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A point-distinguishing edge-coloring problem (**)

1 - Introduction

The point-distinguishing (p.d.) chromatic index of a graph $G=(V, E)$, denoted by $\chi_0(G)$, is the minimum number of colors assignable to E so that no two distinct points are incident with the same color sets of edges.

The problem of characterizing the spanning subgraphs H of a graph G for which

$$\chi_0(G) = 1 + \chi_0(H) \tag{1}$$

was posed in [2]. The following Theorem 2.1 settles this problem.

Moreover, in Propositions 3.1 and 4.2 we determine values of n for which K_n and $K_{n,n}$ do not contain spanning subgraphs satisfying (1), while Propositions 3.2 and 4.4 prove the existence of, and give a construction for similar subgraphs in the remaining cases.

We call a spanning subgraph of G satisfying (1) a (1)-*spanning subgraph.*

We denote by $P(k)$ the power set of $N_k = \{1, 2, ..., k\}$. A set assignment for G is an assignment of one member S_i of $P(k)$ to each vertex of G such that no two vertices are assigned to the same set. Let $\{v\}$ denote the set assigned to the vertex v and $\{v\} \setminus x$, where $x \in \{v\}$, the set assigned to the vertex v but the element x.

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2 - Characterization

In [2] the following relation for a graph G and a spanning subgraph H of G was shown

$$\chi_0(G) \leqslant \chi_0(H) + 1 . \tag{2}$$

We now have the following

Theorem 2.1. *A spanning subgraph H of a graph G satisfies the relation $\chi_0(G) = \chi_0(H) + 1$ if and only if there exists a point-distinguishing χ_0-coloring of G containing a color x such that*

(i) *for every x-colored edge $e = (v, w)$, $\{v\} \setminus x$ and $\{w\} \setminus x$ are non-empty and distinct from the remaining color sets;*

(ii) *H does not contain x-colored edges.*

Proof. Let H be a (1)-spanning subgraph of G; we prove that (i) and (ii) hold.

Let $\chi_0(H) = k$ and let $e_1, e_2, \ldots, e_s$ be the edges removed from G to obtain H, where $e_1 = (v_{i_1}, v_{i_2})$. The sets $\{v_{i_1}\}$, $\{v_{i_2}\}$ are distinct and non-empty in a p.d. k-coloring of H.

If $\{v_{i_1}\} \cap \{v_{i_2}\} \neq \emptyset$ we color e_i with any color common to $\{v_{i_1}\}$ and $\{v_{i_2}\}$. In this way the color sets assigned to v_{i_1} and v_{i_2} in G are not changed with respect to H.

This situation is not possible for every edge e_i, because otherwise $\chi_0(G)$ and $\chi_0(H)$ would be equal. Let $e_j = (v_{j_1}, v_{j_2})$ be an edge such that $\{v_{j_1}\}$, $\{v_{j_2}\}$ are disjoint, obviously non-empty and distinct from the remaining sets. We color every such edge e_j by a new color x; thus (i) and (ii) are satisfied.

Now, we suppose that H is a spanning subgraph of G and let (i) and (ii) hold. We prove that H satisfies (1).

Let $\chi_0(G) = k + 1$. By (i), we see that the color sets obtained by deleting the x-colored edges of G are non-empty and distinct from the remaining ones. By (ii), H does not contain x-colored edges. So H has a p.d. h-coloring, where $h \leqslant k$.

By (2), we see that $\chi_0(H) \geqslant k$. Thus $\chi_0(H) = k$; that is, H satisfies (1).

3 - The case $G = K_n$

Harary and Plantholt [2] showed that

$$\chi_0(K_n) = \lceil \log_2 n \rceil + 1 . \tag{3}$$

This implies that $\chi_0(K_n) = k$, for every n satisfying $2^{k-2} < n < 2^{k-1}$.

Proposition 3.1. *For* $n = 2^{k-1}$ $(k > 3)$, K_n *does not contain* (1)*-spanning subgraphs.*

Proof. By (3), we see that, for $n = 2^{k-1}$, $\chi_0(K_n) = k$. We prove that in this case a p.d. k-coloring of K_n satisfying the condition (i) of Theorem 2.1 does not exist.

Assume that such a coloring exists. Without loss of generality, we can denote $x = k$. In every p.d. coloring of K_n, we have $\{v_i\} \cap \{v_j\} \neq \emptyset$, for $i, j \in \{1, 2, ..., n\}$, because $\{v_i\}$ and $\{v_j\}$ must both contain the color of the edge (v_i, v_j). Thus, if a set $\{v_i\}$ is assigned to the vertex v_i, then no other vertex can be assigned to the complement of $\{v_i\}$ with respect to $N_k = \{1, 2, ..., k\}$.

By condition (i) of Theorem 2.1, there are no color sets $\{v_i\}$, $\{v_j\}$ such that $x \in \{v_i\}$ and $\{v_i\} \setminus x = \{v_j\}$.

In this way we obtain a p.d. k-coloring of K_n for $n = 2^{k-1}$ by assigning to the 2^{k-1} vertices of K_n all the subsets of $P(k-1)$ together with the k-color. So there is a vertex v which is assigned to the monochromatic set $\{k\}$, corresponding to the empty set of $P(k-1)$. This contradicts the condition that $\{v\} \setminus k$ is non-empty.

Proposition 3.2. *For each* n *satisfying* $2^{k-2} < n < 2^{k-1}$ *and* $k > 3$, K_n *contains* (1)*-spanning subgraphs.*

Proof. By (3), for each n satisfying $2^{k-2} < n < 2^{k-1}$, we have $\chi_0(K_n) = k$. As we proved in Proposition 3.1, a k-coloring of K_n, for $2^{k-2} < n < 2^{k-1}$, can be obtained by assigning to the vertices of K_n the subsets of $P(k-1)$, except the empty set, and by coloring the edge (v_i, v_j) with $x = k$, when two sets $\{v_i\}$, $\{v_j\}$ are disjoint.

Because $n > 2^{k-2}$, at least two vertices correspond to disjoint sets of $P(k-1)$; so at least one edge of K_n is x-colored.

Thus the conditions (i) and (ii) of Theorem 2.1 are satisfied and a (1)-spanning subgraphs of G exists.

4 - The case $G = K_{n,n}$

For n-regular complete bipartite graphs with $n \geq 2$ the following bounds were found in [2]

$$\lceil \log_2 n \rceil + 1 \leq \chi_0(K_{n,n}) \leq \lceil \log_2 n \rceil + 2 .$$

These inequalities imply that, for $2^{k-2} \leqslant n \leqslant 2^{k-1}$, $\chi_0(K_{n,n})$ is either k or $k+1$.

In [3], we proved the following

Proposition 4.1. *A* p.d. χ-coloring of $K_{n,n}$ exists if and only if there exists a matrix of order n with elements belonging to $\{1, 2, \ldots, \chi\}$ such that distinct edges correspond to distinct sets.

Let n_0 be the greatest integer n satisfying $2^{k-2} \leqslant n \leqslant 2^{k-1}$, for which $\chi_0(K_{n,n}) = k$.

Proposition 4.2. *For every n satisfying $2^{k-2} \leqslant n \leqslant n_0$, a (1)-spanning subgraph of $K_{n,n}$ does not exist.*

Proof. Suppose that there exists a (1)-spanning subgraph H of $K_{n,n}$, where $2^{k-2} \leqslant n \leqslant n_0$ and $\chi_0(K_{n,n}) = k$.

Then $\chi_0(H) = k-1$ and there are $2n(\geqslant 2^{k-1})$ distinct elements of $P(k-1)$ corresponding to the vertices of H.

The inequality $2n > 2^{k-1}$ is clearly impossible; also the equality is impossible because not every element of $P(k-1)$ can be used. In fact no vertex of H can be assigned to the empty set.

Lemma 4.3. *Let α and β be two elements of $P(k)$ not assigned to the lines of a matrix A corresponding to a k-coloring of $K_{n,n}$.*

Then at least one of α and β is disjoint from some of the sets assigned to the lines of A.

Proof. Suppose that α and β are not disjoint from the sets assigned to the rows of A.

Let $\alpha \cap \beta \neq \emptyset$. By using the procedure given in [3] we can determine a matrix of order $n_0 + 1$ whose lines are the same as A, with the addition of two new non parallel lines corresponding to α and β. Thus we have determined a k-coloring of $K_{n_0+1,\, n_0+1}$; this is a contradiction.

Let $\alpha \cap \beta = \emptyset$. We can suppose that the set $N_k = \{1, 2, \ldots, k\}$ is contained in a line of A (for example a row), because it is not disjoint from any set of $P(k)$ and we could substitute a line of A by N_k. We add α and β to the rows of A, and shift the line corresponding to N_k to the columns. Thus we again obtain a matrix of order $n_0 + 1$ whose lines correspond to elements of $P(k)$, a contradiction.

Proposition 4.4. *For every n satisfying $n_0 < n < 2^{k-1} - 1$, $K_{n,n}$ contains (1)-spanning subgraphs.*

Proof. Let A be a matrix corresponding to a k-coloring of K_{n_0,n_0}; such a matrix exists, by Proposition 4.1.

It was shown in [3] that, for n satisfying $2^{k-1} - [\frac{1}{2}k] \leqslant n \leqslant 2^{k-1}$ and $k \geqslant 3$, $\chi_0(K_{n,n}) = k + 1$. So there are at least two elements α and β of $P(k)$ that are not assigned to the lines of A.

By using the procedure given in [3], it is possible to determine a new row $\bar{r}$ and a new column $\bar{c}$ with respect to the lines of A that correspond to $\alpha \cup \{k+1\}$ and $\beta \cup \{k+1\}$.

In fact, when α (or β) is disjoint from the set η assigned to a line of A, we write $k+1$ at the crossing of the lines corresponding to η and $\alpha \cup \{k+1\}$. Otherwise, if $\alpha \cap \eta \neq \emptyset$, we can write an element of α (or β) so that all the elements of α (or β) are in $\bar{r}$ (or $\bar{c}$) at least once.

In this way we determine a new matrix B of order $n_0 + 1$, whose first n_0 rows and n_0 columns are the same as A, plus $\bar{r}$ and $\bar{c}$.

We can proceed in this way until there are elements of $P(k)$ not yet assigned, with the exception of the empty set. The determined p.d. $(k+1)$-coloring of $K_{n,n}$, where $n_0 < n < 2^{k-1} - 1$, clearly satisfies condition (i) of Theorem 2.1. Moreover, the subgraph obtained by deleting all the $k+1$-colored edges is a (1)-spanning subgraph.

This completes the proof.

References

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[2] F. Harary and M. Plantholt, *The point-distinguishing chromatic index*, Graphs and Applications (Proc.), Wiley-Interscience, New York, 1985 (147-162).

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Sommario

Si determina una caratterizzazione dei sottografi generanti, H, di un grafo G per i quali risulta $\chi_0(G) = 1 + \chi_0(H)$, *ove* $\chi_0(G)$ *è l'indice cromatico con distinzione vertici di un grafo G. In tal modo si ottiene una risposta ad un problema posto da Harary e Plantholt.*

Inoltre, in base a tale risultato, sono determinati i valori di n per i quali K_n *e* $K_{n,n}$ *contengono simili sottografi.*

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