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A nonlinear parabolic system (**)

A TRISTANO MANACORDA per il suo 70° compleanno

1 - Introduction

Let Ω be a bounded domain of R^2 with a regular boundary S of class C^2. In this paper we prove the existence of at least one weak solution in $[0, T]$ for the following problem (Pb):

(1.1) $$\nabla \cdot (\sigma(u) \nabla \varphi) + \Delta \varphi_t = 0 \qquad \text{in } Q_T$$

(1.2) $$\varphi = \varphi_0 \qquad \text{on } \Gamma_T$$

(1.3) $$u_t - \Delta u = \sigma(u)|\nabla \varphi|^2 + \nabla \varphi \cdot \nabla \varphi_t$$

(1.4) $$u = 0 \qquad \text{on } \Gamma_T$$

where Q_T is the cylinder $\Omega \times (0, T)$, S_T the lateral surface i.e. $S_T = \{(x, t); x \in S, t \in [0, T]\}$, and $\Gamma_T = S_T \cup \{(x, t); x \in \Omega, t = 0\}$. Moreover $\varphi_0(x, t) \in C^3(\bar{Q}_T)$ and $\sigma(u) \in C^1(R^1)$ are given functions such that

(1.5) $$\sigma_1 \geqslant \sigma(u) \geqslant \sigma_0 > 0 \qquad \text{for all } u \in R^1 .$$

The interest of problem (Pb) lies in its nonstandard structure and in the quadratic growth in the gradient. For a related problem we refer to [2].

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The notations for the Sobolev spaces will be those of the book [4]. However we recall the main definitions.

$W_2^{1,0}(Q_T)$, $W_2^{1,1}(Q_T)$ and $W_2^{2,1}(Q_T)$ are Hilbert spaces with the following scalar products and the corresponding generalized derivatives in $L^2(Q_T)$

$$(u,\ v)_{W_2^{1,0}(Q_T)} = \int_{Q_T} (uv + \nabla u \cdot \nabla v)\,\mathrm{d}x\,\mathrm{d}t \qquad \nabla u = (u_{x_1},\ u_{x_2})$$

$$(u,\ v)_{W_2^{1,1}(Q_T)} = \int_{Q_T} (uv + \nabla u \cdot \nabla v + u_t v_t)\,\mathrm{d}x\,\mathrm{d}t$$

$$(u,\ v)_{W_2^{2,1}(Q_T)} = \int_{Q_T} (uv + \nabla u \cdot \nabla v + u_t v_t + \sum_{i,j=1}^{2} u_{x_i x_j} v_{x_i x_j})\,\mathrm{d}x\,\mathrm{d}t\,.$$

$V_2(Q_T)$ is the Banach space consisting of all elements of $W_2^{1,0}(Q_T)$ having finite the norm

$$\|v\|_{V_2(Q_T)} = \sup_{(0,T)}\|v(t)\|_{2,\Omega} + \|\nabla v\|_{2,Q_T}\,.$$

W is the space of the elements of $L^2(Q_T)$ with the following generalized derivatives ∇v, v_t and ∇v_t in $L^2(Q_T)$. With the scalar product

$$(u,\ v)_W = \int_{Q_T} (uv + u_t v_t + \nabla u \cdot \nabla v + \nabla u_t \cdot \nabla v_t)\,\mathrm{d}x\,\mathrm{d}t$$

W is an Hilbert space. Finally a dot over the spaces $W_2^{1,0}(Q_T)$, $W_2^{1,1}(Q_T)$, $V_2(Q_T)$ and W will denote the corresponding subspace of the functions vanishing on S_T.

A weak solution of (Pb) will be a couple $(\varphi,\ u)$ such that:

$$(1.6) \qquad \varphi - \varphi_0 \in \dot{W}(Q_T) \qquad \varphi(x,\ 0) = \varphi_0(x,\ 0) \qquad x \in \Omega\,;$$

$$(1.7) \qquad \varphi \in W \cap L^\infty(Q_T) \qquad \int_{Q_T} \nabla \varphi_t \cdot \nabla v dx\,\mathrm{d}t + \int_{Q_T} \sigma(u)\, \nabla\varphi \cdot \nabla v dx\,\mathrm{d}t = 0$$

for all $v \in \dot{W}_2^{1,1}(Q_T)$;

$$(1.8) \qquad u \in \dot{V}_2(Q_T) \qquad -\int_0^T (u,\ w_t)\,\mathrm{d}t + \int_0^T (\nabla u,\ \nabla w)\,\mathrm{d}t = -\int_0^T (\varphi\sigma(u)\,\nabla\varphi + \varphi\nabla\varphi_t,\ \nabla w)\,\mathrm{d}t$$

for all $w \in \dot{W}_2^{1,1}(Q_T)$ such that $w(T,\ 0) = 0$.

(u, v) denotes the scalar product in $L^2(\Omega)$. It is easy to verify by integration by parts that every regular solution of (Pb) is also a weak solution. Vivecersa every regular solution of (1.6), (1.7) and (1.8) is solution of (Pb). The existence of a weak solution will be proved in the next two sections using the Faedo-Galerkin method [3].

2 - Application of the Faedo-Galerkin method

Let $\Psi_k(x) \in C_0^\infty(\Omega)$, $k=1, 2, \ldots$, be a basis of $\dot{W}_2^1(\Omega)$ orthonormal with respect to $L^2(\Omega)$. Let $g^n(t) \in C^3[0, T]$, $j=1, 2, \ldots, n$ such that

$$g_j^n(0)=0. \tag{2.1}$$

Define
$$u^n(x, t)=\sum_{j=1}^n g_j^n(t)\, \Psi_j(x)$$

and consider the problem

$$\varphi^n=\varphi_0 \quad \text{on } S_T \qquad \varphi^n(x, 0)=\varphi_0(x, 0) \qquad x\in\Omega \tag{2.2}$$

$$\varphi^n \in W \qquad \int_{Q_T} \nabla\varphi_t^n \cdot \nabla v \mathrm{d}x\, \mathrm{d}t + \int_{Q_T} \sigma(u^n)\, \nabla\varphi^n \cdot \nabla v\, \mathrm{d}x\, \mathrm{d}t = 0 \tag{2.3}$$

for all $v \in \dot{W}_2^{1,1}(Q_T)$.

First of all we establish certain «a priori» estimates for the solutions of (2.2), (2.3). Put $v=\varphi^n-\varphi_0$ in (2.3), we have

$$\int_\Omega \sigma(u^n)|\nabla\varphi^n|^2\, \mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\varphi^n|^2\, \mathrm{d}x = \int_\Omega \sigma(u^n)\, \nabla\varphi^n \cdot \nabla\varphi_0\, \mathrm{d}x + \int_\Omega \nabla\varphi_t^n \cdot \nabla\varphi_0\, \mathrm{d}x.$$

Integrating between 0 and t we obtain

$$\int_{Q_t} \sigma(u^n)|\nabla\varphi^n|^2\, \mathrm{d}x\, \mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla\varphi^n(x, t)|^2\, \mathrm{d}x$$

$$= \frac{1}{2}\int_\Omega |\nabla\varphi^n(x, 0)|^2\, \mathrm{d}x + \int_{Q_t} \sigma(u^n)\, \nabla\varphi^n \cdot \nabla\varphi_0\, \mathrm{d}x\, \mathrm{d}t + \int_{Q_t} \nabla\varphi_t^n \cdot \nabla\varphi_0\, \mathrm{d}x\, \mathrm{d}t.$$

Integrating by parts in the second integral on the right hand side and then using the Cauchy-Schwartz inequality we have

$$\sup_{(0,T)}\|\nabla\varphi^n(t)\|_{2,\Omega}\leq C_1 \tag{2.4}$$

$$\|\varphi^n\|_{W_2^{1,0}(Q_T)}\leq C_2 \tag{2.5}$$

where the constants C_1 and C_2 do not depend on n.

Choose $v=(\varphi^n-\varphi_0)_t\in\dot W_2^{1,1}(Q_T)$ in (2.3), we get

$$\int_\Omega|\nabla\varphi_t^n|^2\,dx=\int_\Omega\nabla\varphi_t^n\cdot\nabla\varphi_{0t}\,dx+\int_\Omega\sigma(u^n)\,\nabla\varphi^n\cdot\nabla\varphi_{0t}\,dx-\int_\Omega\sigma(u^n)\,\nabla\varphi^n\cdot\nabla\varphi_t^n\,dx\,.$$

Recalling (1.5) and (2.4) we obtain

$$\sup_{(0,T)}\|\nabla\varphi_t^n\|_{2,\Omega}\leq C_3 \tag{2.6}$$

$$\|\nabla_t^n\|_{2,Q_T}\leq C_4\,. \tag{2.7}$$

Moreover by the Poincaré inequality, we have

$$\|\nabla\varphi_t^n\|_{2,Q_T}\leq C_5 \tag{2.8}$$

where C_3, C_4 and C_5 are constants not depending on n.

Lemma 2.1. *If $g_j^n(t)$, $j=1,2,..,n$, are given functions, problem* (2.2), (2.3) *has one and only one solution $\varphi^n(x,t)\in C^3(\bar Q_T)$.*

Dim. First of all we prove the uniqueness. Let $a(x,t)\in C^1(\bar Q_T)$ satisfy $a(x,t)>0$ in $\bar Q_T$. Consider the problem

$$\psi\in\dot W(Q_T)\qquad\psi(x,0)=0\qquad x\in\Omega \tag{2.9}$$

$$\int_{Q_T}\nabla\psi_t\cdot\nabla v\,dx\,dt+\int_{Q_T}a\nabla\psi\cdot\nabla v\,dx\,dt=0\qquad\text{for all } v\in\dot W_2^{1,1}(Q_T)\,. \tag{2.10}$$

Putting $v=\psi$ in (2.10) we find

$$\int_{Q_T}a|\nabla\psi|^2\,dx\,dt+\frac{1}{2}\int_\Omega|\nabla\psi(x,T)|^2\,dx=0\,.$$

This implies that problem (2.9), (2.10) has only one solution and also the uniqueness for problem (2.2), (2.3) if the functions $g_j^n(t)$ are given.

To prove the existence, let us consider the problem

$$\Delta\varphi_t + \sigma'(u)\nabla u\cdot\nabla\varphi + \sigma(u)\nabla\varphi = 0 \quad \text{in } Q_T \tag{2.11}$$

$$\varphi = \varphi_0 \qquad \text{on } \Gamma_T \tag{2.12}$$

in which we omit the index n.

Problem (2.11) and (2.12) can be restated in equivalent form with the following system:

$$\Delta\varphi = v \tag{2.13}$$

$$v_t + \sigma(u)v = -\sigma'(u)\nabla u\cdot\nabla\varphi \tag{2.14}$$

$$\varphi = \varphi_0 \qquad \text{on } S_T \tag{2.15}$$

$$v(x,\ 0) = \Delta\varphi_0(x,\ 0) \qquad x\in\Omega. \tag{2.16}$$

We integrate (2.14) as a first order equation in t. We find by (2.13) and (2.16)

$$\Delta\varphi(x,\ t) = A(x,\ t)[\Delta\varphi_0(x,\ t) + \sum_{i=1}^{n}\int_0^t B_i(x,\ \tau)\,\varphi_{x_i}(x,\ \tau)\,d\tau] \tag{2.17}$$

where
$$A(x,\ t) = \exp[\int_0^t \sigma(u(x,\ s))\,ds]$$

$$B_i(x,\ t) = -\sigma'(u(x,\ t))\,u_{x_i}(x,\ t)\exp[\int_0^t \sigma(u(x,\ s))\,ds].$$

We note that $A(x,\ t)$, $B_i(x,\ t)$ are of class $C^3(\bar{Q}_T)$ by the assumptions made on $g_j^n(t)$. We want to prove that the integrodifferential equation (2.17) with the boundary condition (2.15), has at least one solution. Let

$$\sum = \{v\in W_2^{1,0}(Q_t),\ v=\varphi_0 \text{ on } S_T,\ \|v\|_{W_2^{1,0}(Q_T)}\leqslant C_2\}$$

where C_2 is the constant of (2.5). Define the operator $\varphi = T(w)$, $T:\Sigma\to W_2^{1,0}(Q_T)$

via the linear problem

$$\Delta\varphi(x, t) = A(x, t)[\Delta\varphi_0(x, 0) + \int_0^t \sum_{i=1}^n B_i(x, \tau) w_{x_i}(x, \tau) d\tau] \tag{2.18}$$

$$\varphi = \varphi_0 \qquad \text{on } S_T. \tag{2.19}$$

The right hand side of (2.17) belongs to $W_2^{1,1}(Q_T)$, therefore problem (2.18), (2.19) can be solved and we find $\varphi \in W_2^{2,1}(Q_T)$. Moreover by (2.5), we have $T(\Sigma) \subseteq \Sigma$. Since T is continuous and $T(\Sigma)$ is compact in $W_2^{2,1}(Q_T)$, we can apply the Schauder fixed point theorem.

Hence there exists a solution in $W_2^{2,1}(Q_T)$ of (2.15), (2.17). With the usual «bootstrap» argument we can regularize the solution and conclude that $\varphi(x, t) \in C^3(\bar{Q}_T)$.

Remark 2.1. From (2.15) and (2.17) we obtain $\varphi(x, t)$ and $\nabla\varphi(x, t)$ if the functions $g_j^n(t)$ are known. On the other hand we can get $\varphi_t(x, t)$ and also $\nabla\varphi_t(x, t)$ from (2.11) by solving the problem

$$\Delta\varphi_t = -\sigma'(u)\nabla u \cdot \nabla\varphi - \sigma(u)\Delta\varphi \quad \text{in } Q_T \tag{2.20}$$

$$\varphi_t = \varphi_{0t} \qquad \text{on } S_T. \tag{2.21}$$

Again only the $g_j^n(t)$'s are involved in the right hand side of (2.20).

We want to deduce now that the φ^n's are a priori bounded in the maximum norm. We start by studying the following linear problem

$$\Delta\psi_t + \nabla \cdot (a(x, t)\nabla\psi) = \nabla \cdot F \quad \text{in } Q_T \tag{2.22}$$

$$\psi = 0 \qquad \text{on } \Gamma_T. \tag{2.23}$$

Lemma 2.2. *Let $F = (F_1, F_2) \in C^3(\bar{Q}_T)$ and $a(x, t) \in C^3(\bar{Q}_T)$. Suppose*

$$a_1 \geq a(x, t) \geq a_0 > 0. \tag{2.24}$$

Let $\psi(x, t) \in C^3(\bar{Q}_T)$ be a solution of (2.22) and (2.23). We claim that there exists $\hat{p} > 2$ such that for all $2 \leq p \leq \hat{p}$ the following estimate holds

$$\|\nabla\psi\|_{p, Q_T} \leq C\|F\|_{p, Q_T}. \tag{2.25}$$

The constant C depends only on a_0, a_1, p, and Q_T.

Proof. It is easy to verify that we may assume $a_1 = 1$ without loss of generality. Let us consider the following problem $(\mathrm{Pb})_\lambda$

$$\Delta\psi_t^{(\lambda)} + \nabla\cdot[(1-\lambda)+\lambda a(x,\ t)\,\nabla\varphi^{(\lambda)}] = \nabla\cdot F \qquad \text{in } Q_T \tag{2.26}$$

$$\psi^{(\lambda)} = 0 \qquad \text{on } \Gamma_T. \tag{2.27}$$

We denote by $C(\lambda,\ p)$ the best constant (which may by $+\infty$) for which (2.25) holds true for the solutions of $(\mathrm{Pb})_\lambda$, i.e.

$$C(\lambda,\ p) = \sup_{\|F\|_{p,Q_T}\neq 0}\frac{\|\nabla\psi^{(\lambda)}\|_{p,\,Q_T}}{\|F\|_{p,\,Q_T}}. \tag{2.28}$$

We find easily that $C(0,\ p) < +\infty$ for all $p \geqslant 2$ and

$$\lim_{p\to 2+} C(0,\ p) = C(0,\ 2) \leqslant 1. \tag{2.29}$$

One can also verify by direct calculations, that $C(1,\ 2) < +\infty$. Let us consider in the $(\lambda,\ p)$-plane the values $(\lambda,\ p)$ for which $C(\lambda,\ p) < +\infty$ and

$$\|\nabla\psi\|_{p,\,Q_T} \leqslant C(\lambda,\ p)\|F\|_{p,\,Q_T}. \tag{2.30}$$

If we derive with respect to λ equation (2.26), we find again a problem of the type $(\mathrm{Pb})_\lambda$ i.e.

$$\Delta\psi_{t\lambda} + \nabla\cdot[((1-\lambda)+\lambda a(x,\ t))\,\nabla\psi_\lambda] = \nabla\cdot[(1-a(x,\ t))\,\nabla\psi] \qquad \text{in } Q_t \tag{2.31}$$

$$\psi_\lambda = 0 \qquad \text{on } \Gamma_T. \tag{2.32}$$

Therefore we have, with the same constant $C(\lambda,\ p)$ of (2.30),

$$\|\nabla\psi_\lambda\|_{p,\,Q_T} \leqslant C(\lambda,\ p)\|(1-a)\,\nabla\psi\|_{p,\,Q_T} \leqslant C(\lambda,\ p)(1-a_0)\|\nabla\psi\|_{p,\,Q_T}. \tag{2.33}$$

Put $K = 1 - a_0 < 1$. Since $\dfrac{\mathrm{d}}{\mathrm{d}\lambda}\|\nabla\psi\|_{p,\,Q_T} \leqslant \|\nabla\psi_\lambda\|_{p,\,Q_T}$ we have, recalling (2.29) and (2.32),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\|\nabla\psi\|_{p,\,Q_T} \leqslant KC^2(\lambda,\ p)\|F\|_{p,Q_T}. \tag{2.34}$$

Integrating from λ_1 to λ_2, with $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant 1$, we get by (2.28)

$$C(\lambda_2,\ p) \leqslant C(\lambda,\ p) + K \int_{\lambda_1}^{\lambda_2} C^2(\lambda,\ p)\, d\lambda . \tag{2.35}$$

Using a comparison theorem for ordinary differential equations we may obtain an estimate from above for $C(\lambda, p)$ with the solution of the Cauchy problem

$$\frac{dz}{d\lambda} = Kz^2 \qquad z(0) = C(0,\ 2+\delta) . \tag{2.36}$$

$z(\lambda)$ is defined in $[0,\ 1/K\ C(0,\ 2+\delta))$. Since $C(0,\ 2) \leqslant 1$ and $K<1$ we can find $\hat{\delta}>0$ so small that $1/K\ C(0,\ 2+\hat{\delta})>1$. Put $\hat{p}=2+\hat{\delta}$. Since $C(\lambda,\ \hat{p}) \leqslant z(\lambda)$ it follows that $C(\lambda,\ \hat{p})$ is defined in $[0,\ 1]$. Therefore $C(1,\ \hat{p}) < +\infty$.

Corollary 2.1. *Under the same assumptions of Lemma 2.2 we have*

$$\max_{\bar{Q}_T} |\psi| \leqslant C \|F\|_{p, Q_T} \tag{2.37}$$

where C depends only on a_0, a_1, and Q_T.

Proof. Let $v(x,\ t) = \psi_t(x,\ t)$ and $f(x,\ t) = F - a(x,\ t)\nabla\psi$. For all $t \in [0,\ T]$ we have

$$\Delta v = \nabla \cdot f \qquad \text{in } \Omega \qquad\qquad v=0 \qquad \text{on } S.$$

By standard results [5] we get

$$\|v(x,\ t)\|_{p,\Omega} \leqslant C \|f\|_{p,\Omega} \qquad\qquad p>2.$$

Therefore

$$\|\psi_t\|_{p, Q_T} \leqslant C \|f\|_{p, Q_T} .$$

It follows by Lemma 2.2

$$\|\nabla\psi\|_{p, Q_T} + \|\psi_t\|_{p, Q_T} \leqslant C \|F\|_{p, Q_T} .$$

By Sobolev's imbedding theorem [1] we have (2.37) since $m=2$.

Putting $\psi = \varphi^n - \varphi_0$ we can apply Corollary 2.1 to problem (2.2), (2.3). Therefore we have

$$\|\varphi^n\|_{L^\infty(Q_T)} \leq C_7$$

where C_7 depends only on σ_0, σ_1, Q_T and φ_0.

Ley us consider the system

$$(u_t^n,\ \Psi_k) + (\nabla u^n,\ \nabla \Psi_k) = -(\varphi^n \sigma(u^n) \nabla \varphi^n + \varphi^n \nabla \varphi_t^n,\ \nabla \Psi_k) \qquad k = 1, \ldots, n \tag{2.40}$$

where $\nabla \varphi^n$ and $\nabla \varphi_t^n$ is known via the functions $g_j^n(t)$.

Since $(u_t^n,\ \Psi_k) = g_k^{n'}(t)$, system (2.40) with the initial conditions $g^n(0) = 0$ has locally one and only one solution. To prove that this solution is defined in $[0,\ T]$ and to pass to the limit for $n \to 0$ we need other «a priori» estimate. Let us multiply the k-th equation of (2.40) by $g_t^n(t)$ and then sum all of the equations over k from 1 to n. From the resultant equality integrated with respect to t we obtain

$$\frac{1}{2}\|u(x,\ t)\|_{2,\Omega}^2 + \|\nabla u\|_{2,Q_t}^2 \leq \sup_{Q_t} |\varphi^n| (\sigma_1 \|\nabla \varphi^n\|_{2,Q_t} + \|\nabla \varphi_t^n\|_{2,Q_t}) \cdot \|\nabla u^n\|_{2,Q_T}.$$

By (2.39), (2.7) and (2.5) we have

$$\|u^n\|_{V_2} \leq C_8. \tag{2.41}$$

This in particular implies that the solutions of (2.40) are defined in $[0,\ T]$ and

$$\|\nabla u^n\|_{2,Q_T} \leq C_9. \tag{2.42}$$

By (2.5), (2.7) we can extract from $\{\varphi^n\}$ a subsequence, still denoted $\{\varphi^n\}$, such that:

$$\nabla \varphi^n \to \nabla \varphi \qquad \text{weakly} \qquad \text{in } L^2(Q_T) \tag{2.43}$$

$$\varphi_t^n \to \varphi_t \qquad \text{weakly} \qquad \text{in } L^2(Q_T) \tag{2.44}$$

$$\nabla \varphi_t^n \to \nabla \varphi_t \qquad \text{weakly} \qquad \text{in } L^2(Q_T) \tag{2.45}$$

$$\varphi^n \to \varphi \qquad \text{strongly} \qquad \text{in } L^2(Q_T). \tag{2.46}$$

By (2.41) and (2.42) we can extract from $\{u^n\}$ a subsequence, still denoted $\{u^n\}$, such that

$$(2.47) \qquad u^n \to u \quad \text{weakly} \quad \text{in } L^2(Q_T)$$

$$(2.48) \qquad \nabla u^n \to \nabla u \quad \text{weakly} \quad \text{in } L^2(Q_T).$$

However from (2.41) and (2.42) does not follow immediately the strong convergence in $L^2(Q_T)$ of $\{u^n\}$ which is needed for letting $n \to \infty$ in (2.40). Therefore we shall use the following version of Rellich's theorem [**1**].

Lemma 2.3. *Let* $v^n(x, t) \in V_2(Q_T)$ *and*

$$(2.49) \qquad \|v^n\|_{V_2(Q_T)} \leqslant C.$$

Suppose

$$(2.50) \qquad v^n(x, t) \to v(x, t) \text{ weakly in } L^2(\Omega) \text{ and uniformly with respect to } t \in [0, T].$$

Then it is possible to extract from $\{v^n\}$ *a subsequence, still denoted* $\{v^n\}$*, such that*

$$(2.51) \qquad v^n \to v \quad \text{strongly} \quad \text{in } L^2(Q_T).$$

We want to apply Lemma 2.3 to the sequence $\{u^n\}$. The a priori estimate (2.49) holds by (2.41). By (2.42) the functions $g^n(t)$ are uniformly bounded.

If we integrate (2.40) in $[t, t+h]$ then use (2.39), (2.42), (2.5) and (2.7) we find

$$|g_j^n(t+h) - g_j^n(t)| \leqslant C_k|h|.$$

Therefore for fixed j and $n \geqslant j$ the $g_j^n(t)$ are equicontinuous in $[0, T]$. By the usual diagonal process we can find a subsequence $g_j^{n_m}(t)$ converging to a continuous function $g_j(t)$ for every $j = 1, 2, \ldots$. Let $v(x) \in L^2(\Omega)$ and $v_k(x)$ be a complete orthonormal system of $L^2(\Omega)$. We have $v(x) = \sum_{k=1}^{\infty} (v, v_k)\, v_k$ and

$$\Big|\int_\Omega [u^{n_m}(x, t) - u(x, t)]\, v(x)\, dx\Big| \leqslant \|u^n - u\|_{2\Omega} \Big(\sum_{k=s+1}^{\infty} (v, v^k)^2\Big)^{1/2} + \Big|\sum_{k=1}^{s} (v, v_k)[g_k^{n_m}(t) - g_k(t)]\Big|.$$

Choosing s sufficiently large and recalling (2.41) the first term in the right hand side becomes less than a preassigned $\varepsilon > 0$. By the uniform convergence of the $g_k^{n_m}(t)$ the second summation can be made less than ε for all $t \in [0, T]$. This proves (2.50). Therefore by Lemma 2.3 we can extract from $\{u^n\}$ a subsequence, still denoted $\{u^n\}$, such that

(2.52) $$u^n \to \text{strongly} \qquad \text{in } L^2(Q_T)$$

(2.53) $$\sigma(u^n) \to \sigma(u) \quad \text{strongly} \qquad \text{in } L^p(Q_T) \qquad 1 \leq p < \infty .$$

We let $n \to \infty$ in (2.2) and (2.3). By (2.43), (2.45) and (2.53) we have

(2.54) $$\varphi = \varphi_0 \quad \text{on} \quad S_T \qquad \varphi(x, 0) = \varphi_0(x, 0) \qquad x \in \Omega$$

(2.55) $$\varphi \in W \qquad \int_{Q_T} \nabla \varphi_t \cdot \nabla w dx\, dt + \int_{Q_T} \sigma(u) \nabla \varphi \cdot \nabla w dx\, dt = 0$$

for all $w \in \dot{W}_2^{1,1}(Q_T)$.

We multiply each equation (2.40) by a regular function $d_j^n(t)$ which vanish for $t = T$. Summing up over all j from 1 to $l \leq n$ and integrating the result with respect to t we get, after an integration by parts,

(2.56) $$-\int_0^T (u^n, \Phi_t^l)\, dt + \int_0^T (\nabla u^n, \nabla \Phi^l)\, dt$$
$$= -\int_0^T (\varphi^n \sigma(u^n) \nabla \varphi^n, \nabla \Phi_t^l)\, dt - \int_0^T (\varphi^n \nabla \varphi_t^n, \nabla \Phi^l)\, dt$$

where $\Phi^l(x, t) = \sum_{j=1}^{l} d_j^n(t) \Psi_j(x)$.

We let $n \to \infty$ in the left hand side of (2.56) for l fixed using (2.47) and (2.48). Write

$$\int_{Q_T} [\varphi^n \sigma(u^n) \nabla \varphi^n - \varphi\sigma(u) \nabla \varphi] \cdot \nabla \Phi^l\, dx\, dt$$

$$= \int_{Q_T} [(\varphi^n - \varphi)\, \sigma(u^n) \nabla \varphi^n + (\sigma(u^n) - \sigma(u))\, \varphi \nabla \varphi^n + \varphi\sigma(u)(\nabla \varphi^n - \nabla \varphi)] \cdot \nabla \Phi^l\, dx\, dt .$$

Using (2.46), (2.53) and (2.44) we get

$$\int_{Q_T} \varphi^n \sigma(u^n) \nabla \varphi^n \cdot \nabla \Phi^l\, dx\, dt \to \int_{Q_t} \varphi\sigma(u) \nabla \varphi \cdot \nabla \Phi^l\, dx\, dt .$$

In a similar way we have

$$\int_{Q_T} \varphi^n \nabla\varphi_t^n \cdot \nabla\Phi^l \, dx \, dt \to \int_{Q_T} \varphi\nabla\varphi_t \cdot \nabla\Phi^l \, dx \, dt$$

by (2.43) and (2.46). Then we arrive at

$$(2.57) \qquad -\int_{Q_T} u\Phi^l \, dx \, dt + \int_{Q_T} \nabla u \cdot \nabla\Phi^l \, dx \, dt = -\int_{Q_T} [\varphi\sigma(u)\nabla\varphi + \varphi\nabla\varphi_t] \cdot \nabla\Phi^l \, dx \, dt \, .$$

The functions Φ^l are dense in the subspace of $\dot{W}_2^{1,1}(Q_T)$ of the functions which vanish on $t = T$. Therefore we obtain (1.8) from (2.57). This completes the proof of the existence of a weak solution for problem (Pb).

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Sommario

Si studia un sistema nonlineare di due equazioni di tipo parabolico. Viene dato un teorema di esistenza di soluzioni per il problema misto in un arbitrario intervallo di tempo.

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