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Remarks on a perturbation solution of the Boltzmann equation for electrons in a gas (**)

Alla memoria di ANTONIO MAMBRIANI

1 - Introduction

In a previous paper [3] we considered the motion of a swarm of electrons immersed in a background gas of neutral atoms and we made remarks on the limits of validity of some truncation procedures of solution of the Boltzmann equation governing the electron distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ in the Lorentz limit, under the action of an electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ («slowly variable» with time; see below) and a magnetic field $\boldsymbol{B}(\boldsymbol{r})$

$$(1) \qquad \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial \boldsymbol{r}} - \frac{e}{m}[\boldsymbol{E}(\boldsymbol{r}, t) + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r})] \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v}, t)}{\partial \boldsymbol{v}} = J(f(\boldsymbol{r}, \boldsymbol{v}, t)).$$

The notations in (1) are rather standard. J is the collision term, which, in the case when only elastic collisions are taken in account, is a linear integral operator.

In [3] we started with the assumptions

$$(2) \qquad \tau_m |\partial \ln f/\partial t| \ll 1 \qquad |eE\tau_m/(mv)| \ll 1 \qquad v\tau_m/L \ll 1$$

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where $\tau_m \equiv \tau_m(v)$ is the mean time-of-flight for momentum transfer relevant to an electron of speed v, and L is a macroscopic scale length such that $|\partial \ln f/\partial \boldsymbol{r}| \sim 1/L$. We claimed that assumptions (2), among other things, «*presume situations which change very slowly on the scale of* τ_m». We think that such sentence can be given more details and that the whole problem of the construction of a perturbation solution of (1) can be treated in a modified way. In particular, we shall follow such a procedure that the order of magnitude of each term with respect to the small parameter $\alpha = \sqrt{m/M}$ (i.e. the square root of the ratio between electron and atom masses) appears in a natural way and the method can also be applied in different cases.

2 - The «extended» Boltzmann equation

We start remarking that, as regards the problem of the relaxation towards equilibrium of the electron distribution function, one can distinguish two different time scales: a faster one, which is related to the mean time-of-flight τ_m, and a slower one, which is related to the exchange of energy between an electron and a neutral atom. As the energy lost in the average by an electron in a collision with an atom is of the order of $\alpha^2 = m/M$ [2], it will be convenient to introduce a «slow time» $\tau_2 = \alpha^2 t$. Since we shall consider expansions in the parameter α, it is suitable to introduce also $\tau_1 = \alpha t$. Moreover we set $\tau_0 \equiv t$. We also suppose that the electric field $\boldsymbol{E}$ may depend on τ_2, but not on τ_0 and τ_1.

In order to reach our aim, we substitute to $f(\boldsymbol{r}, \boldsymbol{v}, t)$ in (1) the «extended function» $\widetilde{F}(\boldsymbol{r}, \boldsymbol{v}, \tau_0, \tau_1, \tau_2)$ [4], [5] such that [1]

$$\widetilde{F}(\boldsymbol{r}, \boldsymbol{v}, \tau_0, \tau_1, \tau_2)\Big|_{\substack{\tau_0=t\\ \tau_1=\alpha t\\ \tau_2=\alpha t^2}} \equiv f(\boldsymbol{r}, \boldsymbol{v}, t) \tag{3}$$

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[1] We can generalize without any difficulty this notion of extended function to each integer order k. So we can set

$$f(\boldsymbol{r}, \boldsymbol{v}, t) \to \widetilde{F}(\boldsymbol{r}, \boldsymbol{v}, \tau_0, \tau_1, \dots, \tau_k) \qquad \text{with}$$

$$\widetilde{F}(\boldsymbol{r}, \boldsymbol{v}, \tau_0, \tau_1, \dots, \tau_k)\Big|_{\substack{\tau_i=\alpha^i t\\ i=0,1,\dots k}} \equiv f(\boldsymbol{r}, \boldsymbol{v}, t).$$

For our aim, however, we can limit ourselves to the case $k = 2$.

and set, moreover,

$$\frac{\partial f}{\partial t} \to \frac{\partial \widetilde{F}}{\partial \tau_0} + \alpha \frac{\partial \widetilde{F}}{\partial \tau_1} + \alpha^2 \frac{\partial \widetilde{F}}{\partial \tau_2}. \tag{4}$$

The assumptions $(2)_2$ and $(2)_3$ can be quantified by assuming that, accordingly with [1] and [3], the second and the third term in the l.h.s. of (1) are of the order of α. The fourth term, on the contrary, will not be considered small. Different assumptions are, however, possible, so that the procedure we follow equally applies.

In order to make automatic the specification of the order of each term, it is suitable to introduce a «reduced velocity» $\xi = \alpha \boldsymbol{v}$, a «reduced spatial variable» $\boldsymbol{x} = \alpha^2 \boldsymbol{r}$ and a «reduced temperature» $\Theta = \alpha^2 T$ for the background gas particles. Moreover we set

$$F(\boldsymbol{x}, \xi, \tau_0, \tau_1, \tau_2) = \widetilde{F}(\boldsymbol{x}/\alpha^2, \xi/\alpha, \tau_0, \tau_1, \tau_2). \tag{5}$$

Then eq. (1) is converted into the «extended Boltzmann equation» (EBE)

$$\frac{\partial F}{\partial \tau_0} + \alpha \frac{\partial F}{\partial \tau_1} + \alpha^2 \frac{\partial F}{\partial \tau_2} + \alpha \xi \cdot \frac{\partial F}{\partial \boldsymbol{x}} + \alpha \boldsymbol{a} \cdot \frac{\partial F}{\partial \xi} + \boldsymbol{\omega}_c \times \xi \cdot \frac{\partial F}{\partial \xi} = J(F) \tag{6}$$

where $\boldsymbol{a} = -e\boldsymbol{E}/m$ and $\boldsymbol{\omega}_c = e\boldsymbol{B}/(mc)$.

In our hypotheses we have, moreover (see [1], p. 133)

$$J(F) = J^0(F) + \alpha^2 J^2(F) + O(\alpha^4)$$

where

$$\begin{aligned} J^0(F) &= \int_\Omega [F(\dots, \boldsymbol{\eta}, \dots) - F(\dots, \xi, \dots)]\, q(\xi, \chi)\, d\Omega \\ J^2(F) &= \frac{k\Theta}{2m} \int_\Omega [F(\dots, \boldsymbol{\eta}, \dots) - F(\dots, \xi, \dots)]\, \nabla_\xi^2 q(\xi, \chi)\, d\Omega \\ &+ \frac{\partial}{\partial \xi} \cdot \int_\Omega 2q(\xi, \chi)(\mathbf{I} - \boldsymbol{nn}) \cdot [\xi F(\dots, \boldsymbol{\eta}, \dots) + \frac{k_B \Theta}{m} \frac{\partial F(\dots, \boldsymbol{\eta}, \dots)}{\partial \xi}]\, d\Omega . \end{aligned} \tag{8}$$

As well as in [3] we mean that $\boldsymbol{\eta} = (2\boldsymbol{nn} - \mathbf{I}) \cdot \xi$, where $\mathbf{I}$ is the unit dyadic and $\boldsymbol{n}$ is the unit vector directed along the bisector of the angle between the relative velocities before and after collision. Moreover k_B is the gas Boltzmann constant and ∇_ξ^2 indicates the Laplace operator in the (reduced) velocity space.

Now we expand F in power series of α

$$F = F_0 + \alpha F_1 + \alpha_2 F_2 + \dots . \tag{9}$$

It is just inserting the expansion (9) in (6) that we are gonig to build up our perturbation solution to the given equation.

3 – The perturbation solution of EBE

We start by considering the terms of zero-order in α in both sides of (6). We obtain then

$$\frac{\partial F_0}{\partial \tau_0} + \boldsymbol{\omega}_c \times \xi \cdot \frac{\partial F_0}{\partial \xi} + J^0(F_0) . \tag{10}$$

If we introduce

$$H(\boldsymbol{x},\ \tau_0,\ \tau_1,\ \tau_2) = \int_\xi F_0(\boldsymbol{x},\ \xi,\ \tau_0,\ \tau_1,\ \tau_2) \ln F_0(\boldsymbol{x},\ \xi,\ \tau_0,\ \tau_1,\ \tau_2)\, d\xi \tag{11}$$

we prove without any difficulty (see Appendix) that a sort of «H-theorem» holds for F_0 on the scale of τ_0, namely

$$\frac{\partial H}{\partial \tau_0} \leqslant 0 \tag{12}$$

where the sign of equality is valid if and only if F_0 is an isotropic function in the velocity space. Then F_0 relaxes to an isotropic function on the (fast) scale of τ_0. So we can write F_0 as the sum of a «transient part» F_0^T, which tend to zero when $\tau_0 \to \infty$, and an «asymptotic part» F_0^A (independent, therefore, of τ_0) which results to be isotropic. We can think of a similar decomposition for all F_k's, with $k \geqslant 1$, i.e. we can set $F_k = F_k^A + F_k^T$, where $F_k^T \to 0$ on the scale of τ_0. At this point we can better define the meaning of the expression of Introduction «situations which change very slowly on the scale of τ_m» (i.e. of τ_0), according to assumption $(2)_1$: we mean that $F_k \simeq F_k^A$ where F_k^A is independent of τ_0. Then we shall identity F_k with F_k^A in what follows.

If we now take in account the terms of the first order with respect to α in (6), we obtain

$$\frac{\partial F_0}{\partial \tau_1} + \xi \cdot \frac{\partial F_0}{\partial \boldsymbol{x}} + \boldsymbol{a} \cdot \frac{\partial F_0}{\partial \xi} + \boldsymbol{\omega}_c \times \xi \cdot \frac{\partial F_1}{\partial \xi} = J^0(F_1) \tag{13}$$

where $\partial F_0/\partial \xi = (1/\xi)(\partial F_0/\partial \xi)\,\xi$ (F_0 being isotropic in ξ).

In a quite similar way like in [3] one can obtain that

$$F_1 = G_1 \cdot \xi + \overline{F}_1 \tag{14}$$

with

$$G_1 = -\tau_m \mathbf{M} \cdot A(F_0) \tag{15}$$

where ($\boldsymbol{b} = \boldsymbol{B}/B$):

$$\mathbf{M} = \boldsymbol{b}\boldsymbol{b} + \frac{1}{1+\omega_c^2 \tau_m^2}(\mathbf{I} - \boldsymbol{b}\boldsymbol{b}) + \frac{\omega_c \tau_m}{1+\omega_c^2 \tau_m^2}\boldsymbol{b} \times \mathbf{I} \tag{16}$$

$$A(F_0) = \left(\frac{\partial}{\partial \boldsymbol{x}} + \frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\right) F_0 \tag{17}$$

and $\overline{F}_1$ is the isotropic part of F_1, for which we shall obtain an equation later. Note that G_1 is an isotropic vector with respect to ξ (2).

Taking the isotropic part of (13) we have, moreover,

$$\frac{\partial F_0}{\partial \tau_1} = 0 \tag{18}$$

so that F_0 depends on time only through τ_2.

Turning now to second order terms, we have

$$\frac{\partial F_0}{\partial \tau_2} + \frac{\partial F_1}{\partial \tau_1} + \xi \cdot \frac{\partial F_1}{\partial \boldsymbol{x}} + \boldsymbol{a} \cdot \frac{\partial F_1}{\partial \xi} + \boldsymbol{\omega}_c \times \xi \cdot \frac{\partial F_2}{\partial \xi} = J^0(F_2) + J^2(F_0)\,. \tag{19}$$

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(2) If, according to the fact that spherical harmonics (in velocity space) are eigenfunctions of both $\boldsymbol{\omega}_c \times \xi \cdot (\partial/\partial \xi)$ and J^0, we start with a spherical harmonics expansion of F_1, we see by invoking orthogonality and completeness properties that F_1 can have no form but (14). Analogous conclusions hold for F_k, with $k \geqslant 2$.

If we remember (14) for F_1 and observe that

$$\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{G}_1\cdot\xi)\cdot\xi=\frac{\partial \boldsymbol{G}_1}{\partial \boldsymbol{x}}:(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})+\frac{1}{3}\xi^2\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{G}_1 \tag{20}$$
$$\frac{\partial}{\partial \xi}(\boldsymbol{G}_1\cdot\xi)\cdot\boldsymbol{a}=\frac{1}{\xi}\boldsymbol{a}\frac{\partial \boldsymbol{G}_1}{\partial \xi}:(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})+\frac{1}{3}\xi\boldsymbol{a}\cdot\frac{\partial \boldsymbol{G}_1}{\partial \xi}+\boldsymbol{a}\cdot\boldsymbol{G}_1,$$

the components of $\xi\xi/\xi^2-(1/3)\mathbf{I}$ being the 2-order spherical harmonics [3], we obtain that the first four terms in the l.h.s. of (19) are of the form $P(\xi)+\boldsymbol{Q}(\xi)\cdot\xi+\mathbf{R}(\xi):(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})$ while $J^2(F_0)$ is isotropic in ξ. Then it is natural to seek a solution (see also foot-note (2)) for F_2 in the form

$$F_2=\mathbf{G}_2:\xi\xi+\boldsymbol{F}_2^{(1)}\cdot\xi+\overline{F}_2 \tag{21}$$

where $\mathbf{G}_2$ is an isotropic traceless symmetric 2-tensor so that $\mathbf{G}_2:\xi\xi\equiv\mathbf{G}_2:(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})$, $\boldsymbol{F}_2^{(1)}$ and $\overline{F}_2$ are, respectively, an isotropic vector and an isotropic scalar function.

We have, after some calculations,

$$\boldsymbol{\omega}_c\times\xi\cdot\frac{\partial F_2}{\partial \xi}=-2\boldsymbol{\omega}_c\times\mathbf{G}_2:(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})-\boldsymbol{\omega}_c\times\boldsymbol{F}_2^{(1)}\cdot\xi \tag{22}$$

$$J^0(F_2)=-\nu_2\mathbf{G}_2:(\xi\xi-\frac{1}{3}\xi^2\mathbf{I})-\frac{1}{\tau_m}\boldsymbol{F}_2^{(1)}\cdot\xi \tag{23}$$

where [3]

$$\nu_2(\xi)=\int_\Omega q(\xi,\ \chi)[1-P_2(\cos\chi)]\,d\Omega \tag{24}$$

($P_2(\cos\chi)=(3/2)\cos^2\chi-1/2$ being the second-order Legendre polynomial in $\cos\chi$).

So, if we equate the isotropic parts of each side of (19), we obtain

$$\frac{\partial F_0}{\partial \tau_2}+\frac{\partial \overline{F}_1}{\partial \tau_1}+\frac{1}{3}\xi^2\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{G}_1+\frac{1}{3}\xi\boldsymbol{a}\cdot\frac{\partial \boldsymbol{G}_1}{\partial \xi}+\boldsymbol{a}\cdot\boldsymbol{G}_1=J^2(F_0). \tag{25}$$

As F_0 is independent of τ_1 and $\boldsymbol{G}_1$ can be expressed in terms of F_0 by (15), so must be $\overline{F}_1$, since otherwise the integration of (25) with respect to τ_1 would cause secular behaviour. Thus (25) yields to the basic equation for F_0, namely

$$\frac{\partial F_0}{\partial \tau_2} = \frac{1}{\xi}\Big(\frac{\partial}{\partial \boldsymbol{x}} + \frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)\cdot\Big(\frac{\xi^3 \tau_m}{3}\mathbf{M}\Big)\cdot\Big(\frac{\partial}{\partial \boldsymbol{x}} + \frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)F_0 + \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\Big[\frac{\xi^3}{\tau_m}\Big(F_0 + \frac{k_B\Theta}{m\xi}\frac{\partial F_0}{\partial \xi}\Big)\Big]. \tag{26}$$

If we now look at the coefficients of the 1-order spherical harmonics in (19), we obtain

$$\frac{\partial \overline{\boldsymbol{F}}_1}{\partial \boldsymbol{x}} + \frac{1}{\xi}\boldsymbol{a}\frac{\partial \overline{\boldsymbol{F}}_1}{\partial \xi} - \boldsymbol{\omega}_c \cdot \boldsymbol{F}_2^{(1)} = -\frac{1}{\tau_m}\boldsymbol{F}_2 \qquad \text{i.e.} \qquad \text{(cf. [3])} \tag{27}$$

$$\boldsymbol{F}_2^{(1)} = -\tau_m \mathbf{M}\cdot \boldsymbol{A}(\overline{\boldsymbol{F}}_1). \tag{27$'$}$$

The remainder yields

$$\Big[\Big(\frac{\partial}{\partial \boldsymbol{x}} + \frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)\boldsymbol{G}_1 - 2\boldsymbol{\omega}_c \times \mathbf{G}_2 + \nu_2 \mathbf{G}_2\Big] : \Big(\xi\xi - \frac{1}{3}\xi^2 \mathbf{I}\Big) = 0. \tag{28}$$

Following the same procedure as in [3], we can solve (28) for $\mathbf{G}_2$ in terms of $\boldsymbol{G}_1$ and, therefore, of F_0. The result, in compact form, is

$$\mathbf{G}_2 = \frac{\nu_2 \mathbf{A}^0 - 2\boldsymbol{\omega}\times\mathbf{A}^0 + (2/\omega^2)\,[\boldsymbol{\omega}\times(\mathbf{A}^0\cdot\boldsymbol{\omega})]\,\boldsymbol{\omega} + (2/\nu_2)(\mathbf{A}^0 : \boldsymbol{\omega}\boldsymbol{\omega})\,\mathbf{I}}{\nu_2^2 + 4\omega^2} - \frac{[(\mathbf{A}^0\cdot\boldsymbol{\omega})\cdot\boldsymbol{\omega}]\,(\boldsymbol{\omega}\times\mathbf{I}) + 4\,\{\boldsymbol{\omega}\,[\boldsymbol{\omega}\times(\mathbf{A}^0\cdot\boldsymbol{\omega})] + [\boldsymbol{\omega}\times(\mathbf{A}^0\cdot\boldsymbol{\omega})\,\boldsymbol{\omega}\}}{4\omega^2(\nu_2^2 + \omega^2)} + \frac{(6/\nu_2)(\mathbf{A}^0 : \boldsymbol{\omega}\boldsymbol{\omega})\,\boldsymbol{\omega}\boldsymbol{\omega} + 12\nu_2[(\mathbf{A}^0\cdot\boldsymbol{\omega})\,\boldsymbol{\omega} + \boldsymbol{\omega}(\mathbf{A}^0\cdot\boldsymbol{\omega})]}{(\nu_2^2 + 4\omega^2)(\nu_2^2 + \omega^2)} \tag{29}$$

where we have written $\boldsymbol{\omega}$ instead of $\boldsymbol{\omega}_c$ and we have indicated with $\mathbf{A}^0$ the «symmetric traceless part» of the tensor

$$\mathbf{A} = -\Big(\frac{\partial}{\partial \boldsymbol{x}} + \frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)\boldsymbol{G}_1 \qquad \text{i.e.} \tag{30}$$

$$\mathbf{A}^0 = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) - \frac{1}{3}(\mathbf{A}:\mathbf{I})\,\mathbf{I}\ (^3). \tag{31}$$

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(3) Note that if **B** is any traceless symmetric 2-tensor, then

$$\mathbf{A}:\mathbf{B} = \mathbf{A}^T:\mathbf{B} = \mathbf{A}^0:\mathbf{B} \qquad \text{for each 2-tensor } \mathbf{A}.$$

If we now take in account the third order terms, we get

$$\frac{\partial F_2}{\partial \tau_1}+\frac{\partial F_1}{\partial \tau_2}+\xi\cdot\frac{\partial F_2}{\partial \boldsymbol{x}}+\boldsymbol{a}\cdot\frac{\partial F_2}{\partial \xi}+\boldsymbol{\omega}_c\times\xi\cdot\frac{\partial F_3}{\partial \xi} \tag{32}$$
$$=J^0(F_3)+J^2(F_1)\,.$$

Taking once again the isotropic part, we obtain

$$\frac{\partial \overline{F}_2}{\partial \tau_1}+\frac{\partial \overline{F}_1}{\partial \tau_2}+\frac{1}{3}\xi^2\Big(\frac{\partial}{\partial \boldsymbol{x}}\cdot \boldsymbol{F}_2^{(1)}+\frac{\boldsymbol{a}}{\xi}\cdot\frac{\partial \boldsymbol{F}_2^{(1)}}{\partial \xi}\Big)+\frac{3\boldsymbol{a}}{\xi^2}=J^2(\overline{F}_1)\,. \tag{33}$$

By the same argument as before we conclude that, since $\boldsymbol{F}_2^{(1)}$ can be written in terms of $\overline{F}_1$ by (27), also $\overline{F}_2$ is independent of τ_1, so that (33) yields

$$\frac{\partial \overline{F}_1}{\partial \tau_2}=\frac{1}{\xi}\Big(\frac{\partial}{\partial \boldsymbol{x}}+\frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)\cdot\Big(\frac{\xi^3\tau_m}{3}\mathbf{M}\Big)\cdot\Big(\frac{\partial}{\partial \boldsymbol{x}}+\frac{\boldsymbol{a}}{\xi}\frac{\partial}{\partial \xi}\Big)\overline{F}_1+\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\Big[\frac{\xi^3}{\tau_m}\Big(\overline{F}_1+\frac{k_B\Theta}{m\xi}\frac{\partial \overline{F}_1}{\partial \xi}\Big)\Big] \tag{34}$$

i.e. a self-consistent equation for $\overline{F}_1$. Note that (26) for F_0 is identical to (34) for $\overline{F}_1$.

Then we can consider the terms in (32) containing the spherical harmonics of order 1, 2, 3, and so on for higher order terms in α, obtaining each time a new information and/or a new equation for some F_k.

4 - Conclusive remarks

The procedure we have followed in last sections can be developed at the desired order, supplying the required perturbation solution. One can see that, as remarked in [3], [5], the expansion in spherical harmonics appears as a consequence of the expansion in the small parameter $\alpha=\sqrt{m/M}$ and can justify the truncation procedures which are classically followed.

Although we have made use of some results and some ideas of our previous paper [3], we think that something has been improved now, expecially as regards the mathematical translation of our assumptions.

As a matter of fact, the use of the «extension method» (with different time scales) and the introduction of «reduced» variables and constants allow to obtain the expansion with respect to α in a natural way and can be applied to different physical situations.

Appendix

We are going to prove that the function H introduced in 3 by (11) is never increasing with τ_0, i.e. $\partial H/\partial\tau_0 \leqslant 0$, the equality sign holding if and only if F_0 is isotropic in the velocity space.
We start by multiplying both sides of (10) by $1+\ln F_0$ and integrating over the whole velocity space. We observe that

(A1)
$$\boldsymbol{\omega}_c \times \xi \cdot \frac{\partial F_0}{\partial \xi}(1+\ln F_0) = \boldsymbol{\omega}_c \times \xi \cdot \frac{\partial}{\partial \xi}(F_0 \ln F_0) = \frac{\partial}{\partial \xi} \cdot [(\boldsymbol{\omega}_c \times \xi) F_0 \ln F_0]$$

and the contribution of this term, by divergence theorem and taking account of the behaviour of F_0 at infinity, vanishes. Moreover, introducing the vector $\boldsymbol{l} = \xi - \boldsymbol{\eta}$ and recalling that $q(\xi, \varkappa) = N\xi\sigma(\xi, \varkappa)$, where N is the atom (constant) number density and $\sigma(\xi, \varkappa)$ is the (elastic) differential cross section relevant to collisions between electrons and atoms, we see, by the properties of the Dirac «delta function», that

(A2)
$$\int_\xi J^0(F_0)(1+\ln F_0(\xi))\,\mathrm{d}\xi$$
$$= 2N \int_\xi \int_l \delta(l^2 - 2\xi\cdot \boldsymbol{l})\,\sigma(\xi, \varkappa)[F_0(\xi - \boldsymbol{l}) - F_0(\xi)](1+\ln F_0(\xi))\,\mathrm{d}\boldsymbol{l}\,\mathrm{d}\xi$$
$$= 2N \int_\xi \int_\eta \delta(\eta^2 - \xi^2)\,\sigma(\xi, \varkappa)[F_0(\boldsymbol{\eta}) - F_0(\xi)](1+\ln F_0(\xi))\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\xi$$
$$= 2N \int_\xi \int_\eta \delta(\eta^2 - \xi^2)\,\sigma(\eta, \varkappa)[F_0(\xi) - F_0(\boldsymbol{\eta})](1+\ln F_0(\boldsymbol{\eta}))\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\xi\,.$$

In (A2) we have suppressed the explicit indication of the variables $\boldsymbol{x}$, τ_0, τ_1, τ_2.
Remembering that $\xi = |\xi| = |\boldsymbol{\eta}| = \eta$ and returning to $\boldsymbol{l}$ as an integration variable, we obtain finally

(A3)
$$\int_\xi J^0(F_0)(1+\ln F_0(\xi))\,\mathrm{d}\xi$$
$$= N \int_\xi \int_l \delta(l^2 - 2\xi\cdot\boldsymbol{l})\,\sigma(\xi, \varkappa)\,[F_0(\xi - \boldsymbol{l}) - F_0(\xi)] \ln \frac{F_0(\xi)}{F_0(\xi - \boldsymbol{l})}\,\mathrm{d}\boldsymbol{l}\,\mathrm{d}\xi \leqslant 0.$$

As

$$\int_{\xi} \frac{\partial F_0}{\partial \tau_0}(1 + \ln F_0)\, d\xi = \frac{\partial}{\partial \tau_0} \int_{\xi} F_0 \ln F_0 \, d\xi = \frac{\partial H}{\partial \tau_0} \tag{A4}$$

it follows immediately the validity of our assertions.

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Sunto

Si considera il problema, già affrontato in un precedente lavoro, del moto di uno sciame di elettroni (di massa m) in un gas di fondo neutro monoatomico (le cui particelle hanno massa M), nel limite di Lorentz, sotto l'azione di campi esterni. Viene formulato un metodo perturbativo, valido in diverse situazioni fisiche, di soluzione dell'equazione di Boltzmann per gli elettroni, basato su sviluppi nel parametro $\alpha = \sqrt{m/M}$.

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