-transform (cf. also [11]$_1$). Gurtin's approach to variational questions (e.g., in classical elasticity [9]) starts from a single integro-differential equation that is proved to be equivalent to the motion equation together with the appropriate initial conditions. Finally, Reiss' method [17], in some sense related to Gurtin's, allows to lay down minimizing functionals based on the original time domain rather than on the Laplace-transform domain (cf. [1]).

The quoted principles will be proved in 3, 4 and 5, respectively, after havings stated the basic equations and other preliminary things in 2.

2 - Basic equations. Preliminary results

Throughout this paper, we shall employ the standard (Cartesian) indicial notation. Introduced once for all a fixed, orthonormal frame of reference in the physical space ($\equiv R^3$), vectors and tensors will have components denoted by Latin subscripts (ranging over $\{1, 2, 3\}$). Summation over repeated subscripts is implied. Superposed dots or subscripts preceded by a comma will mean partial derivative with respect to the time or the corresponding coordinates. Moreover, we shall disregard regularity questions, and simply assume a degree of smoothness up to the order requested to make sense everywhere.

The general balance laws can be invoked in the context of linear, generalized thermoelasticity for non-simple materials with voids, to yield the following field equations [2], [11]$_3$

$$\tau_{ij,j} - \mu_{kji,kj} + \rho f_i = \rho \ddot{u}_i$$

(1)

$$h_{i,i} + g + \rho l = \rho k \ddot{\varphi} \qquad q_{i,i} + \rho r = \rho \theta_0 \dot{\eta} \qquad \text{in } \Omega \times (0, +\infty).$$

(1) The last result will be actually achieved in the purely mechanical case.

In these equations, usually referred to as *motion* equations (the former two) and *energy* equation (the latter), Ω stands for the (bounded) domain of R^3 occupied by the elastic body in a fixed, natural reference configuration. We assume the body is homogeneous, and identify it with Ω.

$\mu_{ijk} = \mu_{jik}$ is the so-called *hyperstress* tensor and $\tau_{ij} \equiv T_{ij} + \mu_{kij,k} = \tau_{ji}$, where T_{ij} is the usual (Cauchy) stress tensor $[20]_{1,2}$, [14], $[11]_3$; h_i and g are the equilibrated stress vector and the intrinsic equilibrated body force [16], [4], respectively; q_i and η denote the heat flux and the (specific) entropy. These are the dependent variables of the theory, needing a constitutive equation.

Further, ρ is the *bulk* mass density [16] f_i the usual body force, r the external heat supply, k the equilibrated inertia and θ_0 the (absolute, constant) temperature in the reference configuration.

Finally, u_i and φ, along with the temperature θ (measured from θ_0), are the thermokinetic variables of the theory: u_i represents the displacement field and φ the change in volume fraction field (with respect to the reference one, assumed constant).

In terms of the *constitutive* fields E_{ij}, K_{ijh}, φ, $\varphi_{,i}$, θ, $\dot{\theta}$, $\theta_{,i}$, where $E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $K_{ijh} = u_{h,ij}$, the constitutive equations for (1) read as follows [2]:

$$
\begin{aligned}
\tau_{ij} &= A_{ijkl} E_{kl} + B_{ijpqr} K_{pqr} + M_{ij} \varphi + H_{ijk} \varphi_{,k} + A_{ij} (\theta + \alpha\dot{\theta}) \\
\mu_{ijk} &= B_{pqijk} E_{pq} + C_{ijkpqr} K_{pqr} + R_{ijk} \varphi + L_{ijkr} \varphi_{,r} + C_{ijk} (\theta + \alpha\dot{\theta}) \\
h_i &= H_{ijk} E_{jk} + L_{pqri} K_{pqr} + d_i \varphi + D_{ij} \varphi_{,j} + c_i (\theta + \alpha\dot{\theta}) \\
g &= -M_{ij} E_{ij} - R_{ijh} K_{ijh} - c\varphi - d_i \varphi_{,i} - m(\theta + \alpha\dot{\theta}) \\
q_i &= \theta_0 (b_i \dot{\theta} + k_{ij} \theta_{,j}) \\
\rho\eta &= a - A_{ij} E_{ij} - C_{ijh} K_{ijh} - m\varphi - c_i \varphi_{,i} + d\theta + h\dot{\theta} - b_i \theta_{,i} \,.
\end{aligned}
\tag{2}
$$

The coefficients in the above expressions are all characteristic constants related to the material and thermal properties of the (homogeneous) body. It can be proved [2] that they obey the following symmetry relations

$$
A_{ijkl} = A_{klij} = A_{jikl} \qquad B_{ijpqr} = B_{jipqr} = B_{ijqpr} \qquad C_{ijkpqr} = C_{pqrijk} = C_{ijkqpr}
$$

$$
A_{ij} = A_{ji} \qquad D_{ij} = D_{ji} \qquad C_{ijk} = C_{jik} \qquad H_{ijk} = H_{jik} = H_{ikj}
$$

$$R_{ijk} = R_{jik} \qquad M_{ij} = M_{ji} \qquad L_{ijkr} = L_{jikr} \qquad k_{ij} = k_{ji}\ (^2).$$

In the sequel, we shall assume $b_i = 0$ in $(2)_{5,6}$ [8], [19]. Note that insertion of $(2)_5$ and $(2)_6$ into energy equation $(1)_3$ leads to an evolution equation for θ of wave (hyperbolic) type.

Consider now four pairs of disjoint and complementary subsets of the (smooth) boundary $\partial\Omega$, $\partial_s\Omega$, $\partial_{s+1}\Omega$ ($s = 1,\ 3,\ 5,\ 7$), and let n_i denote the outward unit normal to these. We append to (1) the following system of mixed initial-boundary conditions:

$$u_i = u_i^0 \qquad \dot{u}_i = \dot{u}_i^0 \qquad \varphi = \varphi^0 \qquad \dot{\varphi} = \dot{\varphi}^0 \qquad \theta = \theta^0 \qquad \eta = \eta^0 \qquad \text{in } \Omega \times \{0\}\,; \tag{3}$$

$$\begin{array}{llll} u_i = u_i^* & \text{on } \partial_1\Omega \times (0,\ +\infty) & P_i = P_i^* & \text{on } \partial_2\Omega \times (0,\ +\infty) \\ Du_i = g_i^* & \text{on } \partial_3\Omega \times (0,\ +\infty) & R_i = R_i^* & \text{on } \partial_4\Omega \times (0,\ +\infty) \\ \varphi = \varphi^* & \text{on } \partial_5\Omega \times (0,\ +\infty) & h = h^* & \text{on } \partial_6\Omega \times (0,\ +\infty) \\ \theta = \theta^* & \text{on } \partial_7\Omega \times (0,\ +\infty) & q = q^* & \text{on } \partial_8\Omega \times (0,\ +\infty) \end{array} \tag{4}$$

where $Du_i = u_{i,j} n_j$, $h = h_i n_i$, $q = q_i n_i$, and P_i, R_i are suitably defined vectors so that the power of the surface tractions and hypertractions can be written as $\int_{\partial\Omega} (P_i \dot{u}_i + R_i D\dot{u}_i)\, \mathrm{d}\Sigma$ [20]$_1$, [14], [11]$_3$. P_i and R_i, as well as h and q, are of course determined by u_i, φ, θ through the constitutive equations.

Right-hand members in (3), (4) denote assigned fields: along with f_i, l and r — the external actions — these are the *data* of the initial-boundary value problem (1)-(4). An array of fields $(u_i,\ \varphi,\ \theta)$ meeting the left-side conditions in (4) will be called *thermokinematically admissible*; if $(u_i,\ \varphi,\ \theta)$ meets all equations (1)-(4), for some assignment of the data, it will be called a (regular) *solution* to the problem.

With a view towards the claimed results, we conclude the section by giving an alternative formulation of the problem (1)-(4) in which the initial conditions (3) are incorporated into the field equations.

Let $e(\boldsymbol{x},\ t) = t$, $1(\boldsymbol{x},\ t) = 1\ \forall(\boldsymbol{x},\ t) \in \Omega \times [0,\ +\infty)$, and denote by $a * b$ the *convolution* (in time) of fields on $\Omega \times (0,\ +\infty)$

$$(a * b)(\boldsymbol{x},\ t) = \int_0^t a(\boldsymbol{x},\ t - \tau)\, b(\boldsymbol{x},\ \tau)\, \mathrm{d}\tau = (b * a)(\boldsymbol{x},\ t) \qquad (\boldsymbol{x} \in \Omega).$$

(2) The symmetry of the (linear) *heat-conduction* tensor k_{ij} is one of the major outcomes of generalized thermoelasticity.

Of course, $e = 1 * 1$. Further, define

$$F_i = \rho(e * f_i + u_i^0 + t\dot{u}_i^0) \qquad L = \rho[e * l + k(\varphi^0 + t\dot{\varphi}^0)]$$

(5)

$$R = \rho[(1/\theta_0)\, 1 * r + \eta^0 - (a/\rho)]\,.$$

Following [9], we can easily prove

Theorem 1. *The fields* u_i, φ, θ *form a solution to problem* (1)-(4) *if and only if they satisfy, through* (2), *the boundary conditions* (4) *and the following equations*

$$e * (\tau_{ij,j} - \mu_{kji,kj}) + F_i = \rho u_i \qquad e * (h_{i,i} + g) + L = \rho k \varphi$$

(6)

$$(1/\theta_0)\, 1 * q_{i,i} + R = (\rho\eta - a) \qquad \text{in } \Omega \times [0, +\infty)\,.$$

In the sequel, we shall assume $\theta^0 = 0$ in (3).

3 - A reciprocity principle

Consider now two different sets of data for the problem in concern

$$D^{(\beta)} = \{f_i^{(\beta)},\ l^{(\beta)},\ r^{(\beta)},\ u_i^{0(\beta)},\ \dot{u}_i^{0(\beta)},\ \varphi^{0(\beta)},\ \dot{\varphi}^{0(\beta)},\ \eta^{0(\beta)},\ u_i^{*(\beta)},$$

$$P_i^{*(\beta)},\ g_i^{*(\beta)},\ R_i^{*(\beta)},\ \varphi^{*(\beta)},\ h^{*(\beta)},\ \theta^{*(\beta)},\ q^{*(\beta)}\} \qquad (\beta = 1,\ 2)$$

with the corresponding solutions

(7) $$U^{(\beta)} \equiv (u_i^{(\beta)},\ \varphi^{(\beta)},\ \theta^{(\beta)}) \qquad (\beta = 1,\ 2)\,.$$

We also define $F_i^{(\beta)}$, $L^{(\beta)}$, $R^{(\beta)}$ by means of (5), and $\tau_{ij}^{(\beta)}$, $\mu_{ijk}^{(\beta)}$, $h_i^{(\beta)}$, $g^{(\beta)}$, $q_i^{(\beta)}$, $\eta^{(\beta)}$ by means of (2). We prove the following

Theorem 2 (*Reciprocity Principle*). *Let a non-simple, generalized thermoelastic body with voids* Ω *be subjected to data* $D^{(1)}$ *and* $D^{(2)}$ *with corresponding solutions* $U^{(1)}$, $U^{(2)}$. *Then, the following relation holds*

$$I_{12} = I_{21}$$

where
$$I_{\beta\gamma} = \int_{\Omega} [F_i^{(\beta)} * u_i^{(\gamma)} + L^{(\beta)} * \varphi^{(\gamma)} - (e + \alpha 1) * R^{(\beta)} * \theta^{(\gamma)}] \, d\Omega$$
$$+ \int_{\partial\Omega} e * [P_i^{(\beta)} * u_i^{(\gamma)} + R_i^{(\beta)} * Du_i^{(\gamma)} + h^{(\beta)} * \varphi^{(\gamma)} - (1/\theta_0)(1 * q^{(\beta)} + \alpha q^{(\beta)}) * \theta^{(\gamma)}] \, d\Sigma .$$

Proof. The proof follows standard steps (cf. [11], [18]). By equations (2) and symmetry relations, we easily see that

$$J_{12} \equiv \int_{\Omega} e * [\tau_{ij}^{(1)} * E_{ij}^{(2)} + \mu_{ijh}^{(1)} * K_{ijh}^{(2)} + h_i^{(1)} * \varphi_{,i}^{(2)} - g^{(1)} * \varphi^{(2)} - (\rho\eta^{(1)} - a) * (\theta^{(2)} + \alpha\dot{\theta}^{(2)})] \, d\Omega$$

is equal to J_{21} (recall that $\theta^0 = 0$). By convolution of the field equations (6), which hold for both $U^{(1)}$, $U^{(2)}$, with the fields of the other solution $U^{(2)}$, $U^{(1)}$, respectively, we also realize — through divergence theorem — that

$$I_{\beta\gamma} = J_{\beta\gamma} + \int_{\Omega} [\rho u_i^{(\beta)} * u_i^{(\gamma)} + \rho k \varphi^{(\beta)} * \varphi^{(\gamma)} - (1/\theta_0) \, e * (1 * q_i^{(\beta)} + \alpha q_i^{(\beta)}) * \theta_{,i}^{(\gamma)}] \, d\Omega .$$

The terms on the right-side are all symmetric in β, γ, and thus the thesis follows.

4 - A variational principle

The field equations (6) can be of course written in form of one (five-dimensional) vector equation

$$AU = F \tag{8}$$

where A is a linear operator. We only need to define

$$F = (F_i, \ L, \ -(e + \alpha 1) * R)$$

and, by means of (2),

$$(AU)_i = \rho u_i - e * (\tau_{ij} - \mu_{kji,k})_{,j} \qquad (AU)_4 = \rho k \varphi - e * (h_{i,i} + g) \qquad (i = 1, \ 2, \ 3)$$
$$(AU)_5 = -(e + \alpha 1) * (\rho\eta - a) + (1/\theta_0) \, e * (1 * q_{i,i} + \alpha q_{i,i}) . \tag{9}$$

For U see (7). Recall now the definitions of 3, and call D_A the domain of definition of A. It is a simple matter to prove — a proof based only on (9) and divergence theorem — that $\forall U^{(1)}\ U^{(2)} \in D_A$

$$\int_\Omega (AU^{(1)} * U^{(2)} - AU^{(2)} * U^{(1)})\, \mathrm{d}\Omega \tag{10}$$

$$= \int_{\partial\Omega} e * \{(P_i^{(2)} * u_i^{(1)} - P_i^{(1)} * u_i^{(2)}) + (R_i^{(2)} * Du_i^{(1)} - R_i^{(1)} * Du_i^{(2)}) + (h^{(2)} * \varphi^{(1)}$$

$$- h^{(1)} * \varphi^{(2)}) - (1/\theta_0)[(1 * q^{(2)} + \alpha q^{(2)}) * \theta^{(1)} - (1 * q^{(1)} + \alpha q^{(1)}) * \theta^{(2)}]\}\, \mathrm{d}\Sigma .$$

Note that if $AU^{(1)} = F^{(1)}$ and $AU^{(2)} = U^{(2)}$, we recover from above the reciprocity principle.

Now, if the right-hand member in (10) were zero for each $U^{(1)}$, $U^{(2)}$ belonging to some subdomain D^0 of D_A, a well-known result (e.g., **[10]**) would imply that the functional on D^0

$$\tilde{\Phi}(U) = \tfrac{1}{2} \int_\Omega (AU * U - 2U * F)\, \mathrm{d}\Omega$$

has a stationary point at some $U \in D^0$ if and only if U satisfies equation (8). Thus, a variational principle for our problem in the case of *homogeneous* boundary conditions (4) would be available at once.

Nevertheless, a more general result can be proved as well $[\mathbf{11}]_2$. To this end, let us introduce a *given* (five) vector $W = (w_i,\ \varphi_w,\ \theta_w) \in D_A$ obeying *all* the boundary conditions (4). Assume that $U \in D_A$ meets equations (4), (8). Then, it is clear that $V \equiv U - W \in D_A$ satisfies

$$AV = F' \qquad \text{in } \Omega \times [0,\ +\infty)$$

with $F' = F - AW$ and *homogeneous* boundary conditions.

So, we are led to consider the functional

$$\tilde{\Phi}(V) = \Phi(U) \tag{11}$$

$$= 1/2 \int_\Omega [AU * U + (AW * U - AU * W) - 2U * F - AW * W + 2W * F]\, \mathrm{d}\Omega$$

that can be easily worked out by means of (9) and (10).

We give the final result in form of the following

Theorem 3 (*Variational Principle*). *Let D_A^* the subdomain of all thermokinematically admissible vectors $U = (u_i,\ \varphi,\ \theta) \in D_A$. Then, the func-*

tional on D_A^*

$$\Phi(u_i,\ \varphi,\ \theta) = \frac{1}{2}\int_\Omega\{[e*(\tau_{ij}*E_{ij} + \mu_{ijh}*K_{ijh} + h_i*\varphi_{,i} - g*\varphi) \tag{12}$$

$$-(\rho\eta - a)*(e+\alpha 1)*\theta] + [\rho u_i*u_i + \rho k\varphi*\varphi - (1/\theta_0)\,e*(1*q_i + \alpha q_i)*\theta_{,i}]$$

$$-2[F_i*u_i + L*\varphi - (e+\alpha 1)*R*\theta]\}\,\mathrm{d}\Omega - \int_{\partial_2\Omega}(e*u_i*P_i^*)\,\mathrm{d}\Sigma$$

$$-\int_{\partial_4\Omega}(e*Du_i*R_i^*)\,\mathrm{d}\Sigma - \int_{\partial_6\Omega}(e*\varphi*h^*)\,\mathrm{d}\Sigma + \int_{\partial_8\Omega}(1/\theta_0)\,e*(1*q^* + \alpha q^*)*\theta\,\mathrm{d}\Sigma$$

has a stationary point at, and only at, the solution of the initial-boundary value problem (1)-(4).

We only note that, in writing the functional (12) from (11), we have neglected inessential terms (i.e., with vanishing variations).

Theorem 3 is, actually, a variational principle of Gurtin type [9].

5 - A minimum principle for the purely mechanical problem

In this final section, we assume

$$\theta = q_i = \eta \equiv 0 \qquad \text{in } \Omega\times[0,\ +\infty).$$

Dropping the equations involving θ, q_i and η, we will refer to problem (1)-(4) as *purely mechanical*. An array $(u_i,\ \varphi)$ satisfying the pertinent left-side conditions in (4) will now be called *kinematically admissible*; $\mathcal{K}$ will denote the class of these arrays. Moreover, we keep the notions of *solution* to the problem and of *data* (with clear exclusions).

Recall now the constitutive equations (2). We will have occasion to consider the quadratic form

$$\Pi(X,\ X_i,\ X_{ij},\ X_{ijk}) = \tfrac{1}{2}A_{ijkl}X_{ij}X_{kl} + \tfrac{1}{2}C_{ijkpqr}X_{ijk}X_{pqr} + \tfrac{1}{2}cX^2 \tag{13}$$

$$+\tfrac{1}{2}D_{ij}X_iX_j + B_{pqijk}X_{pq}X_{ijk} + L_{ijkh}X_{ijk}X_h$$

$$+M_{ij}X_{ij}X + H_{ijk}X_{ij}X_k + R_{ijk}X_{ijk}X + d_iX_iX\,.$$

Note that, for $X=\varphi$, $X_i = \varphi_{,i}$, $X_{ij} = E_{ij}$, $X_{ijh} = K_{ijh}$, this form takes the meaning of a *potential energy* density for Ω along $(u_i,\ \varphi)$.

The stationary functional (12) can be used to yield a minimizing functional in the present context. We premise some definitions.

Let $\mathscr{L}$ denote the class of the fields on $\Omega \times (0, +\infty)$, say v, that admit (real) Laplace-transform $\hat{v}$ in Ω

$$\hat{v}(\boldsymbol{x}, s) = \int_{R^+} \exp(-st)\, v(\boldsymbol{x}, t)\, \mathrm{d}t \qquad \forall s \in R^+ .$$

Set also $$\tilde{\mathscr{L}} = \{v \in \mathscr{L}:\ \dot{v},\ v_{,i},\ v_{,ij} \in \mathscr{L}\}.$$

Further, let G be the set of all functions $g: t \in [0, \infty) \to g(t) \in R^+$ such that

$$g(t) = \int_{R^+} \exp(-st)\, G(s)\, \mathrm{d}s \tag{14}$$

for some non-negative (smooth) function G having compact support in R^+ (cf. [17], [5]).

Now, assume the data belong to $\mathscr{L}$ and u_i, φ to $\tilde{\mathscr{L}}$: in this case, we would be entitled to consider the *transformed* functional $\hat{\Phi}(\hat{u}_i, \hat{\varphi})$ and try to prove a minimum principle, in the Laplace-transform domain, of Benthien & Gurtin type [1]. Rather than do this (3), we resort to Reiss' idea [17], and set up the functional

$$\Psi[(u_i, \varphi);\ g] = \int_{R^+} G(s)\, \hat{\Phi}(\hat{u}_i, \hat{\varphi})\, s^2\, \mathrm{d}s .$$

We are thus led to the following (cf. $[3]_2$)

Theorem 4 (*Minimum Principle*). *Let* (u_i, φ) *be a solution to the purely mechanical problem* (1)-(4), *with all data belonging to* $\mathscr{L}$; *let also* u_i, φ *belong to* $\tilde{\mathscr{L}}$. *Given any* $g \in G$, *consider the (well-defined) functional over* $\mathscr{K} \cap \tilde{\mathscr{L}}$

$$\Psi[(v_i, \psi);\ g] = \int_{R^+ \times R^+} g(t+\tau)\{\int_{\Omega} [(\tfrac{1}{2}A_{ijkl}F_{ij}(t)\, F_{kl}(\tau) + \tfrac{1}{2}D_{ij}\psi_{,i}(t)\, \psi_{,j}(\tau)$$

$$+ \tfrac{1}{2}C_{ijkpqr}S_{ijk}(t)\, S_{pqr}(\tau) + \tfrac{1}{2}c\psi(t)\, \psi(\tau) + B_{pqijk}F_{pq}(t)\, S_{ijk}(\tau)$$

(3) The minimum character is lost when going back to the time domain by inverse transform.

$$+L_{ijkh}\psi_{,h}(t)S_{ijk}(\tau)+M_{ij}F_{ij}(t)\psi(\tau)+H_{ijk}\psi_{,k}(t)F_{ij}(\tau)+R_{ijk}S_{ijk}(t)\psi(\tau)$$

$$+d_i\psi_{,i}(t)\psi(\tau))+\tfrac{1}{2}(\rho\dot{v}_i(t)\dot{v}_i(\tau)+\rho k\dot{\psi}(t)\dot{\psi}(\tau))-(\rho f_i(t)v_i(\tau)$$

$$+\rho l(t)\psi(\tau))]\,\mathrm{d}\Omega\}\,\mathrm{d}t\,\mathrm{d}\tau-\int_{R^+\times R^+}g(t+\tau)\{\int_{\partial_2\Omega}P_i^*(t)v_i(\tau)\,\mathrm{d}\Sigma$$

$$+\int_{\partial_4\Omega}R_i^*(t)Dv_i(\tau)\,\mathrm{d}\Sigma+\int_{\partial_6\Omega}h^*(t)\psi(\tau)\,\mathrm{d}\Sigma\}\,\mathrm{d}t\,\mathrm{d}\tau+\int_{R^+}g(t)\,\mathrm{d}t\int_{\Omega}\rho\{[v_i(0)$$

$$-u_i^0]\dot{v}_i(t)-\dot{u}_i^0v_i(t)+k[\psi(0)-\varphi^0]\dot{\psi}(t)-k\dot{\varphi}^0\psi(t)\}\,\mathrm{d}\Omega$$

$$+g(0)\int_{\Omega}\rho\{v_i(0)[\tfrac{1}{2}v_i(0)-u_i^0]+k\psi(0)[\tfrac{1}{2}\psi(0)-\varphi^0]\}\,\mathrm{d}\Omega$$

where $F_{ij}=\frac{1}{2}(v_{i,j}+v_{j,i})$, $S_{ijk}=v_{k,ij}$. *Then, provided the quadratic form* (13) *is positive definite, it results*

$$\Psi[(v_i,\ \psi);\ g]\geq\Psi[(u_i,\ \varphi);\ g]\qquad\forall(v_i,\ \psi)\in\mathscr{H}\cap\tilde{\mathscr{L}}$$

and equality holds only if $(v_i,\ \psi)=(u_i,\ \varphi)$.

Proof. The main steps are as follows (for more details see $[3]_2$). Evaluate first $\Delta\Psi=\Psi[(v_i,\ \psi);\ g]-\Psi[(u_i,\ \varphi);\ g]$. Then, consider that $(u_i,\ \varphi)$ is a solution, and multiply equations $(1)_1$ and $(1)_2$, at time t, by (v_i-u_i) and $(\psi-\varphi)$ at time τ, respectively. Integrate over Ω and use the divergence theorem (with extensive recourse to integration by parts and reversal of the order of integration between space and time); of course, $v_i-u_i=0$ on $\partial_1\Omega\times(0,\ +\infty)$, $\mathrm{D}v_i-\mathrm{D}u_i=0$ on $\partial_3\Omega\times(0,\ +\infty)$ and $\psi-\varphi=0$ on $\partial_5\Omega\times(0,\ +\infty)$.

By insertion of $g(t)$ as in (14), Laplace-transform of the various fields will appear, so that we finally get

$$\Delta\Psi=\int_{R^+}G(s)\,\mathrm{d}s\{\int_{\Omega}[\Pi(\hat{\psi}-\hat{\varphi},\ \hat{\psi}_{,i}-\hat{\varphi}_{,i},\ \hat{F}_{ij}-\hat{E}_{ij},\ \hat{S}_{ijh}-\hat{K}_{ijh})$$

$$+\tfrac{1}{2}\rho s^2(\hat{v}_i-\hat{u}_i)^2+\tfrac{1}{2}\rho ks^2(\hat{\psi}-\hat{\varphi})^2]\,\mathrm{d}\Omega\}\,.$$

Clearly, $\Delta\Psi$ is (strictly) positive, unless $(u_i,\ \varphi)=(v_i,\ \psi)$. Conversely, $\Delta\Psi=0$ implies $(\hat{v}_i-\hat{u}_i)(s)=(\hat{\psi}-\varphi)(s)\equiv0$ over the support of G (containing an interval), and then over the whole of R^+ [5]. By the uniqueness of Laplace-transform, we thus get $v_1-u_1=\psi-\varphi\equiv0$ in $\Omega\times(0,\ +\infty)$.

References

[1] G. BENTHIEN and M. E. GURTIN, *A principle of minimum transformed energy in linear elastodynamics*, J. Appl. Mech. 37 (1970), 1147.

[2] M. CIARLETTA and A. SCALIA, *Thermoelasticity of non-simple materials with voids*, Pre-print n. 28/88 Ist. Fis. Mat. Inf. Univ. Salerno (to appear on Rev. Roumaine Sci. Tech. Sér. Méc. Appl.).

[3] M. CIARLETTA and E. SCARPETTA: $[\bullet]_1$ *Some results in a generalized thermodynamic theory for elastic materials with voids: constitutive relations and uniqueness*, Rev. Roumaine Sci. Techn. Méch. Appl. 2 (1989), 113; $[\bullet]_2$ *A minumum principle in the dynamics of elastic materials with voids*, Atti Accad. Naz. Lincei, Rend. 82 (in press).

[4] S. C. COWIN and J. W. NUNZIATO, *Linear elastic materials with voids*, J. of Elasticity 13 (1983), 125.

[5] M. FABRIZIO, C. GIORGI and A. MORRO, *Minimum principle, convexity and thermodynamics in viscoelasticity*, pre-print n. 10/88, Univ. Cattolica Brescia.

[6] D. GRAFFI, *Sui teoremi di reciprocità nei fenomeni non stazionari*, Atti Accad. Sci. Ist. Bologna (11) 10 (1963), 33.

[7] A. E. GREEN and N. LAWS, *On the entropy production inequality*, Arch. Rat. Mech. Anal. 45 (1972), 47.

[8] A. E. GREEN and K. A. LINDSAY, *Thermoelasticity*, J. of Elasticity 2 (1972), 1.

[9] M. E. GURTIN, *Variational principles for linear elastodynamics*, Arch. Rat. Mech. Anal. 16 (1964), 34.

[10] I. HLAVACEK, *Variational formulation of the Cauchy problem for equations with operator coefficients*, Aplikace Math. 16 (1971), 46.

[11] D. IESAN: $[\bullet]_1$ *Sur la theorie de la thermoelaticité micropolaire coupleé*, C. R. Acad. Sc. Paris 265 A (1967), 271; $[\bullet]_2$ *On some reciprocity and variational theorems in linear dynamic theories of continuum mechanics*, Mem. Accad. Sci. Torino (4) 17 (1974); $[\bullet]_3$ *Thermoelasticity of non-simple materials*, J. Thermal Stresses 6 (1983), 167.

[12] G. LEBON, *A generalized theory of thermoelasticity*, J. Tech. Phys. 23 (1982), 37.

[13] H. W. LORD and Y. SHULMAN, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids 15 (1967), 299.

[14] R. D. MINDLIN and N. N. ESHEL, *On first strain-gradient theories in linear elasticity*, Int. J. Solids Struct. 4(1968), 109.

[15] I. MULLER, *The coldness, a universal function in thermoelastic bodies*, Arch. Rat. Mech. Anal. 41 (1971), 319.

[16] J. W. NUNZIATO and S. C. COWIN, *A non-linear theory of elastic materials with voids*, Arch. Rat. Mech. Anal. 72 (1979), 175.

[17] R. REISS, *Minimum principles for linear elastodynamics*, J. of Elasticity 8 (1978), 35.

[18] E. RUSU, *On some theorems in a generalized theory of linear thermoelasticity*, Bul. Inst. Pol. din Iasi, XX, 3-4 (1974), 75.

[19] E. SUHUBI, *Thermoelastic solids*, in Continuum Physics, 2, Academic Press, New York, 1975.

[20] R. A. TOUPIN: $[\bullet]_1$ *Elastic materials with couple-stress*, Arch. Rat. Mech. Anal. **11** (1962), 385; $[\bullet]_2$ *Theories of elasticity with couple-stress*, Arch. Rat. Mech. Anal. **17**(1964), 85.

Summary

In the context of a temperature-rate dependent formulation of thermodynamics for non-simple elastic solids with voids, we prove reciprocity and variational theorems for the mixed initial-boundary value problem. A minimum principle is also established in the isothermal (purely mechanical) case.

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