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Study of a relaxed degenerate Dirichlet problem (**)

1 - Introduction

In this work we deal with pointwise continuity at the points at which the solution of the problem

$$(1.1) \qquad \mathrm{L}u + \mu u = \sigma \quad \text{in } \Omega \qquad u = g \quad \text{on } \partial\Omega$$

vanishes (L is a degenerate second order differential operator, μ a Borel measure and σ a Radon measure. For the definition of Radon measure see [1]).

The problem (1.1) is a relaxed Dirichlet problem in $\Omega \subset R^n$, $n \geqslant 3$. The estimate of the modulus of continuity will be carried out by a structural estimate of the ratio $V(r)/V(R_0)$, $0 \leqslant r \leqslant R_0$, on two concentric balls of the function of r

$$(1.2) \qquad V(r) = \underset{B_r}{\operatorname{osc}} |u|^2 + \int_{B_r} |Du|^2 G_\Sigma(x, y)\, w \, \mathrm{d}x + \int_{B_r} |u|^2 G_\Sigma(x, y) \, \mathrm{d}\mu$$

where $G_\Sigma(x, y)$ is the Green function of the operator L in a fixed large ball $\Sigma = (x : |x| \leqslant R_0)$ centred in the origin, containing the closure of Ω.

The estimate will be given in terms of the so called Wiener modulus of the Borel measure μ defined in **7** and of $\|\sigma\|_{K_n(B_r)}$, the norm of the Radon measure σ in the Kato space $\mathrm{K}_n(B_r)$, as indicated in **6**, by a method developped by Dal Maso, Mosco in the work [3].

Our purpose is to extend the results obtained in [3], to the case of homogeneous degenerate differential operators.

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2 - Notation and preliminaries

In this section we introduce the degenerate elliptic operator and point out the main properties of problem (1.1).

We denote by L the operator

$$\mathrm{L} = -\Sigma_{i,j}\,\mathrm{D}_j(a_{ij}(x)\,\mathrm{D}_i) \tag{2.1}$$

where D_i denotes the derivative with respect to the variable x_i and $\mathrm{D} = \mathrm{grad}$ the usual gradient; $a_{ij}(x)$ $\forall i, j = 1, 2, \ldots, n$ is a symmetric matrix of measurable functions defined in Ω such that

$$\frac{1}{c} w(x)|\xi|^2 \leqslant \Sigma_{i,j}\, a_{ij}(x)\,\xi_i\,\xi_j \leqslant c w(x)|\xi|^2 \qquad (c > 1)\,. \tag{2.2}$$

The weight $w(x)$ will be a non negative function, defined in R^n such that $w(x)$, $w^{-1}(x) \in L^1\mathrm{loc}(R^n)$ and for which the following (A_2) conditions holds

$$\sup_{|B_r|}\Big[\frac{1}{|B_r|}\int_{B_r} w(x)\,\mathrm{d}x\Big]\Big[\frac{1}{|B_r|}\int_{B_r} w^{-1}(x)\,\mathrm{d}x\Big] \leqslant C \tag{A_2}$$

where $|B_r|$ is the Lebesgue measure of the ball B_r and the supremum is taken over all Euclidean balls $B_r \subset R^n$.

We shall use the additional assumption

$$\frac{w(B_r)}{w(B_R)} \leqslant C\Big[\frac{|B_r|}{|B_R|}\Big]^{(2+a)/n} \qquad r < R \qquad a > 0\,. \tag{2.3}$$

The Green function of L in Σ, $G_\Sigma(x, y)$, is defined as the distributional solution of the problem

$$\mathrm{L}G_\Sigma(x, y) = \delta x \quad \text{in } \Omega \qquad\qquad G_\Sigma(x, y) = 0 \quad \text{on } \partial\Omega \tag{2.4}$$

where δ_x is the Dirac distribution at x; $G_\Sigma(x, y)$ has a singularity at x and that satisfies the properties

$$G_\Sigma(x, y) = G_\Sigma(y, x) \qquad\qquad G_\Sigma(x, y) \geqslant 0$$

$$G_\Sigma(x,\ y) \in H^1(\Sigma \setminus B_r(y),\ w) \qquad B_r(y) \subset \Sigma \quad r > 0$$

$$G_\Sigma(x,\ y) \in H^{1,p}(\Sigma,\ w) \qquad 1 \leq p \leq 2n/(2n-1)$$

moreover $G_\Sigma(x,\ y)$ is Hölder continuous in $B_r \setminus \{x\}$, $r > 0$.

3 - Functional framework

We introduce, here, the spaces which are the functional framework of our problem in this paper.

We denote by $L^2(\Omega,\ w)$ the space of square integrable functions respect to the weight $w(x)$ equipped by the norm

$$\|\Phi\|_{L^2(\Omega,w)} = [\int_\Omega |\Phi|^2 w(x)\,\mathrm{d}x]^{1/2} \qquad \forall \Phi \in L^2(\Omega,\ w) \tag{3.1}$$

and by $H^1(\Omega,\ w)$ the closure of $C^\infty(\Omega)$ w. r. to the norm

$$\|\Phi\|_{H^1(\Omega,w)} = (\|\Phi\|^2_{L^2(\Omega,w)} + \sum_i \|\mathrm{D}_i \Phi\|^2_{L^2(\Omega,w)})^{1/2}. \tag{3.2}$$

We define $H_0{}^1(\Omega,\ w)$ the closure of $C_0^\infty(\Omega)$ in the $H^1(\Omega,\ w)$ norm. On $H_0{}^1(\Omega,\ w)$ we can choose the following equivalent norm

$$\|\Phi\|_{H_0{}^1(\Omega,w)} = [\int_\Omega |\mathrm{D}\Phi|^2 w\,\mathrm{d}x]^{1/2} \qquad \forall \Phi \in H_0{}^1(\Omega,\ w). \tag{3.3}$$

We can introduce, now, a bilinear form associated to the operator L on $H_0{}^1(\Omega,\ w)$

$$\mathrm{D}(u,\ v) = \Sigma_{i,j} \int_\Omega a_{i,j}(x)\,\mathrm{D}_i u \mathrm{D}_j v \mathrm{d}x \qquad \forall u,\ v \in H_0{}^1(\Omega,\ w). \tag{3.4}$$

This bilinear form is well defined and coercive on $H_0{}^1(\Omega,\ w)$.

We observe that $\mathrm{D}(u,\ v)$ can be also defined for $u \in W^{1,1}(\Omega,\ w)$ and $v \in C_0^\infty(\Omega)$.

Putting together the properties of $\mathrm{D}(u,\ v)$ and those of $G_\Sigma(x,\ y)$ we have

$$\mathrm{D}(G_\Sigma(x,\ y),\ \varphi(y)) = \varphi(x) \qquad \forall \varphi \in C_0^\infty(\Sigma). \tag{3.5}$$

We shall indicate by $L^2(\Omega,\ \mu)$ the class of functions that are square integrable with respect to the Borel measure μ, i.e.,

$$\int_{\Omega} |\Phi|^2 \mathrm{d}\mu < +\infty \qquad \forall \Phi \in L^2(\Omega,\ \mu). \tag{3.6}$$

Let $M_0(\Omega)$ be the equivalence class of non-negative Borel measures on Ω vanishing on every Borel set of null capacity (see 4). We suppose $\mu \in M_0(\Omega)$. We observe that μ can be $+\infty$ on some large subset of Ω.

If σ is a Radon measure, we denote by $\langle v,\ \sigma\rangle$

$$\langle v,\ \sigma\rangle = \int_{\Omega} v \mathrm{d}\sigma \qquad \forall v \in H^1(\Omega). \tag{3.7}$$

The weak formulation of problem (1.1) which we consider in our problem, is the following

$$\begin{gathered} \mathrm{D}(u,\ v) + \int_{\Omega} uv\, \mathrm{d}\mu = \langle v,\ \sigma\rangle \\ \forall u,\ v \in H^1(\Omega,\ w) \cap L^2(\Omega,\ \mu) \qquad u - g \in H^1{}_0(\Omega,\ w) \cap L^2(\Omega,\ \mu). \end{gathered} \tag{3.8}$$

4 - Capacity and its properties

We introduce here the notion of capacity (see [**6**]) associated to the operator L of a set E with respect to Ω, $E \subset \Omega$, by the relationship

$$\operatorname{cap}(E,\ \Omega) = \inf\{D(v,\ v)\} \qquad v \in H^1{}_0(\Omega,\ w) \qquad v \geqslant 1 \text{ on a neighbourhood of } E. \tag{4.1}$$

If we take $\Omega \equiv \Sigma$, mentioned above, we will write

$$\operatorname{cap}(E,\ \Sigma) = \operatorname{cap} E.$$

The principal features of this type of capacity can be found in [**5**]. We recall the estimate

$$\operatorname{cap}(B_r,\ B_{2r}) \cong w(B_r)/r^2 \tag{4.2}$$

where $\cong$ stands for an equality except for a multiplicative constant indipendent from x and r.

An estimate that will be used in this paper is the following

$$\frac{\operatorname{cap} B_{qr}}{\operatorname{cap} B_{pr}} \leqslant 1 + Cx^{2n}(1+x)^{2n} \qquad x = \frac{q}{p} > 1 . \tag{4.3}$$

The proof of relation (4.3) is the same of that of [2].

The following relation between the capacity of a ball and the Green function is proved in [5]

$$G_\Sigma(x,\ y) = \frac{1}{\operatorname{cap} B_r(y)} \qquad r = |x-y| . \tag{4.4}$$

Then in S_R, $r = B_R \setminus B_r \subset \Omega$, the Green function can be estimated by

$$\frac{C}{\operatorname{cap} B_R(y)} \leqslant G_\Sigma(x,\ y) \leqslant \frac{C}{\operatorname{cap} B_r(y)} \tag{4.5}$$

with C a constant independent from x, r and R.

Finally we observe that the assumption (2.3) doesn't allow the esistence of points with positive capacity.

5 - μ-capacity and his proprieties

Def. 5.1. We say that $E \subset R^n$ is *μ-admissible* in Ω if E is a Borel subset of Ω and there exists a function $\emptyset \in H^1(\Omega,\ w) \cap L^2(\Omega,\ \mu)$ with $\emptyset - 1 \in H^1{}_0(\Omega,\ w)$.

Let μ_E a measure such that

$$\mu_E(T) = \mu_E = \mu(E \cap T) \qquad T \subset \Omega \tag{5.1}$$

i.e. μ_E is the restriction of μ to the set E.

If E is μ-admissible in Ω, there exists a function $\emptyset_E$, called the *μ-capacity potential* of the set E defined as the unique solution of the problem [3]

$$\mathrm{L}\emptyset_E + \mu_E \emptyset_E = 0 \quad \text{in } \Omega \qquad \emptyset_E = 1 \quad \text{on } \partial\Omega . \tag{5.2}$$

The μ-capacity of E respect to Ω is defined by

$$\operatorname{cap}_\mu(E,\ \Omega) = \mathrm{D}(\emptyset_E,\ \emptyset_E) + \int_\Omega |\emptyset_E|^2 \, d\mu_E \tag{5.3}$$

for every μ-admissible set E.

We observe that $0 \leqslant \emptyset_E \leqslant 1$ [3].

If E is μ-admissible then $\emptyset_E \in L^2(\Omega, \mu_E)$ and so $0 \leqslant \mathrm{cap}_\mu(E, \Omega) < +\infty$.

The proprieties of such type of capacity are analogous to those given in [3] for the usual elliptic case.

We are concerned with the case in which the coefficients of the operator L are simmetric. In such case it's easy to see an equivalent definition of capacity is given by the infimum problem

$$(5.4) \qquad \mathrm{cap}_\mu(E, \Omega) = \inf\{(\mathrm{D}(v, v) + \int_\Omega |v|^2 \mathrm{d}\mu_E \qquad v - 1 \in H^1{}_0(\Omega, w) \cap L^2(\Omega, \mu_E))\}$$

for every μ-admissible E.

6 - Kato space

Let Ω an open space, σ a Radon measure such that σ, $|\sigma| \in H^{-1}(\Omega, w)$. We define the functional $\langle \Phi, \sigma \rangle$ as in (3.7).

Def. 6.1. We will denote as $K_n(\Omega)$ the space of Radon measures σ on Ω such that

$$(6.1) \qquad \lim_{r \to 0^+} \sup_{x \in \Omega} \int_{\Omega \cap B_r} G_\Sigma(x, y)\, \mathrm{d}|\sigma|\,(y) = 0 .$$

We introduce a norm on $K_n(\Omega)$ defined by

$$(6.2) \qquad \|\sigma\|_{K_n(\Omega)} = \sup_{x \in \Omega} \int_\Omega G_\Sigma(x, y)\, d|\sigma| .$$

We can prove, as on [1], that $K_n(\Omega)$ is a Banach space with the norm (6.2).

Proposition 6.2. *The following properties hold for the norm on* $K_n(\Omega)$

$$(6.3) \qquad |\sigma|(\Omega) \leqslant k\|\sigma\|_{K_n(\Omega)} . \qquad\qquad (6.4) \qquad \lim_{r \to 0^+} \|\sigma\|_{K_n(B_r)} .$$

Proof. The relation (6.3) is easily proved taking into account the relation (4.4). Let $\Omega \subseteq B_R(y) \subset \Sigma$, R fixed, then the Green function can be evaluated as in

(4.5); so from (6.2)

(6.4) $$\|\sigma\|_{K_n(\Omega)} \geqslant \frac{|\sigma|}{\operatorname{cap} B_R(y)}.$$

This relation implies (6.3) with $k = \operatorname{cap} B_R(y)$.

The relation (6.4) follows immediately from the definition of Kato space (6.1).

The main result of this section is the following

Theorem 6.3. *If $\sigma \in K_n(\Omega)$ and u is a local weak solution of the problem*

(6.5) $$\mathrm{L}u = \sigma \quad in\ \Omega \qquad u = g \quad on\ \partial\Omega$$

then $u \in C^0(\Omega)$.

The proof of this result will be given in 8.

7 - Regular and Wiener points

An important result of this section is the following theorem (Poincaré inequality).

Theorem 7.1. *Let $u \in H^1(\Omega, w) \cap L^2(\Omega, \mu)$ $B_{2r} \subset \Omega$ $0 < q < 1$ and $S_{r,qr} = B_r \setminus B_{qr}$, then*

(7.1) $$\int_{S_{r,qr}} |u|^2 w \, dx \leqslant \frac{K w(B_{2r})}{\operatorname{cap}_\mu(S_{r,qr}, B_{2r})} \Big(\int_{S_{2r,qr/2}} |\mathrm{D}u|^2 w dx + \int_{S_{2r,qr/2}} |u|^2 d\mu \Big)$$

with K a constant independent from x and r.

The proof of this result will be given in 8.

Def. 7.2. A point $x_0 \in R^n$ is a *regular Dirichlet point* for μ if every local weak solution of (3.8) is continuous and vanishes at x_0.

A sufficient condition for the regularity of a point can be given in terms of a function $\omega_\mu(x_0; r, R)$ $0 \leqslant r \leqslant R$ associated to the measure μ at a given point x_0.

Def. 7.3. For every $0 \leqslant \theta \leqslant R_0$, we put

(7.2) $$\delta(\theta) = \frac{\mathrm{cap}_\mu(B_\theta,\ B_{2\theta})}{\mathrm{cap}(B_\theta,\ B_{2\theta})}. \qquad (7.3) \qquad \omega_\mu(x_0;\ r,\ R) = \exp[-\int_r^R \delta(\theta)\, \mathrm{d}\theta/\theta].$$

The function $\omega_\mu(x_0;\ r,\ R)$ is said *Wiener modulus* of μ at x_0.

Remark 7.4. The following properties derived from the properties of the capacity

(7.4) $$0 \leqslant \delta(\theta) \leqslant 1. \qquad (7.5) \qquad \frac{r}{R} \leqslant \omega_\mu(x_0;\ r,\ R) \leqslant 1.$$

Theorem 7.5. *If $V(r)$ is the quantity defined in* (1.2), *then there exist two constants k, β depending only on the elliptic constants and the dimension of the space, such that*

(7.6) $$V(r) \leqslant k(\omega^\beta{}_\mu(x_0;\ r,\ R_0)\, V(R_0) + \|\sigma\|^2_{K_n(B_r)})$$

with $0 \leqslant r < R_0$, R_0 constant.

From Theorem 7.5, easily follows

Theorem 7.6. *If x_0 a Wiener point for measure μ and the operator* L, *then x_0 is a regular Dirichlet point.*

8 - Proofs

Proof Theorem 6.3. Let

(8.1) $$E(x,\ \Omega) = \int_\Omega G_\Sigma(x,\ y)\, \mathrm{d}\sigma(y).$$

We can see that $E(x,\ \Omega)$ is a weak solution of equation (6.5) with $g = 0$ and $|E(x,\ B_r))| \leqslant \|\sigma\|_{K_n(B_r(x))}$.

We will prove that $E(x,\ \Omega)$ is continuous at $x_0 \in \Omega$.

We take a sequence of points $(x_k) \in B_R(x_0) \subset \Omega$ such that $\lim_{k \to +\infty} x_k = x_0$. We will prove

$$\lim_{k\to+\infty} E(x_k,\ \Omega) = E(x_0,\ \Omega).$$

We have that

$$0 \leqslant |E(x_k,\ \Omega) - E(x_0,\ \Omega)| \leqslant |E(x_k,\ \Omega\setminus B_r(x_0)) - E(x_0,\ \Omega\setminus B_r(x_0))|$$

$$+ |E(x_k,\ B_r(x_0))| + |E(x_0,\ B_r(x_0))|$$

$$\leqslant |E(x_k,\ \Omega\setminus B_r(x_0)) - E(x_0,\ \Omega\setminus B_r(x_0))| + 2\|\sigma\|_{K_n(B_r(x_0))}.$$

From (6.4) we obtain the continuity of $E(x,\ \Omega)$ at x_0. We observe now that, if

(8.2) $$p(x) = u(x) - E(x,\ \Omega)$$

where $u(x)$ is the weak solution of (6.5) and $E(x,\ \Omega)$ is the function defined in (8.1), then $p(x)$ is the solution of the following problem

(8.3) $$\mathrm{L}p(x) = 0 \quad \text{in } \Omega \qquad p(x) = g \quad \text{on } \partial\Omega.$$

The De Giorgi-Nash-Moser theorem holds also in the homogeneous degenerate case [4], then $p(x)$ is continuous at x_0. So $u(x) = E(x,\ \Omega) + p(x)$ is continuous at $x_0 \in \Omega$. As the point x_0 is choosen in an arbitrary way, we have proved the theorem.

Proof Theorem 7.1. We need the following lemmas.

Lemma 8.1. *If* $u \in H^1(\Omega,\ w)$, *then*

(8.4) $$\int_{B_r} |u - \bar{u}|^2 w\,\mathrm{d}x \leqslant C_p r^2 \int_{B_r} |\mathrm{D}u|^2 w\,\mathrm{d}x$$

with C_p *is a constant indipendent from* x *and* r *and* $\bar{u}$ *the average of* u *on a set* $E \subset B_r$ [4].

Lemma 8.2. *Let* $u \in H^1(\Omega,\ w) \cap L^2(\Omega,\ \mu)$, *then the following inequality holds*

(8.5) $$|\bar{u}|^2 \leqslant \frac{C}{\mathrm{cap}_\mu(S_{r,qr},\ B_{2r})}\ \Big[\int_{S_{2r,qr/2}} |\mathrm{D}u|^2 w\,\mathrm{d}x + \int_{S_{2r,qr/2}} |u|^2\,\mathrm{d}\mu\Big].$$

Proof. If $u=0$, the relation is obvious. We suppose then $\bar{u} <> 0$. From the definition of μ-capacity (5.4) we have

$$\mathrm{cap}_\mu(S_{r,qr}, B_{2r}) \leq \int_{B_{2r}} |\Phi|^2 w \, dx + \int_{S_{r,qr}} |\Phi|^2 d\mu .$$

Let Φ defined as $\Phi = 1 + \tau \frac{u-\bar{u}}{\bar{u}}$ with u defined on B_r $u \in H^1(B_r, w) \cap L^2(B_r, \mu)$ and with B_1, B_2 two subsets of $S_{2r,qr/2}$, diffeomorfics to a sphere such that $S_{2r,qr/2} \subset B_1 u B_2$ and $E = B_1 \cap B_2$ and with $\bar{u}$ we will denote the average of u on the set E. C_1 and C_2 are two constants in general different from C_p of Lemma 8.1. We define the function τ as $\tau =$ (0 out of $S_{2r,qr/2}$, and 1 on $S_{r,qr}$) and $0 \leq \tau \leq 1$ on B_{2r}, $|D\tau| < C/r$ on B_{2r}.

Taking into account these properties we obtain

$$\mathrm{cap}_\mu(S_{r,qr}, B_{2r}) \leq \frac{K}{|u|^2} \Big(\int_{S_{2r,qr/2}} |Du|^2 w \, dx + \int_{S_{2r,qr/2}} |u|^2 d\mu \Big)$$

from which (8.5) follows.

We return now to the Theorem 7.1.

From the fact that sets B_1 and B_2 are diffeomorphic to a ball on they, it holds a Poincaré inequality

$$\int_{B_1} |u-\bar{u}|^2 w \, dx \leq C_1 r^2 \int_{B_1} |Du|^2 w \, dx \tag{8.6}$$

and analogous for B_2. We have

$$\int_{B_1} |u|^2 w \, dx \leq r^2 \int_{B_1} |u-\bar{u}|^2 w \, dx + |\bar{u}|^2 w(B_1) \leq C_1 r^2 \int_{B_1} |Du|^2 w \, dx + |\bar{u}|^2 w(B_1)$$

and analogous for B_2. Adding the two formulas for B_1 and B_2, letting $K = \max(C_1, C_2)$ and remembering that $w(B_1)$, $w(B_2) \leq w(B_{2r})$ because they are subsets of B_{2r}, we get

$$\int_{S_{r,qr}} |u|^2 w dx \leq K r^2 \int_{S_{r,qr}} |Du|^2 w \, dx + |\bar{u}|^2 w(B_{2r}) . \tag{8.7}$$

As in [3], we obtain

$$r^2 \leq C_3 \frac{w(B_{2r})}{\mathrm{cap}_\mu(S_{r,qr}, B_{2r})} . \tag{8.8}$$

Putting together the (8.5), (8.7) and (8.8) we obtain easily relation (7.1).

Proof Théorème 7.5. To prove the théorème we need the following results.

Lemma 8.3. *For every* $0<q<1$, $k>0$, *we have*

$$V(qR) \leq \frac{k}{R^2} \int_{S_{R,qR}} G_\Sigma(x, y)|u|^2 w \mathrm{d}x + k\|\sigma\|^2_{K_n(B_R)}.$$

We recall also the following result due to [3].

Lemma 8.4. *Let* $R>0$, $0<q<1$ *and* $0<r\leq qR$, $\mathrm{D}(x)$ *a measurable function* $(r, R)\to(0, 1)$ *and* $F(x)$ *a non decreasing function* $(r, R)\to(0, +\infty)$. *We suppose that there exists a constant* $k>0$ *such that*

$$\mathrm{D}(q\theta) \leq \mathrm{D}(\theta)/(1+kF(\theta)) \tag{8.9}$$

for every $r/R<\theta<R_0$. *Then*

$$\mathrm{D}(r) \leq C\mathrm{D}(R_0)\exp(-\beta \int_r^{R_0} F(\theta)\,\mathrm{d}\theta/\theta) \tag{8.10}$$

where $C=\exp(k/(1+k))$ *and* $\beta = k/(1+k)|\log q|$.

Lemma 8.5. *Let*

$$\delta_q(\theta) = \frac{\mathrm{cap}_\mu(S_{\theta,q\theta}, B_{2\theta})}{\mathrm{cap}(B_\theta, B_{2\theta})} \qquad \delta(\theta) = \frac{\mathrm{cap}_\mu(B_\theta, B_{2\theta})}{\mathrm{cap}(B_\theta, B_{2\theta})} \tag{8.11}$$

then

$$\int_r^{R_0} \delta_q(\theta)\,\mathrm{d}\theta/\theta \geq (C-1)\int_r^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta + C\log|q| \qquad C>1. \tag{8.12}$$

Proof Lemma 8.5. As in [3], we have

$$\mathrm{cap}_\mu(B_\theta, B_{2\theta}) \leq \mathrm{cap}_\mu(B_{q\theta}, B_{2q\theta}) + \mathrm{cap}_\mu(S_{q\theta,q\theta}, B_\theta).$$

We divide now for $\operatorname{cap}(B_\theta, B_{2\theta})$ and from the definition (8.11)

$$\delta(\theta) \geqslant \frac{\operatorname{cap}_\mu(B_{q\theta}, B_{2q\theta})}{\operatorname{cap}(B_\theta, B_{2\theta})} - \delta_q(\theta) = \delta(q\theta)\, \frac{\operatorname{cap}(B_{q\theta}, B_{2q\theta})}{\operatorname{cap}(B_\theta, B_{2\theta})} - \delta_q(\theta).$$

We now estimate the rapport between the capacities.

Using relation (2.3) we get $(q>1)$

$$C = \frac{\operatorname{cap}(B_{q\theta}, B_{2q\theta})}{\operatorname{cap}(B_\theta, B_{2\theta})} \leqslant \frac{w(B_{q\theta})}{q^2 w(B_\theta)} \leqslant q^{-2}\Big(\frac{|B_{q\theta}|}{|B_\theta|}\Big)^{(2+\alpha)/n}$$

$$= q^{-2}(q^n\,\theta^n/\theta^n)^{(2+\alpha)/n} = q^\alpha$$

where we have used an estimate of the Lebesgue measure of a ball. We have that $C \geqslant 1$ for q sufficiently small. Hence with the relation between the capacities we obtain the following relation between the δ-functions $\delta(\theta) \geqslant C\delta(q\theta) - \delta_q(\theta)$ and then $\delta_q(\theta) \geqslant C\delta(q\theta) - \delta(\theta)$. Integrating from r to R_0

$$\int_r^{R_0} \delta_q(\theta)\,\mathrm{d}\theta/\theta \geqslant C\int_r^{R_0} \delta(q\theta)\,\mathrm{d}\theta/\theta - \int_r^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta$$

$$= C\int_{qr}^{qR_0} \delta(\theta)\,\mathrm{d}\theta/\theta - \int_r^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta$$

$$= C\int_r^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta - C\int_{qR_0}^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta + \int_{qr}^{r} \delta(\theta)\,\mathrm{d}\theta/\theta - \int_r^{R_0} \delta(\theta)\,\mathrm{d}\theta/\theta.$$

Taking into account that $0 \leqslant \delta(\theta) \leqslant 1$, we have (8.12).

Proof Theorem 7.5. Let

$$V(qR) > 2k\|\sigma\|^2_{K_n(B_R)} \tag{8.13}$$

so that

$$V(qR) \leqslant \frac{2k}{R^2} \int_{S_{R,qR}} G_\Sigma(x, y)|u|^2 w\,\mathrm{d}x.$$

By using relation (4.5) and the Poincaré inequality (7.1) we have

$$N(q, R) = kw(B_{2R})/R^2/\operatorname{cap} B_{qR} = k\operatorname{cap}(B_R, B_{2R})/\operatorname{cap} B_{qR}$$

$$V(qR/2) \leqslant V(qR) \leqslant \frac{N(q,\ R)}{\operatorname{cap}_\mu(S_{R,qR}\ ,\ B_{2R})} \Big(\int_{S_{2R,qR/2}} |Du|^2 w \, dx + \int_{S_{2R,qR/2}} |u|^2 \, d\mu \Big) . \quad \text{So}$$

$$V(qR/2) \leqslant \frac{\operatorname{cap}(B_R,\ B_{2R})}{\operatorname{cap} B_{qR} \operatorname{cap}_\mu(S_{R,qR}\ ,\ B_{2R})} \Big(\int_{S_{2R,qR/2}} |Du|^2 w \, dx + \int_{S_{2R,qR/2}} |u|^2 \, d\mu \Big)$$

$$= \frac{K(q)}{\delta_q(R)} \Big(\int_{S_{2R,qR/2}} G_\Sigma |Du|^2 w \, dx + \int_{S_{2R,2R/2}} G_\Sigma |u|^2 \, d\mu \Big)$$

where relations (4.3) and (4.5) are been used. Adding $V(qR/2)$ on both sides we obtain $(1 + k\delta_q(R))\, V(qR/2) \leqslant V(2R)$ with k a constant depending only on q and n. So

$$V(qR/2) \leqslant \frac{1}{1 + k\delta_q(R)} V(R) .$$

From Lemma 8.4

$$V(r) \leqslant KV(R_0) \exp\Big(-\beta \int_r^{R_0} \delta_q(x) \, dx\Big)$$

and from Lemma 8.5

$$V(r) \leqslant KV(R_0) \exp\Big(-\beta \int_r^{R_0} \delta(\theta) \, d\theta/\theta\Big)$$

where K and β are constants that can vary in each passage but that depend only on q and n.

From the definition of $\omega_\mu(x_0;\ r,\ R_0)$, we have

(8.14) $$V(r) \leqslant K\omega^\beta_\mu(x_0;\ r,\ R_0)\, V(R_0) \qquad r \leqslant qR_0/4$$

where R_0 is a suitable constant.

If the assumption (8.13) does not hold, we have

(8.15) $$V(r) \leqslant 2k\|\sigma\|^2_{K_n(B_r)} .$$

Then (8.14) and (8.15) can be summarized in the relation

$$V(r) \leqslant K\omega^\beta_\mu(x_0;\ r,\ R_0)\, V(R_0) + 2k\|\sigma\|^2_{K_n(B_r)}\, r \leqslant qR_0/4 .$$

Proof Theorem 7.6. We have $0 \leqslant |u(x)|^2 \leqslant V(r)$. Hence

$$0 \leqslant \lim_{x \to x_0} |u(x)|^2 \leqslant \lim_{r \to 0} V(r) = 0\,.$$

By taking account the properties of $\omega_\mu(x_0;\, r,\, R)$ and $\|\sigma\|_{K_\mu(B_r)}$ we have

$$\lim_{x \to x_0} |u(x)|^2 = 0\,.$$

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Summary

We give a sufficient Wiener's type criterion for the regularity of a point for a relaxed Dirichlet problem relative to a degenerate elliptic second order operator.

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