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Models for TAI (**)

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1 - Introduction. Interpretation of the language of TAI

A short presentation of the leading ideas of this paper is in $[2]_2$.

We assume the consistency of ZF. As a consequence of a result of Gödel, also ZF + GCH is consistent, and we place us in this theory in order to construct a model for our TAI presented in $[2]_1$ (1), starting from ideas given in [3].

Let HF the (classical) set of all hereditarily finite (classical) sets, let $\langle \mathbf{V}, \mathscr{E} \rangle$ be the ultrapower of $\langle \mathrm{HF}, \in \rangle$, in the sense of [1], over some non-trivial suitable ultrafilter on ω. By Łos theorem, $\mathbf{V}$ is a model for the axiomatic system $\mathrm{ZF}_{\mathrm{fin}}$, i.e. ZF minus the infinity axiom.

Def. 1. (a) Let M_0 be the set $\mathbf{V}$ and consider $\mathscr{P}(\mathbf{V})$, the (classical) set of all subsets of $\mathbf{V}$. Let $\mathbf{T} = \{X \in \mathscr{P}(\mathbf{V}) | (\exists x \in \mathbf{V})(\forall y \in \mathbf{V})(y \in X \equiv \langle y, x \rangle \in \mathscr{E})\}$. (b) With $\boldsymbol{\emptyset} \in \mathbf{V}$ we denote the equivalence class of the constant function from ω to HF, taking the value $\emptyset$, usually indicated with $\emptyset^*$.

Def. 2. Define, inductively, $M_0 = \mathbf{V}$, $M_1 = (\mathscr{P}(\mathbf{V}) \cup \mathbf{V}) - \mathbf{T} = (\mathscr{P}(M_0) \cup M_0) - \mathbf{T}$, etc., $M_{n+1} = (\mathscr{P}(M_n) \cup M_n) - \mathbf{T}$; and set $\mathbf{M} = \bigcup_{n \in \omega} M_n$.

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(1) Symbols, definitions and so on, introduced in $[2]_1$ are used here without specific mentions.

It can be shown, by induction, that for every $n, m \in \omega$, with $n \leq m$, it is $M_n \subseteq M_m$.

Recall from [2]$_1$ that the language of TAI is a one-sorted, first language (without identity) with the symbols: a constant $\emptyset$, two two-place predicates ε and $\doteqdot$, three one-place predicates V, Set, Cls and an *abstraction operator* $\{\ldots|\ldots\}$ which accepts a variable to the left of the stroke and a formula to the right of it. Elements of $\mathbf{M}$, denoted by lower case Latin letters as x, y, etc. are the intended interpretations of objects.

The interpretations of the specific symbols of the language of TAI is defined as follows:

Def. 3. For $x, y \in \mathbf{M}$, $\emptyset^{\mathbf{M}} = \boldsymbol{\emptyset}$; $x \,\varepsilon^{\mathbf{M}} y$ if and only if $(y \notin \mathbf{V} \wedge x \in y$ or $y \in \mathbf{V} \wedge \langle x, y\rangle \in \mathscr{E})$; $x \doteqdot^{\mathbf{M}} y$ if and only if $x = y$; $\mathsf{V}^{\mathbf{M}}(x)$ if and only if $x \in \mathbf{V}$; $\mathsf{Set}^{\mathbf{M}}(x)$ if and only if ($x \in \mathbf{V}$ or $x \notin \mathbf{V} \wedge \{z \in \mathbf{V} | z \,\varepsilon^{\mathbf{M}} x\} \in \mathbf{T} \wedge \{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\}$ is ZF-finite).

Remark 4. We introduce graphically different symbols to stress the difference between linguistic symbols and set-theoretical ones. If $y \in \mathbf{V}$ and $x \,\varepsilon^{\mathbf{M}} y$, we can conclude $x \in \mathbf{V}$, since $\mathscr{E} \subseteq \mathbf{V} \times \mathbf{V}$. Moreover, if $x \in (M_{n+1} - M_n)$ and $y \in (M_{p+1} - M_p)$ and $x \,\varepsilon^{\mathbf{M}} y$, then we can conclude that $n < p$.

Since $\mathbf{V}$ is a model for $\mathrm{ZF_{fin}}$, for every $x, y \in \mathbf{V}$, then $x = y$ if and only if for every $z \in \mathbf{M}$, $z \,\varepsilon^{\mathbf{M}} x \leftrightarrow z \,\varepsilon^{\mathbf{M}} y$.

Remark that for $x \in (\mathbf{M} - \mathbf{V})$ with $\mathsf{Set}^{\mathbf{M}}(x)$, if $\emptyset = \{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\} = \{z | z \,\varepsilon^{\mathbf{M}} x\} - \mathbf{V}$, then $\{z | z \,\varepsilon^{\mathbf{M}} x\} \subseteq \mathbf{V}$. Hence $\{z \in \mathbf{V} | z \,\varepsilon^{\mathbf{M}} x\} = \{z | z \,\varepsilon^{\mathbf{M}} x\} = \{z | z \in x\} = x$, by Def. 3. Therefore $x \in \mathbf{T}$, in contradiction with the hypothesis $x \in \mathbf{M}$. It follows that for every $x \in \mathbf{M}$ such that $x \notin \mathbf{V} \wedge \mathsf{Set}^{\mathbf{M}}(x)$, $\{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\} \neq \emptyset$.

Finally remark that in case $x \in (\mathbf{M} - \mathbf{V})$ and $y \in x$, then $y \in \mathbf{M}$, since x can be considered as a subset of some M_p, thence $y \in M_p$.

Def. 5. (a) Define for every $x \in \mathbf{M}$, $x^{\smile} = \{y \in \mathbf{M} \mid y \,\varepsilon^{\mathbf{M}} x\}$. (b) Extend the operation $^{\smile}$ to $\mathbf{T}$, setting $x^{\smile} = x$ for every $x \in \mathbf{T}$.

With these notations part of the Def. 3 can be written

$$\mathsf{Set}^{\mathbf{M}}(x) \text{ if and only if } x \in \mathbf{V} \vee (x \notin \mathbf{V} \wedge (x^{\smile} \cap \mathbf{V}) \in \mathbf{T} \wedge (x^{\smile} - \mathbf{V}) \text{ is ZF-finite}).$$

Now we prove the following

Proposition 6. (a) *For every* $x \in \mathbf{V}$, $x^{\smile} \in \mathbf{T}$; *in particular*, $\boldsymbol{\emptyset}^{\smile} = \emptyset$. (b) *For every* $x \in (\mathbf{M} - \mathbf{V})$, $x^{\smile} \in \mathbf{M}$ *and* $x^{\smile} = x$. (c) *For every* $x, y \in \mathbf{M}$, $y \,\varepsilon^{\mathbf{M}} x \equiv y \in x^{\smile}$. (d)

For every $x \in \mathbf{M} \cup \mathbf{T}$, *there is a unique* $y \in \mathbf{M}$ *such that* $x\check{} = y\check{}$. *Moreover if* $x\check{} \subseteq \mathbf{V}$, *then* $y \in M_1$. (e) *For every* x, y, *if* $x \subseteq y$ *and* $y \in (\mathbf{M} - \mathbf{V})$, *then there is* $z \in \mathbf{M}$ *such that* $z\check{} = x$. (f) *For every* $n \in \omega$ *and every* x, $y_1, y_2, \ldots, y_n \in (\mathbf{M} \cup \mathbf{T})$, $x\check{} \cup \{y_1, y_2, \ldots, y_n\} \in \mathbf{T}$ *if and only if* $y_1, y_2, \ldots, y_n \in \mathbf{V}$ *and there is* $z \in \mathbf{V}$ *such that* $x = z\check{}$.

Proof. (a), (b) and (c) are trivial consequences of the Def. 5.

(d) If $x \in M_0$, then $x\check{} \subseteq V$ and $x\check{} \in \mathbf{T}$, by (a); moreover $x\check{} = x\check{}$. If $y \in \mathbf{M}$ is such that $x\check{} = y\check{}$, then when $y \in M_0$, for every $z \in \mathbf{V}$, $z\, \varepsilon^{\mathbf{M}} y$ if and only if $z\, \varepsilon^{\mathbf{M}} y$ and, by Remark 4, it means $x = y$. In case $y \in M_p$, with $p \geq 1$, $y\check{} = y$, therefore $x\check{} = y\check{} = y$, hence $y \in \mathbf{T}$, contradicting the assumption $y \in M_p$. In this way we proved that if $x \in M_0$, the same x is the unique $y \in M_0$ such that $x\check{} = y\check{}$. Let now x be an element of $\mathbf{T}$, it follows $x\check{} = x$. By Def. 1, there is $y \in \mathbf{V}$ such that $y\check{} = x$. The previous considerations imply that this y is unique. Suppose now that $y \in M_p$, with $p \geq 1$, hence, by (b), $x\check{} = x$ and for every $y \in M$, $y = y\check{} = x\check{} = x$, since $y \notin M_0$, as proved before. The previous cosiderations give the proof for the second part of the claim.

(e) If $x \subseteq y$ and $y \in (\mathbf{M} - \mathbf{V})$, then let n be such that $y \in (M_{n+1} - M_n)$; thence $x \subseteq M_n$. There are two cases: $x \nsubseteq \mathbf{V}$ or $x \subseteq \mathbf{V}$. In the first, $x \notin \mathbf{T}$, hence $x \in M_{n+1}$. In the second, it should be $x \in M_1$ or $x \in \mathbf{T}$. By (d), there is a unique $z \in M_1$ such that $z\check{} = x$. Remark that in both cases this z is unique.

(f) Let $x\check{} \cup \{y_1, y_2, \ldots, y_n\} \in \mathbf{T}$, then $x\check{} \cup \{y_1, y_2, \ldots, y_n\} \subseteq \mathbf{V}$, hence $y_1, y_2, \ldots, y_n \in \mathbf{V}$ and $x\check{} \subseteq \mathbf{V}$, therefore $x\check{} \in (\mathbf{M} \cup \mathbf{T})$. By (d), there is a unique $z \in M_1$ such that $x\check{} = z\check{}$. Now if $z \in M_0$, then $x\check{} \in \mathbf{T}$ and the claim is proved. If $z \in (M_1 - M_0)$, then by (b), $z = z\check{} = x\check{}$. Hence $x\check{} \notin \mathbf{T}$; by (a), $x \notin \mathbf{V}$; therefore $x \in (\mathbf{M} - \mathbf{V})$. It follows, by (b), $x = x\check{} = z$. The converse is trivial.

Taking account of Proposition 6 (d) and Def. 1, we can give the following

Def. 7. For $x \in \mathbf{T}$, denote by $[\![x]\!]$ the unique element of $\mathbf{V}$ such that for every $z \in \mathbf{V}$, $z\, \varepsilon^{\mathbf{M}} [\![x]\!] \equiv z \in x$.

Proposition 8. (a) *For every* $x \in \mathbf{V}$, $[\![x\check{}]\!] = x$. (b) *For every* $x \in \mathbf{T}$, $x = [\![x]\!]\check{}$.

Proof. (a) and (b) are trivial.

Proposition 9. *For every* $x \in \mathbf{M}$, $\mathsf{Set}^{\mathbf{M}}(x) \equiv (\exists y \in \mathbf{V})(y^{\smile} \subseteq x^{\smile} \wedge (x^{\smile} - y^{\smile})$ *is* ZF-*finite*).

Proof. Let x be such that $\mathsf{Set}^{\mathbf{M}}(x)$, then $x \in \mathbf{V}$ or $x \notin \mathbf{V}$. In case $x \in \mathbf{V}$, it is $x^{\smile} \subseteq x^{\smile}$ and $(x^{\smile} - x^{\smile}) = \emptyset$, therefore $(x^{\smile} - x^{\smile})$ is ZF-finite. Otherwise $x \notin \mathbf{V} \wedge \{z \in \mathbf{V} | z \,\varepsilon^{\mathbf{M}} x\} \in \mathbf{T} \wedge \{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\}$ is ZF-finite. It is $x^{\smile} = x$. Let $y = [\![\{z \in \mathbf{V} | z \,\varepsilon^{\mathbf{M}} x\}]\!]$; it is $y^{\smile} \subseteq x^{\smile}$; moreover $(x^{\smile} - y^{\smile}) = \{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\}$ is ZF-finite. Conversely, suppose $(\exists y \in \mathbf{V})(y^{\smile} \subseteq x^{\smile} \wedge (x^{\smile} - y^{\smile})$ is ZF-finite). Let $(x^{\smile} - y^{\smile}) = \{z_1, z_2, \ldots, z_n\}$, it is impossible that $\{z_1, z_2, \ldots, z_n\} \subseteq \mathbf{V}$, otherwise $x^{\smile} = y^{\smile} \cup \{z_1, z_2, \ldots, z_n\} \in \mathbf{T}$, and, by Proposition 8 (a), $[\![x^{\smile}]\!] = x$, with $x \in \mathbf{V}$. Hence $(\{z_1, z_2, \ldots, z_n\} - \mathbf{V}) \neq \emptyset$. Consider $[\![y^{\smile} \cup (\{z_1, z_2, \ldots, z_n\} \cap \mathbf{V})]\!]$ it is an element of $\mathbf{V}$ and $\{z \in \mathbf{V} | z \,\varepsilon^{\mathbf{M}} x\} = [\![y^{\smile} \cup (\{z_1, z_2, \ldots, z_n\} \cap \mathbf{V})]\!]^{\smile} \in \mathbf{T}$. Moreover $\{z | z \,\varepsilon^{\mathbf{M}} x \wedge z \notin \mathbf{V}\} = (\{z_1, z_2, \ldots, z_n\} - \mathbf{V})$ and it is ZF-finite, thence, $\mathsf{Set}^{\mathbf{M}}(x)$.

To introduce the interpretations for the predicate Cls and of the abstraction operator $\{\ldots|\ldots\}$, we need some considerations and notations more. In the sequel we say that the object $x \in \mathbf{M}$ is a V-set or a class if $\mathsf{V}^{\mathbf{M}}(x)$ or $\mathsf{Cls}^{\mathbf{M}}(x)$, respectively.

In the Mathematics of the working mathematician, only a finite«degree of complexity» is used. We can identify the degree with a sort of rank, assigning to V-sets the degree 0; to classes (in the sense of Λ-classes, specified later) the degree 1; to ordered pair of classes, the degree 3, ...; but we can choose the degree in a different way. However only a finite degree of complexity is used actually. Call m, the maximum complexity degree considered, *the top*. It is $m \in \omega$ and we can assume $m > 5$. One can consider also more complex specific objects, but only in a finite number. These ideas suggest the following interpretation

$$\mathsf{Cls}^{\mathbf{M}}(x) \quad \text{if and only if} \quad (x^{\smile} - M_m) \quad \text{is} \quad \text{ZF-finite}.$$

Also formulae may have a «complexity degree», e.g. the number of « { » nested, or a type of (Quine's) stratification, for example an increasing function of the characters γ and χ defined below, or a measure of the hierarchy: Σ_n, or Π_n. Denote the degree of the formula φ with $\rho(\varphi)$ and assume that for every φ, $\rho(\varphi) > 4$. We do not specify what kind of degree we assume here, since the construction of the model is independent from the choice we assume. For every choice of the top m and the function ρ, satisfying conditions indicated above, we obtain a model.

Def. 10. For each term τ and each formula φ, define two natural numbers γ and χ as follows: $\gamma(\emptyset) = 0$ and $\chi(\emptyset) = 0$; $\gamma(\Phi) = 0$ and $\chi(\Phi) = 0$, for every variable Φ;

$\gamma(\tau \in \tau') = \gamma(\tau) + \gamma(\tau')$ and $\chi(\tau \in \tau') = \chi(\tau) + \chi(\tau')$; $\gamma(\tau \doteq \tau') = \gamma(\tau) + \gamma(\tau')$ and $\chi(\tau \doteq \tau') = \chi(\tau) + \chi(\tau')$; $\gamma(\mathsf{J}(\tau)) = \gamma(\tau)$ and $\chi(\mathsf{J}(\tau)) = \chi(\tau)$, where $\mathsf{J} \in \{\mathsf{V}, \mathsf{Set}, \mathsf{Cls}\}$; $\gamma(\neg \varphi) = \gamma(\varphi)$ and $\chi(\neg \varphi) = 1 + \chi(\varphi)$; $\gamma(\varphi \otimes \varphi') = \gamma(\varphi) + \gamma(\varphi')$ and $\chi(\varphi \otimes \varphi') = 1 + \chi(\varphi) + \chi(\varphi')$, where $\otimes \in \{\wedge, \vee, \rightarrow, \equiv\}$; $\gamma((Q\,\Psi)\,\varphi) = \gamma(\varphi)$ and $\chi((Q\,\Psi)\,\varphi) = 1 + \chi(\varphi)$, where $Q \in \{\forall, \exists\}$; $\gamma(\{\Phi|\varphi(\Phi)\}) = 1 + \gamma(\varphi(\Phi))$ and $\chi(\{\Phi|\varphi(\Phi)\}) = \chi(\varphi(\Phi))$.

The characters γ and χ are involved in the following

Def. 11. Define, by double induction, for each term τ and each formula φ, the interpretations $\tau^{\mathbf{M}}$ and $\varphi^{\mathbf{M}}$, as follows: $\emptyset^{\mathbf{M}} = \varnothing$; $\Phi^{\mathbf{M}} \in \mathbf{M}$; $(\tau \in \tau')^{\mathbf{M}}$ for $\tau^{\mathbf{M}} \varepsilon^{\mathbf{M}} \tau'^{\mathbf{M}}$; $(\tau \doteq \tau')^{\mathbf{M}}$ for $\tau^{\mathbf{M}} \doteq^{\mathbf{M}} \tau'^{\mathbf{M}}$; $(\mathsf{J}(\tau))^{\mathbf{M}}$ for $\mathsf{J}^{\mathbf{M}}(\tau^{\mathbf{M}})$ where $\mathsf{J} \in \{\mathsf{V}, \mathsf{Set}, \mathsf{Cls}\}$; $(\neg \varphi)^{\mathbf{M}}$ for $\neg (\varphi^{\mathbf{M}})$; $(\varphi \otimes \varphi')^{\mathbf{M}}$; $(\varphi \otimes \varphi')^{\mathbf{M}}$ for $\varphi^{\mathbf{M}} \otimes \varphi'^{\mathbf{M}}$, where $\otimes \in \{\wedge, \vee, \rightarrow, \equiv\}$; $((Q\,\Psi)\,\varphi(\Psi))^{\mathbf{M}}$ for $(Qx)\,\varphi^{\mathbf{M}}(x)$, where $Q \in \{\forall, \exists\}$ and x is not present in $\varphi^{\mathbf{M}}(\Psi^{\mathbf{M}})$; $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}}$ is such that $(\{\Phi|\varphi(\Phi)\}^{\mathbf{M}})\check{} = \{x \in M_{\min(\varrho(\varphi(\Phi)),\,m)}|\varphi^{\mathbf{M}}(x)\}$ and x is not present in $\varphi^{\mathbf{M}}(\Phi^{\mathbf{M}})$.

Remark 12. There can be two logically equivalent formulae, eventually with parameters, $\varphi(\Phi)$ and $\psi(\Phi)$, such that $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}} \neq \{\Phi|\psi(\Phi)\}^{\mathbf{M}}$.

Moreover let $\varphi(\Phi)$ be a formula, eventually with parameters, and let $\{x \in M_{\min(\varrho(\varphi(\Phi)),\,m)}|\varphi^{\mathbf{M}}(x)\} \in \mathbf{T}$, then there exists a unique $y \in \mathbf{V}$ such that $(\forall z)(\langle z, y\rangle \in \mathscr{E} \equiv z \in \{x \in M_{\min(\varrho(\varphi(\Phi)),\,m)}|\varphi^{\mathbf{M}}(x)\})$ and in this case $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}} = y$ and $\{\Phi/\varphi(\Phi)\}^{\mathbf{M}} \neq \{\Phi|\varphi(\Phi)\}$. Otherwise it is $\{x \in M_{\min(\varrho(\varphi(\Phi)),\,m)}|\varphi^{\mathbf{M}}(x)\} = \{\Phi|\varphi(\Phi)\}^{\mathbf{M}}$.

2 - Verifications of the Axioms of TAI

The first Axiom of TAI is

A1. $$(\forall\Phi)(\Phi \doteq \Phi).$$

It holds since equality is a reflexive relation.

The second Axiom

A2. $$(\forall\Phi)(\forall\Psi)((\mathsf{Set}(\Phi) \wedge \mathsf{Set}(\Psi)) \rightarrow (\Phi \doteq \Psi \equiv (\forall\Theta)(\Theta \,\varepsilon\, \Phi \equiv \Theta \,\varepsilon\, \Psi)))$$

requires that two sets are equal if and only if they have the same elements. To prove it, there are many cases: $x,\ y \in \mathbf{V}$, $x \in \mathbf{V}$ and $y \notin \mathbf{V}$, $x,\ y \notin \mathbf{V}$. The first and the third are the only possible ones, since the interpretation of $\doteqdot$ is the equality, thence conclusion easily follows.

The third axiom

A3. $$\mathsf{V}(\emptyset) \wedge (\forall \Phi)(\Phi \notin \emptyset)$$

is trivial, since $\emptyset \in \mathbf{V}$ and $\emptyset^{\smile} = \varnothing$, by Proposition 6 (a).

The Axioms 4 and 5 are schemas

A4. For every set-formula $\varphi(\mathsf{x})$

$$(\varphi(\emptyset) \wedge (\forall \mathsf{x})(\forall \mathsf{y})(\varphi(\mathsf{x}) \rightarrow \varphi(\mathsf{x} \,\%\, \mathsf{y}))) \rightarrow (\forall \mathsf{x})\, \varphi(\mathsf{x}).$$

A5. For every set-formula $\varphi(\mathsf{x})$

$$(\exists \mathsf{x})\, \varphi(\mathsf{x}) \rightarrow (\exists \mathsf{x})(\varphi(\mathsf{x}) \wedge (\forall \mathsf{y})(\mathsf{y} \,\varepsilon\, \mathsf{x} \rightarrow \neg\, \varphi(\mathsf{y})))$$

holding for all hereditarily finite sets. By Łos theorem, they hold for elements of **V** too, as it is proved in [3]. Remark that in formulation of Axiom 4 we use the symbol %, whose definition is a consequence of the Axiom 12.

Recall Axiom

A6. For every formula $\varphi(\Phi)$, eventually with parameters

$$\mathsf{Cls}\,(\{\Phi | \varphi(\Phi)\}) \wedge (\forall \Theta)(\Theta \,\varepsilon\, \{\Phi | \varphi(\Phi)\} \rightarrow \varphi(\Theta)).$$

For every formula $\varphi(\Phi)$ the construction of the object $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}}$ gives an element of M_{m+1}, since $\min(\rho(\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)), m) \leq m$, hence $(\{\Phi|\varphi(\Phi)\}^{\mathbf{M}} - M_m)$ is empty, therefore finite. Moreover if $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}} \notin \mathbf{V}$ and $x \,\varepsilon^{\mathbf{M}} \{\Phi|\varphi(\Phi)\}^{\mathbf{M}}$, then $x \in \{y \in M_{\min(\rho(\varphi(\Phi)), m)} | \varphi^{\mathbf{M}}(y)\}$; hence $\varphi^{\mathbf{M}}(x)$. In case that $\{\Phi|\varphi(\Phi)\}^{\mathbf{M}} \in \mathbf{V}$, by Remark 12, $x \,\varepsilon^{\mathbf{M}} \{\Phi|\varphi(\Phi)\}^{\mathbf{M}}$, implies $\varphi^{\mathbf{M}}(x)$.

Before proving the truth in **M** of the Axiom 7, recall that in [2]$_1$ is given $\Lambda(\Phi)$ for $\mathsf{Cls}\,(\Phi) \wedge (\forall \Psi)(\Psi \,\varepsilon\, \Phi \rightarrow \mathsf{V}(\Psi))$. Then

Proposition 13. *For every* $x \in \mathbf{M}$, (a) $(\forall y)(y \,\varepsilon^{\mathbf{M}} x \rightarrow y \in \mathbf{V})$ *if and only if* $x \in M_1$; (b) $\Lambda^{\mathbf{M}}(x)$ *if and only if* $x \in M_1$.

Proof. (a) Suppose $(\forall y \in \mathbf{M})(y \,\varepsilon^{\mathbf{M}}\, x \rightarrow y \in \mathbf{V})$, then there are two cases: $x \in \mathbf{V}$ or $x \notin \mathbf{V}$. In the first $x \in M_1$, by definition of M_1. In the second, if $y \,\varepsilon^{\mathbf{M}}\, x$ then $y \in x$, thence $x \subseteq \mathbf{V}$. Therefore $x \in M_1$, since $x \in \mathbf{M}$. The converse is trivial by definitions and Remark 4.

(b) The interpretation of the predicate Λ is given by $\mathsf{Cls}^{\mathbf{M}}(x) \wedge \wedge (\forall y)(y \,\varepsilon^{\mathbf{M}}\, x \rightarrow \mathsf{V}^{\mathbf{M}}(y))$, hence $\Lambda^{\mathbf{M}}(x)$ implies $(\forall y \in \mathbf{M})(y \,\varepsilon^{\mathbf{M}}\, x \rightarrow y \in \mathbf{V})$, therefore, by (a), $x \in M_1$. Conversely, by definition of m, $(x - M_m)$ is empty and, of course, ZF-finite i.e. $\mathsf{Cls}^{\mathbf{M}}(x)$ holds. Since $x \in M_1$ there are two cases: $x \in \mathbf{V}$ or $x \subseteq \mathbf{V}$; in both, for every $y \in \mathbf{M}$, if $y \,\varepsilon^{\mathbf{M}}\, x$, then $y \in \mathbf{V}$.

The next Axiom is

A7. $$(\forall \Phi)(\mathsf{V}(\Phi) \equiv \mathsf{Set}(\Phi) \wedge \Lambda(\Phi)).$$

For $x \in \mathbf{V}$, by definitions, $\mathsf{Set}^{\mathbf{M}}(x)$ and $x \in M_1$, and, by Proposition 13 (b), $\Lambda^{\mathbf{M}}(x)$. Let now x be such that $\mathsf{Set}^{\mathbf{M}}(x)$ and $\Lambda^{\mathbf{M}}(x)$. Thence $x \in M_1$ and there are two cases $x \in \mathbf{V}$ or $x \notin \mathbf{V}$. In first case the claim is proved. In the second one, by Remark 4, $\{z | z \,\varepsilon^{\mathbf{M}}\, x \wedge z \notin \mathbf{V}\}$ is non-empty, then there exists $z \in (x - \mathbf{V})$, but this is a contradiction, by Def. 2.

The Axiom 8 is the following

A8. $$(\forall \Phi)(\mathsf{Set}(\Phi) \rightarrow \mathsf{Cls}(\Phi)).$$

To verify it, take $x \in \mathbf{M}$ such that $\mathsf{Set}^{\mathbf{M}}(x)$, then, in case $x \in \mathbf{V}$, $(x - M_m)$ is empty. If $x \notin \mathbf{V} \wedge \{z | z \,\varepsilon^{\mathbf{M}}\, x \wedge z \notin \mathbf{V}\}$, is non-empty and ZF-finite, $(x - M_m)$ is ZF-finite too.

The Axiom 9 does not offer difficulties, since it says that equality on Λ-objects is the true equality on the objecs of M_1, and it is extensional

A9. $$(\forall \mathsf{X})(\forall \mathsf{Z})(\mathsf{X} \doteqdot \mathsf{Z} \equiv (\forall \mathsf{x})(\mathsf{x} \,\varepsilon\, \mathsf{X} \equiv \mathsf{x} \,\varepsilon\, \mathsf{Z})).$$

Axioms 10 and 11, are

A10. $$(\forall \mathsf{F})(\mathsf{Count}(\mathsf{F}) \rightarrow (\exists \mathsf{f})(\mathsf{F} \subseteq \mathsf{f})).$$

A11. $$(\forall \mathsf{X})(\forall \mathsf{Z})((\mathsf{Uncount}(\mathsf{X}) \wedge \mathsf{Uncount}(\mathsf{Z})) \rightarrow \mathsf{X} \cong \mathsf{Z}).$$

They are true in the model, as is proved in [3]; here we omit the proof. In the Axiom 10 it is present the symbol $\subseteq$ which has an (obvious) interpretation in **M**: for $x, y \in \mathbf{M}$, $x \subseteq^{\mathbf{M}} y$ is for $(\forall z \in \mathbf{V})(z \in x^{\smile} \rightarrow z \in y^{\smile})$. We assume also that letters F and f, denote functions, but we shall return to these notions after Axiom 12.

The remaining axioms are not present in [4], but their introduction is justified in $[2]_1$.

Before Axiom 12, we can show the following

Proposition 14. (a) *For every* $x, y \in M_1$, $x \subseteq^{\mathbf{M}} y$ *if and only if* $x^{\smile} \subseteq y^{\smile}$. (b) *For every* $x, y \in \mathbf{M}$, *there is* $z \in \mathbf{M}$ *such that* $z^{\smile} = x^{\smile} \cup \{y\}$ *and for every* $w \in \mathbf{M}$, *it is* $w \,\varepsilon^{\mathbf{M}} z$ *if and only if* $w \,\varepsilon^{\mathbf{M}} x \vee w \doteqdot^{\mathbf{M}} y$.

Proof. (a) Trivial. Remark only that hypothesis $x, y \in M_1$ can't be omitted: if $x, y \in (M_2 - M_1)$ and $x \neq y$, $\{x\} = \{x\}^{\smile} \nsubseteq \{y\}^{\smile} = \{y\}$, but $\{x\} \subseteq^{\mathbf{M}} \{y\}$.

(b) The claim is proved by cases and it is divided in two parts: the existence of a suitable object and the properties of it. Consider $x, y \in \mathbf{V}$, then $x^{\smile} \cup \{y\} \in \mathbf{T}$, and, by Proposition 6 (f) and Def. 7, the claim is satisfied assuming $z = [\![x^{\smile} \cup \{y\}]\!]$. If $x \in \mathbf{V}$ and $y \in (M_n - M_0)$, with $n \geq 1$, then $x^{\smile} \cup \{y\} \subseteq M_n$ but $x^{\smile} \cup \{y\} \notin \mathbf{T}$, otherwise $y \in M_0$; therefore $x^{\smile} \cup \{y\} \in M_{n+1}$ and $(x^{\smile} \cup \{y\})^{\smile} = x^{\smile} \cup \{y\}$, by Proposition 6 (b). If $x \in (M_{n+1} - M_n)$ with $n \geq 1$, and $y \in M_0$, *then* $x^{\smile} = x \subseteq M_n$, and there is $w \in x$ such that $w \notin M_0$; therefore $x \cup \{y\} \subseteq M_n$ and $x^{\smile} \cup \{y\} \notin \mathbf{T}$, hence $x^{\smile} \cup \{y\} \in M_{n+1}$ and $(x^{\smile} \cup \{y\})^{\smile} = x^{\smile} \cup \{y\}$, by Proposition 6 (b). Even if $x \in M_1 - M_0$, hence $x^{\smile} = x$, it is $x^{\smile} \cup \{y\} \notin \mathbf{T}$. Suppose $x^{\smile} \cup \{y\} \in \mathbf{T}$. By Proposition 6 (f), $y \in \mathbf{V}$ and there is $u \in \mathbf{V}$ such that $x = u^{\smile}$. By Proposition 6 (a), $x \in \mathbf{T}$, contradiction. In each case we can choose $z \in \mathbf{M}$ such that $w \in z^{\smile}$ if and only if $w \in x^{\smile} \vee w \in \{y\}$, hence $w \,\varepsilon^{\mathbf{M}} z$ if and only if $w \,\varepsilon^{\mathbf{M}} x \vee w \doteqdot^{\mathbf{M}} y$, by Def. 5.

A12. $(\forall \Phi)(\forall \Psi)(\mathrm{Set}(\Phi) \rightarrow (\exists \Sigma)(\mathrm{Set}(\Sigma) \wedge (\forall \Theta)(\Theta \,\varepsilon\, \Sigma \equiv (\Theta \,\varepsilon\, \Phi \vee \Theta \doteqdot \Phi))))$.

This axiom states the existence of the successor of a given set, indicated with operation %. The previous proposition gives part of the verification of the axiom. It remains to prove that the object we obtain is a set. But this is easy, considering Propositions 9 and 14 (b).

Remark 15. As a consequence of the previous axiom, objects such as singletons, or pairs can be constructed, in the sense that there are in **M** objects as $\{x\}^{\mathrm{M}}$, $\{x, y\}^{\mathrm{M}}$. After Axiom 12 we can use also ordered pairs, defined in Kuratowski's style: $\langle x, y\rangle^{\mathrm{M}}$ is the object $\{\{x\}^{\mathrm{M}}, \{x, y\}^{\mathrm{M}}\}^{\mathrm{M}}$. The fundamental properties of these objects hold in **M** as a consequence of the truth of the Axiom 12 in **M**; in particular $z \varepsilon^{\mathrm{M}} \{x\}^{\mathrm{M}}$ if and only if $z \doteqdot^{\mathrm{M}} x$; $z \varepsilon^{\mathrm{M}} \{x, y\}^{\mathrm{M}}$ if and only if $z \doteqdot^{\mathrm{M}} x$ or $z \doteqdot^{\mathrm{M}} y$ and for ordered pairs: $\langle x, y\rangle^{\mathrm{M}} \doteqdot^{\mathrm{M}} \langle x', y'\rangle^{\mathrm{M}}$ if and only if $x \doteqdot^{\mathrm{M}} x'$ and $y \doteqdot^{\mathrm{M}} y'$, hold in the model.

Some other properties of singletons, pairs and ordered pairs are presented in the following

Proposition 16. (a) *For every* $x \in \mathbf{M}$, $(\{x\}^{\mathrm{M}})^{\check{}} = \{x\}$. *Moreover, for every* $x, z \in \mathbf{V}$, $x \varepsilon^{\mathrm{M}} z$ *if and only if* $\{x\}^{\mathrm{M}} \subseteq^{\mathrm{M}} z$. (b) *For every* $x, y \in \mathbf{M}$, $(\{x, y\}^{\mathrm{M}})^{\check{}} = \{x, y\}$. *Moreover, for every* $x, y, z \in \mathbf{V}$, $x \varepsilon^{\mathrm{M}} z \wedge y \varepsilon^{\mathrm{M}} z$ *if and only if* $\{x, y\}^{\mathrm{M}} \subseteq^{\mathrm{M}} z$. (c) *For every* $x, y \in \mathbf{M}$, $\{x\}^{\mathrm{M}} = \{x\}$ *if and only if* $x \notin M_0$; $\{x, y\}^{\mathrm{M}} = \{x, y\}$ *if and only if* $x \notin M_0 \vee y \notin M_0$; $\langle x, y\rangle^{\mathrm{M}} = \langle x, y\rangle$ *if and only if* $x \notin M_0$. (d) *There is* $x \in \mathbf{M}$ *such that* $\{x\}^{\mathrm{M}} \neq \{y | y \doteqdot^{\mathrm{M}} x\}^{\mathrm{M}}$. (e) *For every* $x, y \in \mathbf{M}$, $\{x\}^{\mathrm{M}}, \{x, y\}^{\mathrm{M}} \in M_0$ *if and only if* $x, y \in M_0$; $\{x\}^{\mathrm{M}} \in (M_{i+2} - M_{i+1})$ *if and only if* $x \in M_{i+1} - M_i$; $\{x, y\}^{\mathrm{M}} \in (M_{i+2} - M_{i+1})$ *if and only if* $\{x, y\} \subseteq M_{i+1}$ *and* $\{x, y\} \nsubseteq M_i$. (f) *For every* $x, y \in \mathbf{M}$, $\langle x, y\rangle^{\mathrm{M}} \in M_0$ *if and only if* $x, y \in M_0$; *otherwise* $\langle x, y\rangle^{\mathrm{M}} \in (\mathbf{M} - M_2)$ *if and only if* $x \in (\mathbf{M} - M_0)$ *or* $y \in (\mathbf{M} - M_0)$. *In particular* $\langle x, y\rangle^{\mathrm{M}} \in (M_3 - M_2)$ *if and only if* $\{x, y\} \subseteq M_1$ *and* $\{x, y\} \nsubseteq M_0$. (g) *For every* $x \in \mathbf{M}$, $\mathsf{Set}^{\mathrm{M}}(\{x\}^{\mathrm{M}})$.

Proof. (a)-(b) The objects $\{x\}^{\mathrm{M}}$ and $\{x, y\}^{\mathrm{M}}$, respectively, are such that $(\{x\}^{\mathrm{M}})^{\check{}} = \emptyset^{\check{}} \cup \{x\}$ and $(\{x, y\}^{\mathrm{M}})^{\check{}} = (\{x\}^{\mathrm{M}})^{\check{}} \cup \{y\}$. The remaining part follows from Def. 5 and Proposition 14.

(c) The claim is proved by (a), (b) and Proposition 6 (b).

(d) Take $x \in (M_{m+1} - M_m)$, then $\{x\}^{\mathrm{M}} = \{x\}$, but $\{y | y \doteqdot^{\mathrm{M}} x\}^{\mathrm{M}} = \emptyset$, since there are no elements of $M_{\min(\rho(\Phi \doteqdot \Psi), m)}$, equal to x, since $\min(\rho(\Phi \doteqdot \Psi), m) \leq m$.

(e) By parts (a) and (c), above, and Propositions 6 (a) and 6 (b), the claim regarding singletons and pairs is trivial.

(f) The first claim is a trivial consequence of (e). Suppose that $\langle x, y\rangle^{\mathrm{M}} \in (M_{i+1} - M_i)$ with $i \in \{0, 1\}$, then by (e), $\{x\}^{\mathrm{M}} \in (M_i - M_{i-1})$ or $\{x, y\}^{\mathrm{M}} \in (M_i - M_{i-1})$ or both. Therefore $i = 1$. By new application of point (e), we get $x, y \in M_0$, hence $\langle x, y\rangle^{\mathrm{M}} \in M_0$, contradiction. Therefore if $\langle x, y\rangle^{\mathrm{M}} \notin M_0$, then $\langle x, y\rangle^{\mathrm{M}} \in (\mathbf{M} - M_2)$ if and only if $x \in (\mathbf{M} - M_0)$ or $y \in (\mathbf{M} - M_0)$ or both. The previous considerations prove the last claim.

(g) Trivial, by Def. 3.

Remark 17. For $x, y \in \mathbf{V}$, the sets $\{x\}$, $\{\{x\}\}$, $\{\{\{x\}\}\}$, ..., $\{x, y\}$, $\{\{x, y\}\}$, $\{\{x\}, \{x, y\}\}$ (i.e. $\langle x, y\rangle$) do not belong to $\mathbf{M}$.

Def. 18. For $x \in \mathbf{M}$, define $x^{\S} = \{\langle u, v\rangle \in \mathbf{M} | \langle u, v\rangle^{\mathbf{M}} \varepsilon^{\mathbf{M}} x\}$.

The following three axioms are trivially verified in the model:

A13. $$(\forall \Phi)(\forall \Psi)(\Phi \doteqdot \Psi) \to (\mathsf{Set}(\Psi) \equiv \mathsf{Set}(\Phi))$$

A14. $$(\forall \Phi)(\forall \Psi)(\Phi \doteqdot \Psi) \to (\mathsf{Cls}(\Psi) \equiv \mathsf{Cls}(\Phi))$$

A15. $$(\forall \Phi)(\forall \Psi)(\Phi \doteqdot \Psi) \to (\Lambda(\Psi) \equiv \Lambda(\Phi))$$

since they require that interpretation of the equality $\doteqdot$ be substitutive on the predicates Set, Cls and Λ.

In the Axioms 17 and 18, the predicate $\mathscr{F}$ is used; here we recall from $[2]_1$, that $\mathscr{F}(\Phi)$, to be read Φ is Fregean, is $\mathsf{V}(\Phi) \vee \Lambda(\Phi) \vee (\exists \Psi, \Sigma)(\Lambda(\Psi) \wedge \wedge \Lambda(\Sigma) \wedge \Phi \doteqdot \langle \Psi, \Sigma\rangle)$. Before proving them let us see which relations hold for elements of $\mathbf{M}$ satisfying the interpretation of the predicate $\mathscr{F}$. Some other properties on operator § are collected together in the following

Proposition 19. (a) *For every* $x \in \mathbf{M}$, $\mathscr{F}^{\mathbf{M}}(x)$ *implies* $x \in M_3$. (b) *For every* $z \in \mathbf{M}$, $\mathrm{Rel}(z^{\S})$ *and if* $z^{\S} \neq \emptyset$, *then* $z^{\S} \in (\mathbf{M} - M_3)$. (c) *For every* $z \in \mathbf{M}$, $z = z^{\S}$ *if and only if* $\mathrm{Rel}(z)$ *and* $z \in (\mathbf{M} - M_3)$. (d) *For every* $z \in \mathbf{M}$, $(z^{\S})^{\smile} = z^{\S}$ *and, if* $z^{\S} \neq \emptyset$, *then* $(z^{\S})^{\S} = z^{\S}$.

Proof. (a) For every $x \in \mathbf{M}$, $\mathscr{F}^{\mathbf{M}}(x)$ is $\mathsf{V}^{\mathbf{M}}(x) \vee \Lambda^{\mathbf{M}}(x) \vee (\exists y, z \in \mathbf{M})(\Lambda^{\mathbf{M}}(y) \wedge \wedge \Lambda^{\mathbf{M}}(z) \wedge x \doteqdot^{\mathbf{M}} \langle y, z\rangle^{\mathbf{M}})$. By Def. 5 and Proposition 13 (b), $\mathscr{F}^{\mathbf{M}}(x)$ can be written as $x \in \mathbf{V} \vee x \in M_1 \vee (\exists y, z \in \mathbf{M})(y \in M_1 \wedge z \in M_1 \wedge x = \langle y, z\rangle^{\mathbf{M}})$. By Proposition 16 (f), if $y, z \in \mathbf{V}$, then $\langle y, z\rangle^{\mathbf{M}} \in M_0$, otherwise, $\langle y, z\rangle^{\mathbf{M}} \in M_3$. Therefore $x \in M_3$.

(b) Let $z \in \mathbf{M}$ belong to M_i with $i = 0, 1, 2, 3$, then $z^{\S} = \emptyset$, by Proposition 16 (f); hence, trivially, $\mathrm{Rel}(z^{\S})$. Suppose $z^{\S} \neq \emptyset$; then $\mathrm{Rel}(z^{\S})$, by definition. Moreover there is $\langle x, y\rangle \in z^{\S}$, that means $\langle x, y\rangle \in \mathbf{M}$ and $\langle x, y\rangle^{\mathbf{M}} \in z^{\smile}$, by Def. 18. Therefore, by Remark 17 and Proposition 16 (f), $\langle x, y\rangle \notin M_2$ and $x \notin M_0$ and $y \notin M_0$, then $\langle x, y\rangle^{\mathbf{M}} = \langle x, y\rangle$, by Proposition 16 (c). It follows $z \notin M_0$, hence $\langle x, y\rangle \in z$, therefore $z^{\S} \subseteq z$. By Proposition 6 (e) there exists $u \in \mathbf{M}$, such that $u^{\smile} = z^{\S}$. In this case we can conclude $z^{\S} \nsubseteq \mathbf{V}$, hence $u \notin M_0$. By Proposition 6 (b), $u = u^{\smile} = z^{\S}$, therefore $z^{\S} \in (\mathbf{M} - M_3)$ since its elements do not belong to M_2.

(c) Let $z \in \mathbf{M}$, if $z \in M_i$, with $i = 0, 1, 2, 3$, then as stated before, $z^{\S} = \emptyset$ and $z \in \mathbf{T}$, hence $z \neq z^{\S}$. Therefore if $z = z^{\S}$, it follows $z \in (\mathbf{M} - M_3)$. Trivially if $z = z^{\S}$, it is $\mathrm{Rel}(z)$. Conversely, suppose $\mathrm{Rel}(z)$ and $z \in (\mathbf{M} - M_3)$. For every $\langle x, y\rangle \in \mathbf{M}$, by Proposition 16 (f) and Remark 17, $x \in (\mathbf{M} - M_0)$ and $y \in (\mathbf{M} - M_0)$, therefore, by Proposition 16 (c), $\langle x, y\rangle^{\mathbf{M}} = \langle x, y\rangle$. By Def. 3, $\langle x, y\rangle \, \varepsilon^{\mathbf{M}} z$ if and only if $\langle x, y\rangle \in z$, hence $z = z^{\S}$.

(d) If $z^{\S} = \emptyset$, then $z^{\S} \in \mathbf{T}$ and by Def. 5 (b), $(z^{\S})^{\smile} = z^{\S}$. If $z^{\S} \neq \emptyset$, then by (b) and Proposition 6 (b), $(z^{\S})^{\smile} = z^{\S}$. The remaining part of the claim is trivial consequence of (c).

The next Axiom is

A16. For every formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$,

$$(\forall\Theta)(\varphi(\Theta, \Psi_1, \Psi_2, \ldots, \Psi_n) \wedge \mathscr{F}(\Theta) \rightarrow \Theta \, \varepsilon \, \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\}).$$

For the formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$, let $p = \min(\rho(\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)), m)$, it is $p > 4$. Therefore if $x, y_1, \ldots, y_n \in \mathbf{M}$ and $\varphi^{\mathbf{M}}(x, y_1, y_2, \ldots, y_n) \wedge \mathscr{F}^{\mathbf{M}}(x)$, it is $x \in M_p$, by Proposition 19 (a), and $\varphi^{\mathbf{M}}(x, y_1, y_2, \ldots, y_n)$, thence $x \in \{z \in M_p | \varphi^{\mathbf{M}}(z, y_1, y_2, \ldots, y_n)\}$. By Remark 12 and Def. 5, $x \, \varepsilon^{\mathbf{M}} \{z \in M_p | \varphi^{\mathbf{M}}(z, y_1, y_2, \ldots, y_n)\}^{\mathbf{M}}$.

Axiom 17 states a substitutivity for Fregean objects.

A17. For every formula $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$

$$(\forall\Theta)(\forall\Psi)((\Theta \doteqdot \Psi \wedge \mathscr{F}(\Theta) \wedge \Theta \, \varepsilon \, \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\}) \rightarrow \Psi \, \varepsilon \, \{\Phi | \varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)\})).$$

Let $\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)$ be a formula, and $p = \min(\rho(\varphi(\Phi, \Psi_1, \Psi_2, \ldots, \Psi_n)), m)$; it is $p > 4$. Consider $x, y, y_1, \ldots, y_n \in \mathbf{M}$ such that $x \doteqdot^{\mathbf{M}} y \wedge \mathscr{F}^{\mathbf{M}}(x) \wedge \wedge x \, \varepsilon^{\mathbf{M}} \{z \in M_p | \varphi^{\mathbf{M}}(z, y_1, y_2, \ldots, y_n)\}^{\mathbf{M}}$, by Def. 3, Remark 12 and Proposition 19 (a), it is $x = y \wedge x \in M_3 \wedge \varphi^{\mathbf{M}}(x, y_1, y_2, \ldots, y_n)$. Thence $y \in \{z \in M_p | \varphi^{\mathbf{M}}(z, y_1, y_2, \ldots, y_n)\}$; therefore, $y \, \varepsilon^{\mathbf{M}} \{z \in M_p | \varphi^{\mathbf{M}}(z, y_1, y_2, \ldots, y_n)\}^{\mathbf{M}}$.

From the truth of the Axioms 1-17 we have

Theorem 20. *All the theorems proved in Sections 1-4 of the Chapter I of* [4] *are true in* $\mathbf{M}$.

The last axiom is connected with the Axiom of Choice. To verify it we need some remarks about symbols used in it and their (obvious) interpretations in $\mathbf{M}$.

Def. 21. Given $z \in \mathbf{M}$, define:

(a) for every $x \in \mathbf{V}$, $(\underset{z}{\downarrow}{}^{\mathrm{M}} x)^{\check{}} = \{y \in \mathbf{V} | \langle y, x\rangle^{\mathrm{M}} \in z^{\check{}}\}$;

(b) $((z^*)^{\mathrm{M}})^{\check{}} = \{y \in \mathbf{M} | (\exists x \in \mathbf{V})(y \doteqdot^{\mathrm{M}} \langle x, \underset{z}{\downarrow}{}^{\mathrm{M}} x\rangle^{\mathrm{M}})\}$;

(c) $((z^{-1})^{\mathrm{M}})^{\check{}} = \{w \in \mathbf{M} | (\exists x, y \in \mathbf{M})(w \doteqdot^{\mathrm{M}} \langle x, y\rangle^{\mathrm{M}} \wedge \langle y, x\rangle^{\mathrm{M}} \in z^{\check{}})\}$;

(d) $(\mathsf{dom}^{\mathrm{M}}(z))^{\check{}} = \{x \in \mathbf{M} | (\exists y \in \mathbf{M})(\langle x, y\rangle^{\mathrm{M}} \in z^{\check{}})\}$;

(e) $(\mathsf{rng}^{\mathrm{M}}(z))^{\check{}} = \{x \in \mathbf{M} | (\exists y \in \mathbf{M})(\langle y, x\rangle^{\mathrm{M}} \in z^{\check{}})\}$;

(f) $\mathsf{Rel}^{\mathrm{M}}(z)$ if and only if $(\forall w \in z^{\check{}})(\exists x, y \in \mathbf{M})(w \doteqdot^{\mathrm{M}} \langle x, y\rangle^{\mathrm{M}})$;

(g) $\mathsf{Fnc}^{\mathrm{M}}(z)$ for $\mathsf{Rel}^{\mathrm{M}}(z) \wedge (\forall x, y, w \in \mathbf{M})(\langle x, y\rangle^{\mathrm{M}} \in z^{\check{}} \wedge \langle x, w\rangle^{\mathrm{M}} \in z^{\check{}} \to y \doteqdot^{\mathrm{M}} w)$;

(h) for every $x, y \in \mathbf{M}$, $x \Subset^{\mathrm{M}} y$ if and only if $(\forall w \in \mathbf{M})(w \in x^{\check{}} \to w \in y^{\check{}})$.

Proposition 22. (a) *For every* $z \in M_1$, *and every* $x \in \mathbf{V}$, *it is* $(\underset{z}{\downarrow}{}^{\mathrm{M}} x) \in M_1$ *and* $\underset{z}{\downarrow}{}^{\mathrm{M}} x = \{y \in \mathbf{V} | \langle y, x\rangle \in z^{\S}\}^{\mathrm{M}}$. *If* $z \in M_0$, *then* $(\underset{z}{\downarrow}{}^{\mathrm{M}} x) \in M_0$. (b) *For every* $z \in \mathbf{M}$, $\mathsf{Rel}^{\mathrm{M}}(z)$ *implies* $z \in M_0$ *or* $z \in (M_1 - M_0)$, *or* $z \in (\mathbf{M} - M_3)$. (c) *For every* $z \in \mathbf{M}$, $\mathsf{Rel}^{\mathrm{M}}((z^{-1})^{\mathrm{M}})$. *If* $z \in M_n$, *then* $((z^{-1})^{\mathrm{M}}) \in M_p$ *with* $p \leq n$, *hence* $((z^{-1})^{\mathrm{M}}) \in \mathbf{M}$. *In particular if* $0 \leq n \leq 3$, *then* $p = 0$ *and if* $z \in (M_n - M_3)$, *with* $n > 3$, *then* $3 < p \leq n$. *Moreover* $((z^{-1})^{\mathrm{M}})^{\S} = (z^{\S})^{-1} = \{\langle x, y\rangle \in \mathbf{M} | \langle y, x\rangle \in z^{\S}\}$. (d) *For every* $z \in M_n$, $(\mathsf{dom}^{\mathrm{M}}(z))$, $(\mathsf{rng}^{\mathrm{M}}(z)) \in M_p$, *with* $p \leq n$. *In particular if* $0 \leq n \leq 3$, *then* $p = 0$ *and if* $z \in (M_n - M_3)$, *with* $n > 3$, *then* $0 < p \leq n - 2$. *Moreover if* $(\mathsf{dom}^{\mathrm{M}}(z)) \notin M_1$ (*resp.* $(\mathsf{rng}^{\mathrm{M}}(z)) \notin M_1$), *then* $\mathrm{dom}\,(z^{\S}) = (\mathsf{dom}^{\mathrm{M}}(z))^{\check{}}$ (*resp.* $\mathrm{rng}\,(z^{\S}) = (\mathsf{rng}^{\mathrm{M}}(z))^{\check{}}$). (e) *For every* $z \in \mathbf{M}$, *it is* $((z^*)^{\mathrm{M}}) \in M_1$ *or* $((z^*)^{\mathrm{M}}) \in (M_n - M_3)$, *with* $n > 3$, *moreover* $\mathsf{Fnc}^{\mathrm{M}}((z^*)^{\mathrm{M}})$ *and* $(\forall y \in ((z^*)^{\mathrm{M}})^{\check{}})(\mathscr{F}^{\mathrm{M}}(y))$. (f) *For every* $x, y \in \mathbf{M}$, $x \Subset^{\mathrm{M}} y$ *if and only if* $x^{\check{}} \subseteq y^{\check{}}$; *in particular* $x \,\varepsilon^{\mathrm{M}} y$, *if and only if* $\{x\}^{\mathrm{M}} \Subset^{\mathrm{M}} y$.

Proof. (a) Take $z \in M_1$, then $z^{\check{}} \subseteq \mathbf{V}$. By Proposition 16 (f), for every $\langle y, x\rangle^{\mathrm{M}} \in z$, it is $y, x \in \mathbf{V}$. It follows that $(\underset{z}{\downarrow}{}^{\mathrm{M}} x)^{\check{}} \subseteq \mathbf{V}$. By Proposition 6 (d), there is a unique $v \in M_1$ such that $v^{\check{}} = (\underset{z}{\downarrow}{}^{\mathrm{M}} x)^{\check{}}$. In case $z \in M_0$, fixed x, the set

$u=\{w\,\varepsilon^{\mathbf{M}} z|(\exists x,\ y\in\mathbf{M})(w \doteq^{\mathbf{M}} \langle y,\ x\rangle^{\mathbf{M}}\varepsilon^{\mathbf{M}} z\}\subseteq\mathbf{V}$ and, by comprehension schema, that holds in $\mathbf{V}$, by Łos theorem, $u\in\mathbf{T}$, therefore $[\![\{w\,\varepsilon^{\mathbf{M}} z|(\exists x,\ y\in\mathbf{M})(w \doteq^{\mathbf{M}} \langle y,\ x\rangle^{\mathbf{M}}\varepsilon^{\mathbf{M}} z\}]\!]\in\mathbf{V}$. By the replacement schema for $\mathbf{V}$-sets, whose validity in $\mathbf{M}$ is guaranted by Theorem 20, we get $(\underset{z}{\downarrow}{}^{\mathbf{M}} x)\in M_0$.

(b) Let $z\in\mathbf{M}$ be such that $\mathsf{Rel}^{\mathbf{M}}(z)$. Since elements of $z^{\smile}$ are the «ordered pairs» $\langle x,\ y\rangle^{\mathbf{M}}$, by Proposition 16 (f), $\langle x,\ y\rangle^{\mathbf{M}}\in\mathbf{M}$ or $\langle x,\ y\rangle^{\mathbf{M}}\in(M_p-M_2)$, with $p>2$; hence $z^{\smile}\subseteq\mathbf{V}$ or $z^{\smile}\subseteq M_p$. It follows that $z\in M_0$ or $z\in(M_1-M_0)$ or $z\in(\mathbf{M}-M_3)$.

(c) Let $z\in M_n$ and let $w\in((z^{-1})^{\mathbf{M}})^{\smile}$, then there are $x,\ y\in\mathbf{M}$ such that $w\doteq^{\mathbf{M}}\langle y,\ x\rangle^{\mathbf{M}}$ with $\langle x,\ y\rangle^{\mathbf{M}}\in z^{\smile}$; thence $((z^{-1})^{\mathbf{M}})^{\smile}\subseteq\mathbf{M}$; more precisely, $((z^{-1})^{\mathbf{M}})^{\smile}\subseteq M_n$. Only two cases are possible: $((z^{-1})^{\mathbf{M}})^{\smile}\in\mathbf{T}$, or $((z^{-1})^{\mathbf{M}})^{\smile}\notin\mathbf{T}$, in both we conclude, by Proposition 6 (d) and Def. 2, $((z^{-1})^{\mathbf{M}})\in\mathbf{M}$. Therefore it is $\mathsf{Rel}^{\mathbf{M}}((z^{-1})^{\mathbf{M}})$. By (b), $((z^{-1})^{\mathbf{M}})\in M_0$, or $((z^{-1})^{\mathbf{M}})\in(M_1-M_0)$, or $((z^{-1})^{\mathbf{M}})\in(\mathbf{M}-M_3)$. If $((z^{-1})^{\mathbf{M}})\in M_0$, then $p=0\le n$. If $((z^{-1})^{\mathbf{M}})\in(M_1-M_0)$, it is $n>0$, otherwise $\{w\,\varepsilon^{\mathbf{M}} z|(\exists x,\ y\in\mathbf{M})(w\doteq^{\mathbf{M}}\langle x,\ y\rangle^{\mathbf{M}}\varepsilon^{\mathbf{M}} z\}\in\mathbf{T}$, by comprehension schema, that holds in $\mathbf{V}$, by Łos theorem, and by the replacement schema for V-sets, whose validity in $\mathbf{M}$ is guaranted by Theorem 20. We get $((z^{-1})^{\mathbf{M}})\in M_0$. Therefore $n\ge 1$. In case $((z^{-1})^{\mathbf{M}})\in(M_p-M_3)$, with $p>3$, there exist $x,\ y\in\mathbf{M}$, such that $\langle y,\ x\rangle\in((z^{-1})^{\mathbf{M}})$ with $\langle y,\ x\rangle\in(M_{p-1}-M_2)$. In this case, by Proposition 16 (f), $x\in(M_{p-3}-M_0)$ or $y\in(M_{p-3}-M_0)$. It follows that the element $\langle x,\ y\rangle^{\mathbf{M}}\in z^{\smile}$ is in $(M_{p-1}-M_2)$, therefore $3<p\le n$. Now take $z\in\mathbf{M}$; if $((z^{-1})^{\mathbf{M}})\in M_1$, then $((z^{-1})^{\mathbf{M}})^{\S}=\varnothing$ and $z^{\S}=\varnothing=(z^{\S})^{-1}$, since if there exists $\langle u,\ v\rangle\in\mathbf{M}$ with $\langle u,\ v\rangle\in z^{\smile}$, then $u\notin M_0$, $v\notin M_0$ and $\langle u,\ v\rangle^{\mathbf{M}}=\langle u,\ v\rangle$. Therefore $\langle v,\ u\rangle\in((z^{-1})^{\mathbf{M}})$ and $\langle v,\ u\rangle\notin M_2$, thence $((z^{-1})^{\mathbf{M}})^{\S}=(z^{\S})^{-1}=\{\langle x,\ y\rangle\in\mathbf{M}|\langle y,\ x\rangle\in z^{\S}\}=\varnothing$. Let now z be such that $((z^{-1})^{\mathbf{M}})\notin M_3$, then by Proposition 19 (c), $((z^{-1})^{\mathbf{M}})^{\S}=((z^{-1})^{\mathbf{M}})$. Moreover $z^{\S}\neq\varnothing$, otherwise $((z^{-1})^{\mathbf{M}})\in M_1$, hence $\langle x,\ y\rangle\in((z^{-1})^{\mathbf{M}})$ if and only if $\langle y,\ x\rangle\in z^{\S}$, therefore $((z^{-1})^{\mathbf{M}})^{\S}=(z^{\S})^{-1}=\{\langle x,\ y\rangle\in\mathbf{M}|\langle y,\ x\rangle\in z^{\S}\}$.

(d) Let $x\in(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}$, then $x\in\mathbf{M}$ and there is $y\in\mathbf{M}$ such that $\langle x,\ y\rangle^{\mathbf{M}}\in z^{\smile}$. It follows, by Proposition 16 (f), $\langle x,\ y\rangle^{\mathbf{M}}\in M_0$ or $\langle x,\ y\rangle^{\mathbf{M}}\notin M_2$, therefore $x\in M_0$ or $x\notin M_0$. Hence $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}\subseteq\mathbf{V}$ or $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}\nsubseteq\mathbf{V}$. In the first case $\mathsf{dom}^{\mathbf{M}}(z)\in \mathrm{M}_1$, by Proposition 6 (d). If $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}\nsubseteq\mathbf{V}$, then there is $x\in((\mathsf{dom}^{\mathbf{M}}(z))^{\smile}-\mathbf{V})$ and there is $y\in\mathbf{M}$ such that $\langle x,\ y\rangle^{\mathbf{M}}\in z^{\smile}$, hence $\langle x,\ y\rangle^{\mathbf{M}}\in(M_p-M_2)$, with $p>2$, therefore $z\in(M_{p+1}-M_3)$. In this case, $n\ge p+1$ and for every «ordered pair» $\langle x,\ y\rangle^{\mathbf{M}}$, $x\in M_{n-3}$, hence $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}\subseteq \mathrm{M}_{n-3}$ and $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile}\notin\mathbf{T}$. It follows that $\mathsf{dom}^{\mathbf{M}}(z)\in M_{n-2}$. Suppose $\mathsf{dom}^{\mathbf{M}}(z)\notin M_1$, then $x\in\mathsf{dom}^{\mathbf{M}}(z)$ if and only if there exists y such that $\langle x,\ y\rangle^{\mathbf{M}}\in z^{\smile}$; but $x\notin M_0$ and by Proposition 16 (c), $\langle x,\ y\rangle^{\mathbf{M}}=\langle x,\ y\rangle$.

By Proposition 6 (b), $z^{\smile} = z$, thence $\langle x, y\rangle \in z$ and $x \in \mathrm{dom}(z)$, therefore $\mathsf{dom}^{\mathbf{M}}(z) = \mathrm{dom}(z)$. The proof for $\mathsf{rng}^{\mathbf{M}}(z)$ is very similar.

(e) Let z be such that for every $x \in \mathbf{V}$, $(\underset{z}{\downarrow}^{\mathbf{M}} x) \in M_0$, hence $((z^*)^{\mathbf{M}})^{\smile} \subseteq \mathbf{V}$. It follows by Proposition 6 (d) that $((z^*)^{\mathbf{M}}) \in M_1$. If there is an x such that $(\underset{z}{\downarrow}^{\mathbf{M}} x) \in M_1$, then the element $\langle x, \underset{z}{\downarrow}^{\mathbf{M}} x\rangle^{\mathbf{M}} \in M_3$. In this case $((z^*)^{\mathbf{M}})^{\smile} \subseteq M_3$ and $((z^*)^{\mathbf{M}}) \notin \mathbf{T}$, hence $((z^*)^{\mathbf{M}}) \in (M_n - M_3)$, with $n > 3$; moreover it is $\mathsf{Fnc}^{\mathbf{M}}((z^*)^{\mathbf{M}})$. The remaining part is trivial by definition.

(f) Trivial.

All instruments needed for verification of Axiom 18 are now ready.

A18. $$(\forall \mathsf{R})((\mathsf{Rel}(\mathsf{R}) \wedge \Lambda(\mathsf{R})) \rightarrow (\exists \Phi)(\mathsf{Cls}(\Phi) \wedge \mathsf{Fnc}(\Phi) \wedge \mathsf{dom}(\Phi) \doteqdot \mathsf{dom}((\mathsf{R}^*)^{-1} \wedge \Phi \Subset (\mathsf{R}^*)^{-1}))).$$

Let $z \in \mathbf{M}$ be such that $\mathsf{Rel}^{\mathbf{M}}(z) \wedge \Lambda^{\mathbf{M}}(z)$. By Proposition 13 (b) and 22 (b), for every $x \in M_1$, it is $(\underset{z}{\downarrow}^{\mathbf{M}} x) \in M_1$. Consider now the relation (that is out of the model $\mathbf{M}$): $\mathrm{S} = \{\langle u, w\rangle \in M_1 \times M_0 | (\exists x \in \mathbf{V})(u = (\underset{z}{\downarrow}^{\mathbf{M}} x) \wedge (\underset{z}{\downarrow}^{\mathbf{M}} w) = (\underset{z}{\downarrow}^{\mathbf{M}} x))\}$, but it is easy to show that $\langle u, w\rangle \in \mathrm{S}$ if and only if $\langle u, w\rangle^{\mathbf{M}} \in ((((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}})^{\smile}$, thence $u \in \mathrm{dom}(\mathrm{S})$ if and only if $u \in (\mathsf{dom}^{\mathbf{M}}(((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}})^{\smile}$. By the Axiom of Choice there is a function F (maybe out of the model) such that $\mathrm{dom}(\mathrm{F}) = \mathrm{dom}(\mathrm{S}) \wedge \mathrm{F} \subseteq \mathrm{S}$. Define now an element $y \in \mathbf{M}$ in this way: $y^{\smile} = \{\langle u, w\rangle^{\mathbf{M}} | \langle u, w\rangle \in \mathrm{F}\}$. By Proposition 16 (f), it is $y^{\smile} \subseteq M_3$. Remark that if for every $\langle u, w\rangle \in \mathrm{F}$ it is $u \in M_0$, then $y^{\smile} = \{\langle u, w\rangle^{\mathbf{M}} | \langle u, w\rangle \in \mathrm{F}\} \subseteq M_0$, therefore, by Proposition 6 (d), there exists a unique $y \in M_1$ for which $y = \{\langle u, w\rangle^{\mathbf{M}} | \langle u, w\rangle \in \mathrm{F}\}$. If there are some elements $\langle u, w\rangle \in \mathrm{F}$ such that $u \in (M_1 - M_0)$, for these u's, by Proposition 16 (c), it is $\langle u, w\rangle^{\mathbf{M}} = \langle u, w\rangle$ and $\langle u, w\rangle \in \mathbf{M}$, hence, in this case, $y^{\smile} \nsubseteq \mathbf{V}$, therefore $y^{\smile} \in M_4$ and $y^{\smile} = y$. In both cases, it follows $\mathsf{Cls}^{\mathbf{M}}(y)$, since $(y^{\smile} - M_m)$ is empty. Trivially, by definition, it is $\mathsf{Rel}^{\mathbf{M}}(y)$ and $y \Subset^{\mathbf{M}} (((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}}$. Moreover it is $\mathsf{Fnc}^{\mathbf{M}}(y)$: when $\langle u, w\rangle^{\mathbf{M}} \in y^{\smile}$ and $\langle u, v\rangle^{\mathbf{M}} \in y^{\smile}$, then $\langle u, w\rangle \in \mathrm{F}$ and $\langle u, v\rangle \in \mathrm{F}$, therefore $w = v$. Moreover let $u \in (\mathsf{dom}^{\mathbf{M}}(z))^{\smile}$, therefore $u \in \mathrm{dom}(\mathrm{F})$ and $u \in \mathrm{dom}(\mathrm{S})$, $u \in (\mathsf{dom}^{\mathbf{M}}(((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}})^{\smile}$ and conversely, i.e. $(\mathsf{dom}^{\mathbf{M}}(z))^{\smile} = (\mathsf{dom}^{\mathbf{M}}(((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}})^{\smile}$. By Propositions 22 (e), 22 (c) and 22 (d), and Proposition 6 (d), $\mathsf{dom}^{\mathbf{M}}(z) = \mathsf{dom}^{\mathbf{M}}(((z^*)^{\mathbf{M}})^{-1})^{\mathbf{M}}$.

We can resume all the results proved before in the following

Theorem 23. (a) *The interpretation* **M** *is a model for* TAI. (b) *Every theorem proved in* [4] *is true in* **M**.

3 - Induction and prolongation

In each model constructed as in the previous sections, induction and prolongation properties, extending, respectively, Axioms 4 and 10, can be proved

Theorem 24. *For every formula* $\varphi(\Phi, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n)$, *eventually with parameters which are sets, the sentence*

$$\varphi(\emptyset, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \wedge (\forall\Phi)(\mathrm{Set}(\Phi) \wedge \varphi(\Phi, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n) \\ \rightarrow (\forall\Psi)(\varphi(\Phi \,\%\, \Psi, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n))) \rightarrow (\forall\Phi)(\mathrm{Set}(\Phi) \rightarrow \varphi(\Phi, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n))$$

holds in the model.

Proof. We write simply $\varphi(\Phi)$ instead of $\varphi(\Phi, \mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n)$. Suppose that $\varphi(\emptyset) \wedge (\forall\Phi)(\mathrm{Set}(\Phi) \wedge \varphi(\Phi) \rightarrow (\forall\Psi)(\varphi(\Phi \,\%\, \Psi))$ is true in the model **M**, but there is $x \in \mathbf{M}$ such that $\mathrm{Set}^{\mathbf{M}}(x)$ and $\neg\varphi^{\mathbf{M}}(x)$. Consider $\mathrm{A} = \{n \in \omega \mid \mathrm{card}(x^{\smile} - \mathbf{V}) = n \wedge \mathrm{Set}^{\mathbf{M}}(x) \wedge \neg\varphi^{\mathbf{M}}(x)\}$; $0 \notin \mathrm{A}$, since Axiom 4 holds in **M**. If $x \notin \mathbf{V}$, then $\mathrm{card}(x^{\smile} - \mathbf{V}) > 0$, by Remark 4. Let $p = \min \mathrm{A}$, then $p > 0$ and there is $x \in \mathbf{M}$ such that $\mathrm{card}(x^{\smile} - \mathbf{V}) = p \wedge \mathrm{Set}^{\mathbf{M}}(x) \wedge \neg\varphi^{\mathbf{M}}(x)$ and $\{y_1, \ldots, y_p\}$ is an enumeration of $(x^{\smile} - \mathbf{V})$. Suppose that $x \in (\mathbf{M} - M_0)$. The object $w = (x^{\smile} \cap \mathbf{V})$ is an element of **T**, therefore, by Proposition 6 (d), there exists $z \in \mathbf{V}$ such that $w = z^{\smile}$. The object z is such that $\mathrm{Set}^{\mathbf{M}}(z)$ and $\mathrm{card}(z^{\smile} - \mathbf{V}) = 0$, thence $\varphi^{\mathbf{M}}(z)$ by Axiom 4. If $p = 1$, by Propositions 14 (b) and 6 (d), there is a unique $u \in \mathbf{M}$ such that $u^{\smile} = z^{\smile} \cup \{y\}$ with $y \notin \mathbf{V}$, and $u^{\smile} = x = z^{\smile} \cup \{y\}$, that means $x = z \,\%\, y$; therefore $\varphi^{\mathbf{M}}(x)$. If $p > 1$, applying $p - 1$ times Proposition 14 (b) to z, there is $u \in \mathbf{M}$ such that $u^{\smile} = z^{\smile} \cup \{y_1, \ldots, y_{p-1}\}$, it is $\mathrm{Set}^{\mathbf{M}}(u)$, it is $u \notin \mathbf{V}$ and $x = u \,\%\, y_p$; moreover $u \cap \mathbf{V} = z^{\smile}$ and $\mathrm{card}(u^{\smile} - \mathbf{V}) = p - 1$. By definition of p, $\varphi^{\mathbf{M}}(u)$, therefore, $\varphi^{\mathbf{M}}(x)$. In each case we get a contradiction, hence $\mathrm{A} = \emptyset$.

The model has another interesting feature regarding prolongation of classes.

Proposition 25. *For every* $x \in \mathbf{M}$ *such that* $\mathsf{Cls}^{\mathbf{M}}(x)$ *if there exists* $y \in \mathbf{M}$ *such that* $\mathsf{Set}^{\mathbf{M}}(y)$ *and* $x \Subset^{\mathbf{M}} y$, *then there are* u, $v \in \mathbf{M}_1$ *such that* $u^{\check{}} = x^{\check{}} \cap \mathbf{V}$, $v^{\check{}} = y^{\check{}} \cap \mathbf{V}$ *for which* $\Lambda^{\mathbf{M}}(u)$, $\mathsf{V}^{\mathbf{M}}(v)$ *and* $u \Subset^{\mathbf{M}} v$; *moreover* $(x^{\check{}} - \mathbf{V})$ *is* ZF-finite.

Proof. Suppose $x \in \mathbf{V}$, it is $x^{\check{}} \subseteq \mathbf{V}$, thence $x^{\check{}} \cap \mathbf{V} = x^{\check{}}$. From hypothesis that there is $y \in \mathbf{M}$ such that $\mathsf{Set}^{\mathbf{M}}(y)$ and $x \Subset^{\mathbf{M}} y$, it follows by Proposition 22 (f), that $x^{\check{}} \subseteq y^{\check{}}$, thence $x^{\check{}} \subseteq y^{\check{}} \cap \mathbf{V}$. In case $y \in \mathbf{V}$ and also in case $y \notin \mathbf{V}$, there is, by Propositions 6 (d), $v \in \mathbf{V}$ satisfying $v^{\check{}} = y^{\check{}} \cap \mathbf{V}$. Therefore it can be chosen u as x, obtaining: $\Lambda^{\mathbf{M}}(u)$, $\mathsf{V}^{\mathbf{M}}(v)$ and $u^{\check{}} \subseteq v^{\check{}}$; moreover $(x^{\check{}} - \mathbf{V})$ is the empty set.

In case $x \notin \mathbf{V}$, from $x \Subset^{\mathbf{M}} y$, by Proposition 22 (f), it follows that $(x^{\check{}} - \mathbf{V}) \subseteq (y^{\check{}} - \mathbf{V})$. Therefore $(x^{\check{}} - \mathbf{V})$ is ZF-finite, since $(y^{\check{}} - \mathbf{V})$ is ZF-finite. Moreover $x^{\check{}} \cap \mathbf{V} \subseteq y^{\check{}} \cap \mathbf{V}$; by Propositions 6 (d), there are $u \in M_1$ and $v \in \mathbf{V}$ such that $u^{\check{}} = x^{\check{}} \cap \mathbf{V}$, $v^{\check{}} = y^{\check{}} \cap \mathbf{V}$ and $u \Subset^{\mathbf{M}} v$, from Proposition 22 (f).

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Sunto

Si presentano modelli per la teoria assiomatica TAI, *introdotta in una nota precedente e si prova la proprietà di consistenza di* TAI, *relativamente alla teoria assiomatica degli insiemi* ZF.

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