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Exterior concurrent vector fields on a pseudo-Riemannian manifold endowed with a D-contact structure (**)

Introduction

Riemannian or pseudo-Riemannian manifolds endowed with a *D-contact structure* have been defined by R. Rosca $[4]_1$.

In the present paper we consider a pseudo-Riemannian manifold $M(\mathcal{U}, \eta, \xi, g, \lambda)$ endowed with a D-contact structure, and such that the (1.1)-structure tensor field $\mathcal{U}$, defines on M a *pseudo-Sasakian structure* (R. Rosca $[4]_2$). On the other hand the concept of *exterior concurrent vector field* (abr. e.c.) has also been recently defined by R. Rosca in [4], and M. Petrovic, R. Rosca and L. Verstraelen [2].

We recall that if M is any Riemannian or pseudo-Riemannian oriented C^∞-manifold with soldering form dp, then a vector field Z of M such that (a) $\nabla^2 Z = u \wedge dp$, is defined as exterior concurrent; (in (a) $\nabla^2 Z$ means the *second covariant differential* of Z and u is a certain 1-form). It is proved in this paper that the necessary and sufficient condition that the structure vector field ξ be e.c. is that it defines on *infinitesimal homotety* on M. This implies $\lambda = \text{const}$. In this case if Z is any e.c. vector field of M, then this property is invariant by operating $\mathcal{U}$ to Z. Further, let M_A be a Riemannian normal anti-invariant (K. Yano and M. Kon [5]) submanifold of a manifold M e.c. vector field ξ. If any vector field X of M_A is e.c. then M_A is a *space form* of elliptic type and the normal connection $\nabla^\perp$ associated with the immersion $x: M_A \to M(\mathcal{U}, \eta, \xi, g, \lambda)$ is *flat*.

We study here the case when the structure vector field ξ is e.c.

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1 - Preliminaries

Let $M(\eta, \xi, g)$ be a contact pseudo-Riemannian manifold with η (resp. ξ) as canonical 1-form (resp. canonical vector field) and let ∇ be the covariant differentiation operator defined by the metric tensor g.

Let TM be the *tangent bundle* of M, and let $\Gamma TM = \chi M$ be the set of *sections* of TM. If for any vector field $Z \in \chi M$ one has $g(\nabla_Z \xi, Z) = \lambda g(Z, Z)$ where $\lambda = \operatorname{div} \xi$ (λ = nowhere vanishing scalar field), then the triple (η, ξ, λ) defines a *D-contact structure* (R. Rosca $[4]_1$).

This definition is valid also in the case when g has a Riemannian structure. If we set $A^q(M, TM) = \Gamma \operatorname{Hom}(\Lambda^q TM, TM)$ we notice that the elements of $A^q(M, TM)$ are *vectorial q-form* (W. Poor [3]). In the following we assume that $M(\eta, \xi, \lambda, g)$ is not flat and that TM is *trivial*.

Further we assume that g is of signature $m+1$, m and that M is endowed with a *pseudo-Sasakian structure* (R. Rosca $[4]_2$). At each point $p \in M$ one has the decomposition $T_p(M) = H_p \oplus \mathcal{J}_p$, where T_p, H_p and $\mathcal{J}_p$ are the tangent space, a $2m$-dimensional neutral vector space and a time like line orthogonal to H_p, respectively.

If $\mathcal{U}$ is the (1.1) tensor field of the pseudo-Sasakian structure and S_p (resp. S_p^*) the time like (resp. space-like) components of H_p, then $\mathcal{U}$ ($\mathcal{U}$ is also called the para *complex operator* [1]) defines an *involutive automorphism*. Since $\mathcal{U}^2 = 1$, one has

$$\mathcal{U} e_a = e_a^* \qquad \mathcal{U} S_p = S_p^* \qquad \mathcal{U} S_p^* = S_p \tag{1.1}$$

$$\mathcal{U} Z = 0 \quad \text{for any } Z \in \{\mathcal{J}_p\}. \tag{1.2}$$

The structure tensor fields satisfy

$$\begin{gathered} \mathcal{U}^2 Z = Z - \eta(Z) \qquad g(Z, \xi) = \eta(Z) \qquad \eta(\mathcal{U}Z) = 0 \\ \eta(Z) = 1 \qquad g(\mathcal{U}Z, \mathcal{U}Z') = -g(Z, Z') + \eta(Z)\eta(Z') \\ d\eta(Z, Z') = -2g(\mathcal{U}ZZ') \qquad \nabla_Z \xi = \lambda(Z - \eta(Z)\xi) + \mathcal{U}Z \to \nabla_\xi \xi = 0. \end{gathered} \tag{1.3}$$

Now let $\mathcal{O} = \operatorname{vect}\{e_a, e_{a^*}, e_0 = \xi | a = 1, \ldots, m, a^* = a + m\}$ be a local field of orthonormal frames over M, where $e_a \in S_p$, $e_{a^*} \in S_p^*$, $\xi \in \mathcal{J}_p$.

Next denote by $\mathcal{O}^* = \{\omega^a, \omega^{a^*}, \eta\}$ the associated coframe of $\mathcal{O}$ and by $\omega_B^A = \gamma_{BC}^A \omega^C$ $(A, B, C = 0, 1, \ldots, 2m)$ and Ω_B^A the connection forms and the curvature 2-forms respectively on M. Taking into account the signature of g then with the help of (1.3), one finds that the soldering form $dp \in A^1(M, TM)$ and the structure equations (E. Cartan) are given by [1]

$$dp = \omega^a \otimes e_a - \omega^{a^*} \otimes e_{a^*} + \eta \otimes \xi \tag{1.4}$$

$$\begin{aligned} &\nabla e_a = \omega_a^b \otimes e_b - \omega_a^{b^*} \otimes e_{b^*} + (\omega^{a^*} - \lambda\omega^a) \otimes \xi \\ &\nabla e_{a^*} = \omega_{a^*}^b \otimes e_b - \omega_{a^*}^{b^*} \otimes e_{b^*} + (\omega^a - \lambda\omega^{a^*}) \otimes \xi \qquad \nabla\xi = \lambda\, \mathcal{U}^2 dp + \mathcal{U} dp \end{aligned} \tag{1.5}$$

$$\begin{aligned} &d\omega^a = \omega^b \wedge \omega_b^a - \omega^{b^*} \wedge \omega_{b^*}^a + \eta \wedge (\lambda\omega^a - \omega^{a^*}) \\ &d\omega^{a^*} = \omega^b \wedge \omega_b^{a^*} - \omega^{b^*} \wedge \omega_{b^*}^{a^*} + \eta \wedge (\lambda\omega^{a^*} - \omega^a) \qquad d\eta = 2\Sigma\omega^a \wedge \omega^{a^*} \end{aligned} \tag{1.6}$$

$$\begin{aligned} &d\omega_b^a = \Omega_b^a + \omega_b^c \wedge \omega_c^a - \omega_b^{c^*} \wedge \omega_{c^*}^a + \omega_b^0 \wedge \omega_0^a \\ &d\omega_{b^*}^{a^*} = \Omega_{b^*}^{a^*} + \omega_{b^*}^c \wedge \omega_c^{a^*} - \omega_{b^*}^{c^*} \wedge \omega_{c^*}^{a^*} + \omega_{b^*}^0 \wedge \omega_0^{a^*} \end{aligned} \tag{1.7}$$

where

$$\omega_0^a = \lambda\omega^a - \omega^{a^*} \qquad \omega_0^{a^*} = \lambda\omega^{a^*} - \omega^a. \tag{1.8}$$

In addition since $\mathcal{O}$ defines an $\mathcal{U}$-orthogonal vector basis $[4]_1$, one derives from (1.5) and (1.3)

$$\omega_b^a + \omega_{b^*}^{a^*} = 0 \qquad \omega_{b^*}^a + \omega_b^{a^*} = 0 \tag{1.9}$$

and the above implies

$$\Omega_b^a + \Omega_{b^*}^{a^*} = 0 \qquad \Omega_{b^*}^a + \Omega_b^{a^*} = 0. \tag{1.10}$$

On the other hand let $Z = Z^A e_A \in \chi M$ be any vector field of M and let $R \in \mathrm{End}(\Lambda^2 TM)$ be the curvature operator on M. As is known the *second covariant differential* $\nabla^2 Z = d^\nabla(\nabla Z)$ of $Z(\nabla^2 Z(U, V) = R(U, V)Z;\ U, V \in \chi M)$ is a vectorial 2-form; i.e. $\nabla^2 Z \in A^2(M, TM)$. One denotes by $d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$ the *exterior covariant derivative operator with respect* *to* ∇ [4]$_2$ ($d^{\nabla^2} = d^{\nabla} o d^{\nabla}$ is not always zero unlike d^2). If Z satisfies $\nabla^2 Z = u \wedge dp$ for a certain 1-form $u \in \Lambda^1 M$ then according to the definition or R. Rosca [4]$_{3,4}$ Z is called an *exterior concurrent vector field.* In this case u is called the *concurrence 1-form* associated with X.

2 — Referring to the expression (1.5) of $\nabla\xi$, and to (1.1) one may write

$$\nabla\xi = \lambda dp + \mathcal{U} dp - \lambda\eta \otimes \xi. \tag{2.1}$$

Taking the exterior covariant derivative $\nabla^2 \xi = d^{\nabla}(\nabla\xi)$ of one gets first of all

$$\nabla^2 \xi = (d\lambda + \lambda^2 \eta) \wedge dp + \lambda\eta \wedge \mathcal{U} dp(d\lambda \wedge \eta + \lambda d\eta) \otimes \xi + d^{\nabla}(\mathcal{U} dp). \tag{2.2}$$

On the other hand one has

$$\mathcal{U} dp = \omega^a \otimes e_{a*} - \omega^{a*} \otimes e_a. \tag{2.3}$$

Hence making use of (1.5) and (1.6) a straightforward calculation gives

$$d^{\nabla}(\mathcal{U} dp) = -\lambda\eta \wedge \mathcal{U} dp + \lambda d\eta \otimes \xi - \eta \wedge dp. \tag{2.4}$$

So by (2.4) equation (2.2) moves to

$$\nabla^2 \xi = (d\lambda - f^2 \eta) \wedge dp - (d\lambda \wedge \eta) \otimes \xi \tag{2.5}$$

where we have set

$$f^2 = 1 - \lambda^2. \tag{2.6}$$

Therefore according to (1.11) the necessary and sufficient condition that ξ be exterior concurrent is

$$\lambda = \text{const} \neq 1. \tag{2.7}$$

Hence equation (2.5) moves to

$$\nabla^2 \xi = -f^2 \eta \wedge dp. \tag{2.8}$$

Since by definition $\lambda = \operatorname{div} \xi$, one may say that the necessary and sufficient condition that ξ be e.c. is that ξ defines an *infinitesimal homotety* on M. In the following we shall consider $M(\mathcal{U}, \eta, \xi, g)$ for which ξ satisfies (2.8). Such a manifold will be called a *pseudo-Riemannian manifold endowed with an exterior concurrent D-contact structure* (abr. e.c.D.c.-structure). If Z is any vector field of M one finds by (1.3) the intrinsec equation

$$\mathcal{U}\nabla Z = \nabla\, \mathcal{U}Z + \eta(Z)\, dp + \lambda\eta(Z)\, dp - \eta(Z)\, \eta \otimes \xi \tag{2.9}$$

which is coherent with (2.1). Next since by Cartan's structure Eqs. one has

$$\nabla^2 \xi = \Omega_0^a \otimes e_a - \Omega_0^{a^*} \otimes e_{a^*} \tag{2.10}$$

comparaison with (2.8) gives instantly

$$\Omega_0^a = -f^2 \eta \wedge \omega^a \qquad \Omega_0^{a^*} = f^2 \eta \wedge \omega^{a^*}. \tag{2.11}$$

On the other hand if

$$Z = Z^A e_A \in \chi_M \tag{2.12}$$

is any vector field of M, then by structure eqs. the second covariant derivative, of Z is gives by

$$\begin{aligned} \nabla^2 Z = {} & (\Omega_b^a Z^b + \Omega_{b^*}^a Z^{b^*} + \Omega_0^a Z^0) \otimes e_a + (\Omega_b^{a^*} Z^b - \Omega_{b^*}^{a^*} Z^{b^*} - \Omega_0^{a^*} Z^0) \otimes e_{a^*} \\ & + (\Omega_a^0 Z^a - \Omega_{a^*}^0 Z^{a^*}) \otimes \xi . \end{aligned} \tag{2.13}$$

Assume that Z is e.c., that is satisfies (1.11). Then by (2.11) and (2.13) one derives from (1.11)

$$u = -f^2 \flat(Z). \tag{2.14}$$

In the above

$$\flat(Z) = \flat : TM \to T^* M = \Sigma_A Z^A \omega^A \tag{2.15}$$

means the *musical isomorphism* [3] defined by g. It is worth to out line that equation (2.14) is in accordance with the basic properties of any e.c. vector field.

Further by (1.3) and (2.3) one finds after some calculations

$$\mathcal{U}\nabla^2 Z = \nabla^2 \mathcal{U}Z + \eta(Z)\nabla^2 \xi . \tag{2.16}$$

Hence if Z is e.c., one finds by (1.11), (2.8) and (2.14)

$$\nabla^2 \mathcal{U}Z = f^2 \flat(\mathcal{U}Z) \wedge \mathrm{d}p + f^2 \eta(Z)\eta \wedge \mathrm{d}p . \tag{2.17}$$

Therefore one may write

$$\nabla^2 \mathcal{U}Z = f^2(\flat(\mathcal{U}Z) + \eta(Z)\eta) \wedge \mathrm{d}p \tag{2.18}$$

and the above proves that the property for Z to be e.c. is invariant by operating $\mathcal{U}$ to Z.

Theorem. *Let $M(\mathcal{U}, \eta, \xi, g, \lambda)$ be a pseudo-Riemannian endowed with a D-contact structure. Then the necessary and sufficient condition that the structure vector field ξ be exterior concurrent, is that ξ defines and infinitesimal homotety on M. In this case, if Z is any exterior recurrent vector of M, then this property is invariant by operating $\mathcal{U}$ to Z.*

3 — Let $x: M_A \to M(\mathcal{U}, \eta, \xi, g, \lambda)$ the proper immersion of *normal antinvariant* $[4]_2$ submanifold M_A of dimension m in $M(\mathcal{U}, \eta, \xi, g, \lambda)$.

Since by definition ξ belongs to the normal bundle $T^\perp M_A$ of M_A such a manifold is defined by the completely integrable Pfaff system

$$\omega^{a^*} = 0 \qquad \eta = 0 \tag{3.1}$$

(we have assumed that M_A is endowed a Riemannian metric tensor g_A). In this case the soldering form $\mathrm{d}p_A$ of M_A is given by

$$\mathrm{d}p_A = \omega^a \otimes e_a \Rightarrow g_A = \Sigma_a(\omega^a)^2 \tag{3.2}$$

(we denote the elements induced by x with the same letters).

Assume that every vector field X on M_A is exterior concurrent.

Then according to $[4]_3$, one knows that M_A is a *space form*.

By (2.13), (2.14) and (3.2) an easy calculation gives for the curvature forms Ω^a_b

of M_A, the following expressions

$$\Omega_b^a = f^2 \omega^a \wedge \omega^b . \tag{3.3}$$

As is known the above proves that M_A is a space form $M(f^2)$ of *elliptic* type. Denote by $T_{p_A}(M_A)$ (resp. $T_{p_A}^{\perp}(M_A)$) the tangent space (resp. the normal space) at $\forall p_A \in M_A$. Since M_A is an antinvariant submanifold of M, then as is known $\mathcal{U} T_{p_A}(M_A) \subseteq T_{p_A}^{\perp}(M_A)$. It follows then by (2.18), that one has

$$\nabla^2 \mathcal{U} X = 0 \tag{3.4}$$

for any tangent vector X of M_A ($\mathcal{U} X$ is obviously normal).

The above equations implies that the curvature tensor $R^{\perp}$ in the normal bundle of M_A vanishes identically. One says in this case (K. Yano and M. Kon [5]) that the connection $\nabla^{\perp}$ associated with $x : M_A \rightarrow M(\mathcal{U}, \eta, \xi, g, \lambda)$ is flat.

Next by the last equation (1.5) and by (3.1) one may write

$$\nabla \xi = \lambda(\omega^a \otimes e_a) + \omega^{a^*} \otimes e_{a^*} . \tag{3.5}$$

But ξ being a *normal section* of M_A, the *second fundamental quadritic form* l_ξ associated with ξ is as known $l_\xi = -\langle dp_A, \nabla \xi \rangle$. Hence by (3.5) and (3.2) one finds $l_\xi = -\lambda g_A$, and this expression proves that ξ is an *umbilical section.*

Theorem. *Let M_A be a Riemannian normal anti-invariant and m-dimensional submanifold of the manifold $M(\mathcal{U}, \eta, \xi, g, \lambda)$ discussed in* 2. *If any vector field X of M_A is exterior concurrent, then M_A is space-form of elliptic type and the normal connection ∇ associated with the immersion $x : M_A \rightarrow M(\mathcal{U}, \eta, \xi, g, \lambda)$ is flat. Further, the structure vector field ξ defines an umbilical section on M_A.*

References

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Riassunto

In questo lavoro si considera una pseudo varietà Riemanniana $M(\mathcal{U}, \eta, \xi, g, \lambda)$ *dotata di una D-struttura di contatto (R. Rosca) e tale che* (1.1) *tensore* U *definisce una struttura pseudo Sasakiana. Si dimostra che la condizione necessaria e sufficiente perché il vettore di struttura* ξ *sia «exterior concurrent» (R. Rosca) è che* ξ *definisca una omotetia infinitesimale sulla* M. *In questa situazione vengono studiate alcune sottovarietà anti-invarianti di* M.

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