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On best approximation in the sense of Lumer and applications (**)

1 - Introduction

Let E be a linear space over real or complex number field K. A mapping $[\,,\,]$ of $E\times E$ into K is a semi-inner product in the sense of Lumer or L-semi-inner product on E, for short, if the following conditions (P1)-(P4) are satisfied (see for example [6] and [3]):

(P1) $[x,\ x]\geq 0$ for all $x\in E$ and $[x,\ x]=0$ implies $x=0$;

(P2) $[\lambda x,\ y]=\lambda[x,\ y]$ and $[x,\ \lambda y]=\bar{\lambda}[x,\ y]$ for all $\lambda\in K$ and $x,\ y\in E$;

(P3) $[x+y,\ z]=[x,\ z]+[y,\ z]$ for all $x,\ y,\ z\in E$;

(P4) $|[x,\ y]|^2\leq [x,\ x][y,\ y]$ for all $x,\ y\in E$.

It is easy to see that the mapping $E\ni x\mapsto [x,\ x]^{1/2}\in \mathrm{R}_+$ is a norm on E and if E is a normed space, then every L-semi-inner product on E which generates its norm is of the form $[x,\ y]=\langle \tilde{J}(y),\ x\rangle$ for all $x,\ y\in E$, where $\tilde{J}$ is a section of normalized dual mapping [9]. It is also known that a normed linear space E is smooth iff there exists a unique L-semi-inner product which generates its norm or if and only if there exists a continuous L-semi-inner product which generates

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the norm, i.e., a L-semi-inner product satisfying condition (see [3])

$$\lim_{t\to 0} \operatorname{Re}[y, x+ty] = \operatorname{Re}[y, x] \quad \text{for all } x, y \in E.$$

Let two elements in E and $[\,,\,]$ a L-semi-inner product on E which generates the norm. The element x is said to be *Lumer-orthogonal over* y or *L-orthogonal over* y, for short, if $[y, x]=0$. We denote this fact by xLy $[2]_{1,2}$. For some properties of L-semi-inner products, L-orthogonality, representation of continuous linear functionals in terms of semi-inner product we send to $[2]_{1,2}$, [3], [7], [9] and [11].

2 - Characterizations of best approximation element in the sense of Lumer

Now, we recall some concepts and results in best approximation theory that will be used in the sequel.

Let E be a normed space and x, y two elements in E. The vector x is called *orthogonal in the sense of Birkhoff* over y if $\|x+\lambda y\| \geq \|x\|$ for all $\lambda \in K$. We denote this $x \perp_B y$.

If G is a nondense linear subspace in E and $\mathscr{P}_G(x_0) := \{g_0 \in G,\ \|x_0 - g_0\| = \inf_{g\in G} \|g - x_0\|\}$, denotes the set of best approximation element refering to $x_0 \in E \setminus \bar{G}$ in G, then the following simple characterization lemma in terms of Birkhoff orthogonality holds (see Lemma 1.14 from [10]).

Lemma 1. *Let E, G, x_0 be as above and $g_0 \in G$. Then $g_0 \in \mathscr{P}_G(x_0)$ if and only if $x_0 - g_0 \perp_B G$.*

For other characterizations of best approximation element in normed spaces see [10] and $[2]_2$.

The following result is also valid (see $[2]_2$, Lemma 1.1).

Lemma 2. *Let E be a smooth normed linear space and x, y two elements in E. Then xLy if and only if $x \perp_B y$.*

In virtue of this fact we can introduce the following concept.

Def. 1. Let E be a normed linear space, $[\,,\,]$ be a L-semi-inner product on E which generates its norm, G a nondense linear subspace of E, $x_0 \in E \setminus \bar{G}$ and

$g_0 \in G$. The vector g_0 is called *the best approximation element of* x_0 *in* G *in the sense of Lumer* relative to semi-inner product $[\,,]$, or *L-best approximation element* of x_0, for short, if $x_0 - g_0 L G$.

Proposition 1. *Let* E, $[\,,]$, G, x_0 *and* g_0 *be as above and denote* $\mathscr{P}_G^L(x_0)$ *the set of L-best approximation elements referring to* x_0 *in* G, *then* $\mathscr{P}_G^L(x_0) \subseteq \mathscr{P}_G(x_0)$. *If, in addition, we suppose that* E *is smooth, then* $\mathscr{P}_G^L(x_0) = \mathscr{P}_G(x_0)$.

Proof. Let $g_0 \in \mathscr{P}_G^L(x_0)$. Then $[g, x_0 - g_0] = 0$ for all $g \in G$. On the other hand, we have

$$\|x_0 - g_0\|^2 = [x_0 - g_0, x_0 - g_0] = [x_0, x_0 - g_0] = [x_0 - g, x_0 - g_0]$$
$$\leq \|x_0 - g\|\,\|x_0 - g_0\| \quad \text{for all } g \in G$$

what implies $\|x_0 - g_0\| \leq \|x_0 - g\|$ for all $g \in G$, i.e., $g_0 \in \mathscr{P}_G(x_0)$.

The second part of proposition follows by Lemma 1 and Lemma 2.

Let G be a linear subspace and denote

$$G^L := \{w \in E \mid wLg \quad \text{for all } g \in G\}.$$

It is easy to see that $O \in G^L$, $G \cap G^L = \{0\}$ and $\alpha \in K$, $x \in G^L$ imply $\alpha x \in G^L$.

Proposition 2. *Let* E *be a normed linear space,* $[\,,]$ *be a L-semi-inner product on it,* G *be a nondense linear subspace in* E, $x_0 \in E \setminus \bar{G}$ *and* $g_0 \in G$. *Then* $g_0 \in \mathscr{P}_G^L(x_0)$ *if and only if there exists an element* $w_0 \in G^L$ *such that*

$$x_0 = g_0 + w_0. \tag{1}$$

The proof is obvious by the definition of L-best approximation and we omit the details.

From the above proposition we have the next

Corollary. *Let* E, $[\,,]$, G *and* x_0 *be as in Proposition* 2. *Then the following statements are equivalent:*

(i) $\mathscr{P}_G^L(x_0)$ *contains at least one [at most one (a unique)] element;*

(ii) *there exists at least one [at most one (a unique)] element* $g_0 \in G$ *and at least one [at most one (a unique)]* $w_0 \in G^L$ *such that* (1) *holds.*

Remark 1. If E is a smooth normed linear space then $g_0 \in \mathscr{P}_G(x_0)$ iff there exists $w_0 \in G^L$ such that (1) holds.

The following result is important in the sequel.

Proposition 3. *Let* E, $[\,,\,]$ *be as above and* f *be a nonzero continuous linear functional on* E, $x_0 \in E \setminus \mathrm{Ker}(f)$ *and* $g_0 \in \mathrm{Ker}(f)$. *Then* $g_0 \in \mathscr{P}^L_{\mathrm{Ker}(f)}(x_0)$ *if and only if the following representation holds*

$$f(x) = \|f\| \; [x, \; \lambda_0(x_0 - g_0)/\|x_0 - g_0\|] \tag{2}$$

for all $x \in E$ *where* $\lambda_0 := \overline{f(x_0)}/|f(x_0)|$.

Proof. Let $g_0 \in \mathscr{P}^L_{\mathrm{Ker}(f)}(x_0)$ and put $w_0 := x_0 - g_0 \neq 0$. Then $w_0 \in \mathrm{Ker}(f)^L$. Since $f(x)\,w_0 - f(w_0)\,x \in \mathrm{Ker}(f)$ for all $x \in E$, hence $[f(x)\,w_0 - f(w_0)\,x, \; w_0] = 0$ what implies $f(x) = [x, \; \overline{f(x_0)}(x_0 - g_0)/\|x_0 - g_0\|^2]$ for all $x \in E$ and $\|f\| = |f(x_0)|/\|x_0 - g_0\|$, then (2) holds. Conversely, if (2) is valid, then $x_0 - g_0 L\, \mathrm{Ker}(f)$, i.e., $g_0 \in \mathscr{P}^L_{\mathrm{Ker}(f)}(x_0)$.

Corollary. *Let* f *and* x_0, *be as above. Then the following statements are equivalent:*

(i) $\mathscr{P}^L_{\mathrm{Ker}(f)}(x_0)$ *contains at least one [at most one (a unique)] element;*

(ii) *there exists at least one [at most one (a unique)] element* $g_0 \in \mathrm{Ker}(f)$ *such that the representation* (2) *holds.*

Remark 2. If E is smooth, then $g_0 \in \mathscr{P}_{\mathrm{Ker}(f)}(x_0)$ if and only if the representation (2) holds.

By the use of Proposition 3 we can prove the second characterization of L-best approximation element.

Proposition 4. *Let* G *be a closed linear subspace in* E, $[\,,\,]$ *be a L-semi-inner product which generates its norm,* $x_0 \in E \setminus G$ *and* $g_0 \in G$. *Then* $g_0 \in \mathscr{P}^L_G(x_0)$

if and only if for all functional $f \in (G \oplus \mathrm{Sp}(x_0))^*$ *such that* $\mathrm{Ker}(f) = G$, *then the following representation holds*

$$f(x) = \|f\|_{G \oplus \mathrm{Sp}(x_0)}[x,\ \lambda_0(x_0 - g_0)/\|x_0 - g_0\|] \tag{3}$$

for all $x \in G \oplus \mathrm{Sp}(x_0)$, *where* λ_0 *is as above.*

The following result also holds.

Corollary. *Let* G *and* x_0 *be as above. Then* $\mathscr{P}_G^L(x_0)$ *contains at least one* [*at most one* (*a unique*)] *element if and only if for all* $f \in (G \oplus \mathrm{Sp}(x_0))^*$ *such that* $\mathrm{Ker}(f) = G$ *there exists at least one* [*at most one* (*a unique*)] *element* $g_0 \in G$ *such that* (3) *holds.*

Remark 3. If E is smooth, then $g_0 \in \mathscr{P}_G(x_0)$ iff the representation (3) is valid.

3 - Characterizations of semitchebychefian, proximinal and tchebychefian subspaces in the sense of Lumer

Firstly, we recall these concepts in the classic sense.

A linear subspace G, $G \neq E$, is called *proximinal* [*semitchebychefian* (*tchebychefian*)] *in* E if for every $x_0 \in E$ the set $\mathscr{P}_G(x_0)$ contains at least one [at most one (a unique)] element.

Def. 2. Let E be a normed linear space and $[\,,\,]$ be a L-semi-inner product which generates its norm. The linear subspace G, $G \neq E$, is called *proximinal* [*semitchebychefian* (*tchebychefian*)] *in the sense of Lumer relative to* $[\,,\,]$ if $\mathscr{P}_G^L(x_0)$ contains at least one [at most one (a unique)] element for all x_0 in E.

The following theorem of characterization holds.

Theorem 1. *Let* G *be a closed linear subspace in* E, $[\,,\,]$ *be a* L*-semi-inner product on it which generates the norm. Then* G *is semitchebychefian* [*proximinal* (*tchebychefian*)] *in the sense of Lumer if and only if for all* $x \in E$ *there exists at most one* [*at least one* (*a unique*)] *element* $x' \in G$ *and at most one* [*at least one* *(a unique)] element* $x'' \in G^L$ *such that*

$$x = x' + x'' \tag{4}$$

and we denote this $E = G \boxplus G^L [E = G + G^L (E = G \oplus G^L)]$.

The proof is obvious by the definitions of semitchebychefian, proximinal and tchebychefian linear subspaces in Lumer's sense and from Corollary of Proposition 2. We omit the details.

Remark 4. If E is smooth then G is semitchebychefian [proximinal (tchebychefian)] iff $E = G \boxplus G^L [E = G + G^L (E = G \oplus G^L)]$.

It is known that a finite-dimensional linear subspace in a normed linear space is proximinal. We shall improve this result.

Proposition 5. *Let E be a normed space and $[,]$ be a L-semi-inner product which generates its norm. Then every finite-dimensional linear subspace in it is proximinal in the sense of Lumer.*

Proof. Let G_n be a n-dimensional linear subspace in E and $x_0 \in E \setminus G_n$. Put $G_{n+1} = G_n \oplus \mathrm{Sp}(x_0)$. Then G_n can be ragarded as a hyperplane in G_{n+1}.

On the other hand, let $\{x_2, \ldots, x_{n+1}\}$ be a base in G_n and $x_1 \in G_{n+1} \setminus G_n$ such that $\{x_1, x_2, \ldots, x_{n+1}\}$ is also a base in G_{n+1}. We construct the vectors

$$e_1 = x_1/\|x_1\| \quad e_2 = x_2 - [x_2, e_1] e_1, \ldots, e_{n+1} = x_{n+1} - \sum_{i=1}^{n} [x_{n+1}, e_i] e_i .$$

It is easy to see that

$$[e_2, e_1] = [e_3, e_1] = \ldots = [e_{n+1}, e_1] = 0$$

and since

$$x_1 = \|x_1\| e_1 \quad x_2 = [x_2, e_1] e_1 + e_2 \ldots \quad x_{n+1} = \sum_{i=1}^{n} [x_{n+1}, e_1] e_i + e_{n+1}$$

we have $\{e_1, e_2, \ldots, e_{n+1}\}$ is a base in G_{n+1} and $\{e_2, \ldots, e_{n+1}\}$ is also a base in G_n. Then $[u, e_1] = 0$ for all $u \in G_n$ and since $e_1 = \lambda_0 x_0 + u_0$ with $\lambda_0 \in K \setminus \{0\}$ and

$u_0 \in G_n$, we obtain: $[u, x_0 - v_0] = 0$ for all $u \in G_n$, where $v_0 := -\frac{1}{\lambda_0} u_0 \in G_n$, i.e., $x_0 - v_0 L G_n$ what is equivalent to $v_0 \in \mathscr{P}_G^L(x_0)$ and the proposition is proven.

Consequences: I. Let E be a normed linear space, $[\,,\,]$ be a L-semi-inner product on it which generates the norm and G be a finite-dimensional linear subspace in it. Then

$$E = G + G^L .$$

II. Let L be a smooth (and strict convex) normed space and G be its linear subspace. If $S_G := \{g \in G \| \|g\| \le 1\}$ is weakly sequentially compact in E then the following decomposition holds

$$E = G + G^L (E = G \oplus G^L) . \tag{5}$$

If E is reflexive (and strict convex) then for all closed linear subspace G in E the decomposition (5) is valid (see also $[2]_2$).

The proof of first statement follows by Klee's theorem (see [5] or [10], Corollary 2.1) and by the above theorem. The second assertion is obvious.

III. Let E be a normed linear space and suppose that E^* endowed with the canonical norm is smooth (and strict convex) normed space. If F is a linear subspace in E^* and F is $\sigma(E^*, E)$-closed or $S_F := \{h \in F \| \|h\| \le 1\}$ is compact in $\sigma(E^*, E)$ or S_F is weak* sequentially compact in E^*, then the following decomposition holds

$$E^* = F + F^L \qquad (E^* = F \oplus F^L) . \tag{6}$$

The proof follows by Phelps' theorems (see [8], p. 239 or [10], Corollary 2.5 and Theorem 2.2) by Klee's theorem (see [5] or [10], Theorem 2.3) and by Theorem 1 for the smooth case (see also $[2]_2$).

The following result establishes a connection between proximinal [semitchebychefian (tchebychefian)] linear subspace in the sense of Lumer and the representation of continuous linear functional on a normed linear space in terms of L-semi-inner products.

Theorem 2. *Let f be a nonzero continuous linear functional on normed space E and $[\,,\,]$ be a L-semi-inner product which generates its norm. Then the following statements are equivalent:*

(i) Ker(f) *is proximinal [semitchebychefian (tchebychefian)] in the sense of Lumer;*

(ii) *there exists at least one [at most one (a unique)] element $u_f \in E$, $\|u_f\| = 1$ such that the following representation holds*

$$f(x) = \|f\| \; [x, \; u_f] \qquad \text{for all } x \in E\,. \tag{7}$$

Firstly, we shall prove the following lemma what is important in themselves too.

Lemma 3. *Let H be a closed linear hyperplane containing the null element and $[\,,\,]$ be a L-semi-inner product which generates the norm of E. Then H is proximinal in the sense of Lumer if and only if there exists a nonzero element u in X such that uLH.*

Proof. If H is proximinal in the sense of Lumer and $x_0 \in E \setminus H$, then there exists an element $g_0 \in H$ such that $g_0 \in \mathscr{P}^L_H(x_0)$ and putting $u := x_0 - g_0$ we have uLH and $u \neq 0$.

Conversely, assume that $x_0 \in E \setminus H$, $u \in E$, uLH and $u \neq 0$ and let f be a nonzero continuous linear functional on X such that $H = \mathrm{Ker}(f)$. If we choose $g_0 := x_0 - f(x_0)/f(u)\,u$ ($f(u) \neq 0$ because $u \notin H$) we have $g_0 \in \mathrm{Ker}(f)$ and since

$$[y, \; x_0 - g_0] = (\overline{f(x_0)/f(u)})[y, \; u] = 0 \qquad \text{for all } y \in H$$

we deduce $g_0 \in \mathscr{P}^L_H(x_0)$, i.e., H is proximinal in the sense of Lumer and the lemma is proved.

Proof of the Theorem 2. (i) $\Rightarrow$ (ii).(a). Let Ker(f) be proximinal in the sense of Lumer. Then by Lemma 3 there exists $w_0 \in E \setminus \mathrm{Ker}(f)$ such that $w_0 L$ Ker(f). By an argument similar to that in the proof of Proposition 3 we have

$$f(x) = [x, \; \overline{f(w_0)}\, w_0/\|w_0\|^2] \quad \text{for all } x \in E \qquad \|f\| = |f(w_0)|/\|w_0\|\,.$$

Now, let $\lambda_0 := \overline{f(w_0)}/|f(w_0)| \in K$ and put $u_f := \lambda_0 w_0/\|w_0\|$ then we obtain representation (7).

(ii) $\Rightarrow$ (i).(a). Suppose that $u_f \in E$, $\|u_f\| = 1$ verifies (7). Then $u_f L \operatorname{Ker}(f)$ and by Lemma 3 it follows that $\operatorname{Ker}(f)$ is proximinal in the sense of Lumer.

(i) $\Rightarrow$ (ii).(b). Assume that $\operatorname{Ker}(f)$ is semitchebychefian in the sense of Lumer and suppose, by absurd, that there exists two distinct elements u_f, $v_f \in E$, $\|u_f\| = \|v_f\| = 1$ such that they satisfy (10). Then u_f, $v_f \in \operatorname{Ker}(f)^L$. Now, let $x \in E \setminus \operatorname{Ker}(f)$ and put

$$y_0 := x - f(x)\, u_f/f(u_f) \qquad y_0' := x - f(x)\, v_f/f(v_f) .$$

Then $f(y_0) = f(y_0') = 0$, i.e., y_0, $y_0' \in \operatorname{Ker}(f)$.

On the other hand, for all $y \in \operatorname{Ker}(f)$ we have

$$[y,\ x - y_0] = (\overline{f(x)/f(u_f)})[y,\ u_f] = 0$$

and a similar relation for y_0'. Consequently, $x - y_0$, $x - y_0'$ $L \operatorname{Ker}(f)$, i.e., y_0, $y_0' \in \mathscr{P}^L_{\operatorname{Ker}(f)}(x)$. Now, if we assume that $y_0 = y_0'$ we derive $u_f/f(u_f) = v_f/f(v_f)$ and since $f(u_f) = f(v_f) = \|f\|$ one gets $u_f = v_f$. In conclusion, $y_0 \neq y_0'$ and since y_0, $y_0' \in \mathscr{P}^L_{\operatorname{Ker}(f)}(x)$ we obtain a contradiction to the fact that $\operatorname{Ker}(f)$ is semitchebychefian in the sense of Lumer and the implication is proven.

(ii) $\Rightarrow$ (i).(b). Assume that (7) holds for a unique element $u_f \in E$, $\|u_f\| = 1$ and suppose, by absurd, that there exists $x_0 \in E \setminus \operatorname{Ker}(f)$ and two distinct elements g_0 and g_0' in $\mathscr{P}^L_{\operatorname{Ker}(f)}(x_0)$. As above, we obtain

$$f(x) = [x,\ \overline{f(x_0)}(x_0 - g_0)/\|x_0 - g_0\|^2] \quad x \in E \qquad \|f\| = |f(x_0)|/\|x_0 - g_0\|$$

and a similar representation for g_0'. Put

$$u_f := \overline{f(x_0)}(x_0 - g_0)/(|f(x_0)|\|x_0 - g_0\|) \qquad v_f := \overline{f(x_0)}(x_0 - g_0')/(|f(x_0)|\|x_0 - g_0'\|) .$$

Then $\|u_f\| = \|v_f\| = 1$ and u_f, v_f satisfy (7). Now, if we assume that $u_f = v_f$, we derive $(x_0 - g_0)/\|x_0 - g_0\| = (x_0 - g_0')/\|x_0 - g_0'\|$ and since $\|x_0 - g_0\| = |f(x_0)|/\|f\| = \|x_0 - g_0'\|$ we obtain $g_0 = g_0'$. Consequently, there exists two distinct elements u_f, $v_f \in E$, $\|u_f\| = \|v_f\| = 1$ and they satisfy (7), what produce a contradiction and the proof is finished.

(i) $\Leftrightarrow$ (ii).(c). The statement: $\operatorname{Ker}(f)$ is tchebychefian in Lumer's sense if and only if there exists a unique element $u_f \in E$, $\|u_f\| = 1$ such that (7) holds, follows by the above arguments.

The next corollary contains a characterization of proximinal [semitchebychefian (tchebychefian)] linear subspaces in the sense of Lumer in normed linear spaces in terms of continuous linear functionals.

Corollary. *Let G be a closed linear subspace in normed linear space E, $G \neq E$, and $[\,,]$ be a L-semi-inner product which generates its norm. Then the following statements are equivalent:*

(i) *G is proximinal [semitchebychefian (tchebychefian)] in the sense of Lumer;*

(ii) *for all $x_0 \in E \setminus G$ and for any $f \in (G \oplus \mathrm{Sp}(x_0))^*$ such that $\mathrm{Ker}(f) = G$, there exists at least one [at most one (a unique)] element $u_{x_0,f} \in G \oplus \mathrm{Sp}(x_0)$, $\|u_{x_0,f}\| = 1$ with the property: $f(x) = \|f\|_{G \oplus \mathrm{Sp}(x_0)}[x, u_{x_0,f}]$ for all $x \in F \oplus \mathrm{Sp}(x_0)$.*

The proof follows by the previous theorem for the normed linear space $E_{x_0} := G \oplus \mathrm{Sp}(x_0)$. We omit the details.

The following consequences are interesting in themselves too.

Consequences: I. Let E be a normed space, $[\,,]$ be a L-semi-inner product which generates its norm and G be a finite-dimensional linear subspace in E. Then for all nonzero continuous linear functional f on E there exists at least one element $u_{G,f} \in G$, $\|u_{G,f}\| = 1$ such that

$$f(x) = \|f\| \; [x, u_{G,f}] \qquad \text{for all } x \in G.$$

II. Let E be a smooth (and strict convex) normed space and f be a nonzero continuous linear functional on it. If $S_{\mathrm{Ker}(f)} = \{h \in \mathrm{Ker}(f) \mid \|h\| \leq 1\}$ is weakly sequentially compact in E, then there exists an (a unique) element $u_f \in E$, $\|u_f\| = 1$ such that (7) holds.

Finally, if we assume that E is reflexive (and strict convex) then for all $f \in E^* \setminus \{0\}$ there exists an (a unique) element $u_f \in E$, $\|u_f\| = 1$ such that (7) holds (see also $[2]_2$).

The proof of first statement follows by Klee's Theorem (see [5] or Corollary 3.1 from [10]) and by the above theorem. The second statement is obvious.

III. Let E be a normed linear space and suppose that E^* endowed with the canonical norm is smooth (and strict convex). If $\Phi \in E^{**} \setminus \{0\}$ satisfies the conditions $\mathrm{Ker}(\Phi)$ is $\sigma(E^*, E)$-closed or $S_{\mathrm{Ker}(f)}$ is compact in $\sigma(E^*, E)$ or $S_{\mathrm{Ker}(f)}$ is weak* sequentially compact in E^*, then there exists a (a unique) functional $f_\Phi \in E^*$, $\|f_\Phi\| = 1$ such that the following representation holds

$$\Phi(f) = \|\Phi\| \, [f, f_\Phi]^* \qquad \text{for all } f \in E^*$$

where $[\,,\,]^*$ is the unique semi-inner product which generates the norm of E^* (see also $[2]_2$).

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Résumé

Dans cet article on introduit la notionne de la meilleure approximation dans le sense de Lumer et on donne quelques caractérisations et applications.

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