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Acceleration waves in fluid mixtures (**)

1 - Introduction

The growth and decay of acceleration waves have been studied extensively by some authors in the context of the theory of singular surfaces (see [6] and references therein) and in particular for various types of mixtures of solids [2], [4]₃ or fluids [4]₁,₂, [5], [7], [10]. Among the main results concerning fluid mixtures, we remark that, under the hypothesis of common temperatures, the existence of acceleration waves has been proved only for mixtures initially at rest. The presence of exchanges of momentum and energy turns out to increase the critical amplitudes [4]₂, [5] thus stabilizing the wave evolution. In addition, in the case of ideal fluid mixtures, the wave front may drive apart the temperatures of different constituents [5]. The underlying hypothesis of these works is that propagation occurs through a state of thermal equilibrium in which each constituent is at rest and has a common temperature. Such a state is dynamically compatible with the theory if the external sources of momentum and energy are absent.

In this paper we extend the previous model including external sources of energy and allowing each constituent to have a peculiar temperature in the unperturbed state. We introduce response functions which depend on densities, temperatures and velocities of all the constituents, i.e. a non-ideal mixture model. The basic equations of this model are described in 2. In 3 we show the existence of acceleration waves for mixtures with several temperatures and

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derive compatibility conditions for the two cases of omothermal waves and of waves into non-conducting mixtures. Such conditions generalize those obtained in [4]$_2$ for common temperatures and that obtained in [5] for ideal mixtures. The evolution equations for both types of waves are derived in 4, exploiting a general result obtained in [7]. Suitable restrictions on the coefficients of the evolution equations are obtained in 5 on the basis of the second law and the properties of the resulting solutions are discussed in 6 for the significant case of non-conducting mixtures.

2 - Basic equations for mixtures of fluids

Let $\mathcal{M}$ represent a mixture of ν fluids, occupying a time dependent region $\mathcal{R}(t)$. We assume that every place $x \in \mathcal{R}(t)$ be simultaneously occupied by all fluids so that, denoting with X_α the position of a particle of the α-th constituent in a fixed configuration, we may write $x = \chi_\alpha(X_\alpha, t)$ for every $\alpha = 1, ..., \nu$. The functions χ_α are supposed to be C^3 at least; their partial time derivatives represent the velocities v_α of each constituent. Following the standard notations, we denote with a backward prime affixed to a quantity, say $\varphi_\alpha(x, t)$ its material time derivative, i.e. $\grave{\varphi}_\alpha = \dfrac{\partial \varphi_\alpha}{\partial t} + (\nabla \varphi_\alpha) \cdot v_\alpha$. We assign to each constituent an absolute temperature $\vartheta_\alpha(x, t)$ and a mass density $\rho_\alpha(x, t)$ and denote with ε_α, T_α, h_α respectively the internal energy density per unit mass, the stress tensor and the heat flux. Each fluid is allowed to exchange linear momentum and energy with all the other constituents, respectively at the rates m_α and l_α. These last quantities obey the total conservation laws in the form $\sum_\alpha m_\alpha = 0$, $\sum_\alpha l_\alpha = 0$.

The balance equations for mass, linear momentum and energy relative to the α-th fluid may be written as follows (see for example [3])

$$(2.1) \qquad \grave{\rho}_\alpha + \rho_\alpha \nabla \cdot v_\alpha = 0$$

$$(2.2) \qquad \rho_\alpha \grave{v}_\alpha - \nabla \cdot T_\alpha - \rho_\alpha b_\alpha = m_\alpha$$

$$(2.3) \qquad \rho_\alpha \grave{\varepsilon}_\alpha - T_\alpha : (\nabla v_\alpha) + \nabla \cdot h_\alpha - \rho_\alpha r_\alpha + m_\alpha \cdot v_\alpha = l_\alpha$$

where b_α and r_α represent respectively the mechanical force density and the power density per unit mass due to the external supplies. Looking at non-ideal

fluid mixtures, we pose

$$m_\alpha = m_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1, ..., \vartheta_\nu, \ \boldsymbol{v}_1, ..., \boldsymbol{v}_\nu)$$

(2.4)

$$l_\alpha = l_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1, ..., \vartheta_\nu, \ \boldsymbol{v}_1, ..., \boldsymbol{v}_\nu) \,.$$

Likewise, if ψ_α denotes the partial free energy per unit mass, we assume

(2.5) $$\psi_\alpha = \psi_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1, ..., \vartheta_\nu) \qquad \varepsilon_\alpha = \varepsilon_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1, ..., \vartheta_\nu) \,.$$

To complete the set of constitutive relations we put

$$\boldsymbol{T}_\alpha = \boldsymbol{T}_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1 ..., \vartheta_\nu, \ \boldsymbol{v}_1 ..., \boldsymbol{v}_\nu, \ \boldsymbol{\nabla}\rho_1, ..., \boldsymbol{\nabla}\rho_\nu, \ \boldsymbol{\nabla}\vartheta_1, ..., \boldsymbol{\nabla}\vartheta_\nu)$$

(2.6)

$$\boldsymbol{h}_\alpha = \boldsymbol{h}_\alpha(\rho_1, ..., \rho_\nu, \ \vartheta_1 ..., \vartheta_\nu, \ \boldsymbol{v}_1 ..., \boldsymbol{v}_\nu, \ \boldsymbol{\nabla}\rho_1, ..., \boldsymbol{\nabla}\rho_\nu, \ \boldsymbol{\nabla}\vartheta_1, ..., \boldsymbol{\nabla}\vartheta_\nu) \,.$$

Eqs. (2.4)-(2.6) should be compared with those adopted in analogous approaches for ideal fluid mixtures with several temperatures (see [5], [8], [9]) and for non-ideal mixtures at common temperatures (see [4]$_2$).

Some restrictions on the form of (2.4)-(2.6) may be achieved on the basis of the second law of thermodynamics. According to Benach and Müller [1] we write

(2.7) $$\sum_\alpha \left\{ \rho_\alpha \dot{\eta}_\alpha + \boldsymbol{\nabla} \cdot \frac{\boldsymbol{h}_\alpha}{\vartheta_\alpha} - \frac{\rho_\alpha r_\alpha}{\vartheta_\alpha} \right\} \geqslant 0$$

where η_α is the partial entropy density per unit mass. Using (2.3) and the thermodynamic relations $\varepsilon_\alpha = \psi_\alpha + \vartheta_\alpha \eta_\alpha$, eq. (2.7) may be written as

(2.8) $$\sum_\alpha \frac{1}{\vartheta_\alpha} \left\{ -\rho_\alpha (\dot{\psi}_\alpha + \eta_\alpha \dot{\vartheta}_\alpha) + \frac{p_\alpha}{\rho_\alpha} \dot{\rho}_\alpha \right.$$

$$\left. + \langle \boldsymbol{T}_\alpha \rangle : (\boldsymbol{\nabla} \boldsymbol{v}_\alpha) - \frac{1}{\vartheta_\alpha} (\boldsymbol{\nabla}\vartheta_\alpha) \cdot \boldsymbol{h}_\alpha - \boldsymbol{m}_\alpha \cdot \boldsymbol{v}_\alpha + l_\alpha \right\} \geqslant 0$$

where $p_\alpha = -\frac{1}{3} tr \boldsymbol{T}_\alpha$, $\langle \boldsymbol{T}_\alpha \rangle = \boldsymbol{T}_\alpha + p_\alpha \boldsymbol{1}$. Since $\langle \boldsymbol{T}_\alpha \rangle$, $\boldsymbol{h}_\alpha$, $\boldsymbol{m}_\alpha$, l_α are assumed to be independent on $\dot{\rho}_\alpha$ and $\dot{\vartheta}_\alpha$; we have

(2.9) $$p_\alpha = \rho_\alpha \vartheta_\alpha \sum_\beta \frac{\rho_\beta}{\vartheta_\beta} \frac{\partial \psi_\beta}{\partial \rho_\alpha} \qquad \eta_\alpha = -\frac{\vartheta_\alpha}{\rho_\alpha} \sum_\beta \frac{\rho_\beta}{\vartheta_\beta} \frac{\partial \psi_\beta}{\partial \vartheta_\alpha} \,.$$

The independence of $\langle \boldsymbol{T}_\alpha \rangle$ on $\boldsymbol{\nabla} \boldsymbol{v}_\beta$ implies that $\langle \boldsymbol{T}_\alpha \rangle = 0$ ($\alpha = 1, ..., \nu$). Hence, the

second law (2.8) reduces to the following two dissipative inequalities

$$(2.10) \qquad \sum_\alpha \frac{1}{\vartheta_\alpha^2} (\boldsymbol{h}_\alpha \cdot \nabla \vartheta_\alpha) \leqslant 0$$

$$(2.11) \qquad \sum_\alpha \frac{1}{\vartheta_\alpha} (l_\alpha - \boldsymbol{m}_\alpha \cdot \boldsymbol{v}_\alpha) \geqslant 0 .$$

We introduce now the arbitrary state σ of the mixture as the set of variables $\rho_1, ..., \rho_\nu,\ \vartheta_1, ..., \vartheta_\nu,\ \boldsymbol{v}_1, ..., \boldsymbol{v}_\nu,\ \nabla \rho_1, ..., \nabla \rho_\nu, ..., \nabla \vartheta_1, ..., \nabla \vartheta_\nu;$ more concisely $\sigma = (\rho_\beta,\ \vartheta_\beta,\ \boldsymbol{v}_\beta,\ \nabla \rho_\beta,\ \nabla \vartheta_\beta)\ (\beta = 1, ..., \nu)$ and assume that in the unperturbed state $\sigma = \sigma^+$ every constituent has the same uniform velocity $\boldsymbol{v}_\alpha^+ = \boldsymbol{v}$ and different uniform temperatures ϑ_α^+ peculiar to each constituent. In particular, we choose the frame of reference in such a way that $\boldsymbol{v} = 0$ and write $\sigma^+ = (\rho_\beta^+,\ \vartheta_\beta^+,\ 0,\ 0,\ 0)$.

In addition, we suppose that $\boldsymbol{b}_\alpha = 0\ (\alpha = 1, ..., \nu)$ so that, owing to (2.2) we get

$$(2.12) \qquad \boldsymbol{m}_\alpha^+ = 0 .$$

Some restrictions due to inequalities (2.10) and (2.11) will be derived in 5. Here we note that, owing to the conditions $\nabla \vartheta |_{\sigma^+} = 0$, inequality (2.10) implies

$$(2.13) \qquad \boldsymbol{h}_\alpha^+ = 0 .$$

Eq. (2.13) is also valid at any state σ which differs from σ^+ by non-vanishing density gradients. Explicitly

$$(2.14) \qquad \boldsymbol{h}_\alpha(\rho_\beta^+,\ \vartheta_\beta^+,\ 0,\ 0,\ 0) = \boldsymbol{h}_\alpha(\rho_\beta^+,\ \vartheta_\beta^+,\ 0,\ \nabla \rho_\beta,\ 0)$$

and consequently

$$(2.15) \qquad \frac{\partial \boldsymbol{h}_\alpha}{\partial \nabla \rho_\beta} = 0 \qquad\qquad (\alpha,\ \beta = 1, ..., \nu)$$

ahead and behind the wave.

3 - Propagation of acceleration waves

Consider a singular surface $\sum$ propagating into $\mathscr{R}$; let $\boldsymbol{n}$ and u_n be respectively the normal to $\sum$ and the speed of displacement of $\sum$ along $\boldsymbol{n}$. Given

a function $\varphi(\boldsymbol{x},\ t)$ in $\mathscr{R} \times t$, we denote by $[\varphi] = \varphi^- - \varphi^+$ the jump of φ across $\sum$. If $[\varphi] = 0$ the well known geometric and kinematic compatibility conditions reduce to (see for example [6])

$$(3.1) \qquad [\boldsymbol{\nabla}\varphi] = ([\boldsymbol{\nabla}\varphi] \cdot \boldsymbol{n})\,\boldsymbol{n}$$

$$(3.2) \qquad [\frac{\partial \varphi}{\partial t}] = - u_n[\boldsymbol{\nabla}\varphi] \cdot \boldsymbol{n}\,.$$

An acceleration wave $\sum$ propagating into the mixture is defined as a singular surface across which

$$(3.3) \qquad [\vartheta_\alpha] = 0 \quad [\boldsymbol{v}_\alpha] = 0 \quad s_\alpha := [\frac{\partial \boldsymbol{v}_\alpha}{\partial t}]$$

for any α. Hence, if $\sum$ propagates through σ^+, the balance equations (2.1)-(2.3) on $\sum$ yield

$$(3.4) \qquad [\rho_\alpha] = [p_\alpha] = 0 \quad \rho_\alpha^+ u_n[\varepsilon_\alpha] + [\boldsymbol{h}_\alpha] \cdot \boldsymbol{n} = 0\,.$$

Because of the constitutive relations (2.5) and the continuity of ρ_α and ϑ_α across $\sum$, the last of (3.4) reduces to

$$(3.5) \qquad [\boldsymbol{h}_\alpha] \cdot \boldsymbol{n} = 0\,.$$

In view of (2.14) we note that a sufficient condition we have $[\boldsymbol{h}_\alpha] = 0$ is $[\boldsymbol{\nabla}\vartheta_\alpha] = 0$ ($\alpha = 1, ..., \nu$); that is peculiar of omothermal acceleration waves. Hence eq. (3.5) allows us to state that a mixture of conducting fluids admits the propagation of omothermal acceleration waves. Obviously (3.5) is also satisfied if $\boldsymbol{h}_\alpha = 0$ identically for every α. This allows us to distinguish the following two cases:

(a) *Propagation of omothermal acceleration waves through a mixture of conducting fluids.*

In this case

$$(3.6) \qquad [\boldsymbol{\nabla}\vartheta_\alpha] = \boldsymbol{0}\,.$$

On using eq. (3.1) and (3.4) we get

$$(3.7) \qquad [\frac{\partial \rho_\alpha}{\partial t}] = \rho_\alpha^+ \frac{s_\alpha \cdot n}{u_n} \qquad [\nabla \rho_\alpha] = -\rho_\alpha^+ \frac{s_\alpha \cdot n}{u_n^2} n$$

$$(3.8) \qquad [\nabla v_\alpha] = -\frac{s_\alpha \otimes n}{u_n}.$$

Taking the jumps of both sides of (2.2) and making use of (3.6), (3.7) and (3.8), we obtain

$$(3.9) \qquad \rho_\alpha^+ u_n s_\alpha = \sum_\beta \rho_\beta^+ (\frac{\partial p_\alpha}{\partial \rho_\beta})^+ \frac{s_\beta \cdot n}{u_n} n$$

which shows that only longitudinal acceleration waves are admitted in the mixture, i.e. $s_\alpha = s_\alpha n$. Eq. (3.9) may be written in a more suitable form introducing the partial chemical potentials as follows

$$(3.10) \qquad \mu_\alpha = \vartheta_\alpha \frac{\partial}{\partial \rho_\alpha} \sum_\gamma \frac{\rho_\gamma}{\vartheta_\gamma} \psi_\gamma.$$

Taking into account the first eq. of (2.9) and (3.10), eq. (3.9) becomes

$$(3.11) \qquad \rho_\alpha^+ u_n \frac{s_\alpha}{\vartheta_\alpha^+} = \rho_\alpha^+ \sum_\beta (\sum_\gamma \frac{\rho_\gamma}{\vartheta_\gamma} \frac{\partial \psi_\gamma}{\partial \rho_\alpha} \delta_{\alpha\beta} - \frac{\rho_\beta}{\vartheta_\alpha} \frac{\partial \psi_\alpha}{\partial \rho_\beta})^+ \frac{s_\beta}{u_n} + \sum_\beta \rho_\alpha^+ \rho_\beta^+ (\frac{\partial (\mu_\alpha / \vartheta_\alpha)}{\partial \rho_\beta})^+ \frac{s_\beta}{u_n}.$$

Now, the determinant of the matrix of indices α, β

$$\rho_\alpha \sum_\gamma \frac{\rho_\gamma}{\vartheta_\gamma} \frac{\partial \psi_\gamma}{\partial \rho_\alpha} \delta_{\alpha\beta} - \frac{\rho_\alpha \rho_\beta}{\vartheta_\alpha} \frac{\partial \psi_\alpha}{\partial \rho_\beta}$$

vanshes identically. As a consequence, the first term in the right hand side of (3.11) vanishes and we obtain

$$(3.12) \qquad \sum_\beta \{\rho_\alpha^+ \rho_\beta^+ (\frac{\partial (\mu_\alpha / \vartheta_\alpha)}{\partial \rho_\beta})^+ - \frac{\rho_\beta^+}{\vartheta_\beta^+} u_n^2 \delta_{\alpha\beta}\} s_\beta = 0.$$

Hence we speeds u_n must obey the propagation condition

(3.13) $$\det\{\Omega_{\alpha\beta} - \chi_{\alpha\beta} u_n^2\} = 0$$

where

$$\Omega_{\alpha\beta} = \rho_\alpha^+ \rho_\beta^+ \left(\frac{\partial(\mu_\alpha/\vartheta_\alpha)}{\partial\rho_\beta}\right)^+ \qquad \chi_{\alpha\beta} = \frac{\rho_\beta^+}{\vartheta_\beta^+}\delta_{\alpha\beta} \cdot$$

We note that, in view of (3.10), $\Omega_{\alpha\beta}$ is symmetric. Then eq. (3.13) admits real roots for u_n if and only if $\Omega_{\alpha\beta}$ is positive definite.

(b) *Propagation of acceleration waves through a mixture of non-conducting fluids.*

In this case $\mathbf{h}_\alpha = 0$ and $[\nabla\vartheta_\alpha] \neq 0$. Then, taking the jumps of both sides of (2.3), using (2.9) and applying (3.1) and (3.2) to the temperatures ϑ_α, we obtain

(3.14) $$u_n^2 \sum_\beta \frac{\rho_\alpha^+}{\vartheta_\alpha^{+2}}\left(\frac{\partial\varepsilon_\alpha}{\partial\vartheta_\beta}\right)^+ [\nabla\vartheta_\beta] \cdot \mathbf{n} = -\sum_\beta \rho_\beta^+ \sum_\gamma \frac{\rho_\gamma^+}{\vartheta_\gamma^+}\left(\frac{\partial^2\psi_\gamma}{\partial\vartheta_\alpha\partial\rho_\beta}\right)^+ \mathbf{s}_\beta \cdot \mathbf{n} .$$

For sake of brevity we introduce the matrices

$$C_{\alpha\beta} := \frac{\rho_\alpha^+}{\vartheta_\alpha^{+2}}\left(\frac{\partial\varepsilon_\alpha}{\partial\vartheta_\beta}\right)^+ = \left(\frac{\rho_\alpha}{\vartheta_\alpha^2}\frac{\partial\psi_\alpha}{\partial\vartheta_\beta} + \frac{\rho_\beta}{\vartheta_\beta^2}\frac{\partial\psi_\beta}{\partial\vartheta_\alpha} - \sum_\gamma \frac{\rho_\gamma}{\vartheta_\gamma}\frac{\partial^2\psi_\gamma}{\partial\vartheta_\alpha\partial\vartheta_\beta} - 2\frac{\delta_{\alpha\beta}}{\vartheta_\alpha}\sum_\gamma \frac{\rho_\gamma}{\vartheta_\gamma}\frac{\partial\psi_\gamma}{\partial\vartheta_\alpha}\right)^+$$

$$X_{\alpha\beta} := \sum_\gamma \frac{\rho_\gamma^+}{\vartheta_\gamma^+}\left(\frac{\partial^2\psi_\gamma}{\partial\vartheta_\alpha\partial\rho_\beta}\right)^+ .$$

Then, on supposing $C_{\alpha\beta}$ invertible, eq. (3.14) yields

(3.15) $$[\nabla\vartheta_\beta] \cdot \mathbf{n} = -\frac{1}{u_n^2}\sum_{\gamma,\delta}\rho_\delta^+ C_{\gamma\delta}^{-1} X_{\beta\gamma}\mathbf{s}_\gamma \cdot \mathbf{n} .$$

Now, taking the jumps of both sides of (2.2), we get

(3.16) $$\rho_\alpha^+ u_n \mathbf{s}_\alpha = \sum_\beta \rho_\beta^+\left(\frac{\partial p_\alpha}{\partial\rho_\beta}\right)^+\frac{\mathbf{s}_\beta \cdot \mathbf{n}}{u_n}\mathbf{n} - \sum_\beta \left(\frac{\partial p_\alpha}{\partial\vartheta_\beta}\right)^+([\nabla\vartheta_\beta] \cdot \mathbf{n})\mathbf{n} .$$

⸱Exploiting the first of (2.9) and (3.15), with a procedure similar to that used in deriving (3.12), eq. (3.16) becomes

$$(3.17) \qquad \sum_{\beta} \{\rho_\alpha^+ \rho_\beta^+ (\frac{\partial(\mu_\alpha/\vartheta_\alpha)}{\partial\rho_\beta})^+ + \rho_\alpha^+ \rho_\beta^+ \sum_{\gamma,\delta} C_{\gamma\delta}^{-1} X_{\gamma\alpha} X_{\delta\beta} - u_n^2 \delta_{\alpha\beta} \frac{\rho_\beta^+}{\vartheta_\beta^+}\} s_\beta = 0 \;.$$

The propagation condition, in the case at hand, is

$$(3.18) \qquad\qquad \det\{\Omega_{\alpha\beta} + A_{\alpha\beta} - \chi_{\alpha\beta} u_n^2\} = 0$$

where

$$A_{\alpha\beta} = \rho_\alpha^+ \rho_\beta^+ \sum_{\gamma,\delta} C_{\gamma\delta}^{-1} X_{\gamma\alpha} X_{\delta\beta} \;.$$

The matrix $A_{\alpha\beta}$ is evidently symmetric. It follows that a necessary and sufficient condition to have real roots for u_n is that $\Omega_{\alpha\beta} + A_{\alpha\beta}$ be positive definite. We remark that in theories with common temperatures $\Omega_{\alpha\beta}$ reduces to the usual acoustic matrix [4]$_2$ and the quantities $\vartheta_\alpha^2 C_{\alpha\beta}$ reduce to the specific heats at constants volume $(c_\alpha)_v$ relative to each constituent of the mixture [5].

In ending this section we observe that it is possible to write

$$(3.19) \qquad\qquad s_\beta = z_\beta s$$

where z_β are solutions of (3.12) or of (3.17) and $s = s(t)$. Since the z_β's are defined up to an arbitrary factor, we introduce the normalization condition

$$(3.20) \qquad\qquad \sum_\alpha \frac{\rho_\alpha^+}{\vartheta_\alpha^+} z_\alpha^2 = 1 \;.$$

4 - Evolution equations for amplitudes

It has been shown by Doria and Bowen [7] that, making use of the geometric and kinematic compatibility conditions, the amplitudes s_α of the longitudinal waves in fluids must obey the following equations

$$(4.1) \qquad 2\frac{\delta s_\alpha}{\delta t} - \frac{s_\alpha}{u_n}\frac{\delta u_n}{\delta t} = u_n \bar{K} s_\alpha + [\frac{\partial^2 v_\alpha}{\partial t^2}] \cdot \boldsymbol{n} - u_n^2 [\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot v_\alpha)]\cdot\boldsymbol{n}$$

where $\partial/\partial t$ denotes the displacement derivative, i.e. the rate of change relative to an observer moving with the surface, and where $\bar{K}$ is the mean curvature of the propagating surface. It is the purpose of this section to obtain the explicit form of (4.1) in both cases of omothermal waves and non-conducting mixtures. We first work out the expression of $[\partial^2 v_\alpha/\partial t^2]$ by deriving (2.2) with respect to time and taking the jumps of both sides of the resulting equation. Owing to (3.3), (3.7), (3.8) we obtain

$$(4.2) \qquad \rho_\alpha^+ [\frac{\partial^2 v_\alpha}{\partial t^2}] = [\frac{\partial m_\alpha}{\partial t}] - [\nabla \frac{\partial p_\alpha}{\partial t}].$$

In deriving (4.2) we have used the hypothesis $b_\alpha = 0$. In the following we shall need the jumps $[\nabla(\partial \rho_\beta/\partial t)]$ which can be obtained from evaluating the gradient of (2.1) and taking the jumps of both sides. The result is

$$(4.3) \qquad [\nabla \frac{\partial \rho_\beta}{\partial t}] = -2\rho_\beta^+ \frac{s_\beta^2}{u_n^3} n - \rho_\beta^+ [\nabla(\nabla \cdot v_\beta)].$$

We distinguish again the two cases:

(a) *Omothermal waves.*

In this case the jumps $[\nabla \vartheta_\alpha]$ and $[\partial \vartheta_\alpha/\partial t]$ vanish and the last term in the right hand side of (4.2) may be obtained by noting that $p_\alpha = p_\alpha(\rho_\beta, \vartheta_\beta)$,

$$(4.4) \qquad [\nabla \frac{\partial p_\alpha}{\partial t}]$$

$$= -\sum_{\gamma,\beta} \rho_\beta^+ \rho_\gamma^+ (\frac{\partial^2 p_\alpha}{\partial \rho_\beta \partial \rho_\gamma})^+ \frac{s_\beta s_\gamma}{u_n^3} n + \sum_\beta (\frac{\partial p_\alpha}{\partial \rho_\beta})^+ [\nabla \frac{\partial \rho_\beta}{\partial t}] + \sum_\beta (\frac{\partial p_\alpha}{\partial \vartheta_\beta})^+ [\nabla \frac{\partial \vartheta_\beta}{\partial t}].$$

The last term in the right hand side of (4.4) may be evaluated taking the jumps of both sides of (2.3). In view of the constitutive hypotheses (2.6) and the results (2.15), (3.1), (3.2), (3.6)-(3.8), we can write

$$(4.5) \qquad \sum_\beta ((\frac{\partial h_\alpha}{\partial \nabla \vartheta_\beta})^- \cdot n) \cdot [\frac{\partial}{\partial t} \nabla \vartheta_\beta]$$

$$= \sum_\beta \{\rho_\alpha^+ \rho_\beta^+ (\frac{\partial \varepsilon_\alpha}{\partial \rho_\beta})^+ - p_\beta^+ \delta_{\alpha\beta} + \frac{\rho_\beta^+}{u_n} (\frac{\partial h_\alpha}{\partial \rho_\beta})^- \cdot n + ((\frac{\partial h_\alpha}{\partial v_\beta})^- \cdot n) \cdot n\} s_\beta.$$

For present and next purposes it is convenient to introduce the quantities

$$(4.6) \qquad H_{i\alpha j\beta} := \frac{1}{\vartheta_\beta^2} \left\{ \frac{\partial \boldsymbol{h}_\alpha}{\partial \boldsymbol{\nabla} \vartheta_\beta} \right\}_{ij} \qquad H_{\alpha\beta} := (H_{i\alpha j\beta} n_i n_j)^- \qquad i, j = 1, 2, 3 .$$

Assuming $H_{\alpha\beta}$ to be invertible, eq. (4.5) yields

$$(4.7) \qquad \qquad [\frac{\partial}{\partial t} \boldsymbol{\nabla} \vartheta_\alpha]$$

$$= \sum_{\beta,\gamma} \frac{H_{\alpha\beta}^{-1}}{\vartheta_\beta^2} \left\{ \rho_\beta^+ \rho_\gamma^+ (\frac{\partial \varepsilon_\beta}{\partial \rho_\gamma})^+ - p_\gamma^+ \delta_{\beta\gamma} + \frac{\rho_\gamma^+}{u_n} ((\frac{\partial \boldsymbol{h}_\beta}{\partial \rho_\gamma})^- \cdot \boldsymbol{n}) + \boldsymbol{n} \cdot ((\frac{\partial \boldsymbol{h}_\beta}{\partial \boldsymbol{v}_\gamma})^- \cdot \boldsymbol{n}) \right\} s_\gamma \, \boldsymbol{n} .$$

Now we substitute eqs. (4.7) and (4.3) into (4.4) and, in turn, substitute (4.4) into (4.2). Taking into account eq. (2.4) and the propagation condition (3.9), eq. (4.2) may be finally substituted into (4.1). In writing the final expression we observe that $\partial u_n / \partial t = 0$ since $u = u_n(\sigma^+)$. The result is

$$(4.8) \qquad \qquad \rho_\alpha^+ \frac{\delta s_\alpha}{\delta t} = \frac{1}{2} \rho_\alpha^+ \bar{K} u_n s_\alpha + \vartheta_\alpha^+ \left\{ \sum_\beta \Lambda_{\alpha\beta} s_\beta + \sum_{\beta,\gamma} \Gamma_{\alpha\beta\gamma} s_\beta s_\gamma \right\} \qquad \text{where}$$

$$(4.9) \qquad \qquad 2\Lambda_{\alpha\beta} = \frac{\rho_\beta^+}{u_n \vartheta_\alpha^+} (\frac{\partial \boldsymbol{m}_\alpha}{\partial \rho_\beta})^+ \cdot \boldsymbol{n} + \frac{1}{\vartheta_\alpha^+} \boldsymbol{n} \cdot ((\frac{\partial \boldsymbol{m}_\alpha}{\partial \boldsymbol{v}_\beta})^+ \cdot \boldsymbol{n})$$

$$- \sum_{\gamma,\delta} (\frac{\partial p_\alpha}{\partial \vartheta_\gamma})^+ \frac{H_{\gamma\delta}^{-1}}{\vartheta_\alpha^+ \vartheta_\delta^{+2}} \left\{ \rho_\delta^+ \rho_\beta^+ (\frac{\partial \varepsilon_\delta}{\partial \rho_\beta})^+ - p_\beta^+ \delta_{\delta\beta} + \frac{\rho_\beta^+}{u_n} (\frac{\partial \boldsymbol{h}_\delta}{\partial \rho_\beta})^- \cdot \boldsymbol{n} + \boldsymbol{n} \cdot ((\frac{\partial \boldsymbol{h}_\delta}{\partial \boldsymbol{v}_\beta})^- \cdot \boldsymbol{n}) \right\}$$

$$(4.10) \qquad \qquad 2\Gamma_{\alpha\beta\gamma} = \frac{1}{u_n^3} \frac{\rho_\gamma^+}{\rho_\beta^+} (\frac{\partial}{\partial \rho_\gamma} (\rho_\alpha \rho_\beta^2 \frac{\partial (u_\alpha / \vartheta_\alpha)}{\partial \rho_\beta}))^+ .$$

Multiplying (4.8) by $z_\alpha / \vartheta_\alpha^+$, summing over α and taking into account (3.19) and (3.20), we obtain

$$(4.11) \qquad \qquad \frac{\delta s}{\delta t} = (\bar{K} u_n + \Lambda) s + \Gamma s^2$$

where $\qquad \qquad \Lambda = \sum_{\alpha,\beta} \Lambda_{\alpha\beta} z_\alpha z_\beta \qquad \qquad \Gamma = \sum_{\alpha,\beta,\gamma} \Gamma_{\alpha\beta\gamma} z_\alpha z_\beta z_\gamma .$

(b) *Waves in non-conducting mixtures.*

In this case we have $\boldsymbol{h}_\alpha = 0$ $(\alpha = 1, \ldots, \nu)$. In deriving the jumps $[\boldsymbol{\nabla} \dfrac{\partial p_\alpha}{\partial t}]$ we must retain the terms proportional to $[\boldsymbol{\nabla} \vartheta_\alpha]$ which have been suppressed in (4.4) and evaluate again the jumps $[\boldsymbol{\nabla}(\partial \vartheta_\alpha / \partial t)]$. To this aim we take the gradient of (2.3) and work out the jumps of both sides of the resulting equation. Making use of (3.15) and (4.3) we obtain

$$(4.12) \qquad\qquad [\boldsymbol{\nabla} \frac{\partial \vartheta_\alpha}{\partial t}] \cdot \boldsymbol{n}$$

$$= -\frac{1}{u_n^2} \sum_\beta \frac{C_{\alpha\beta}^{-1}}{\vartheta_\beta^{+2}} \{ \sum_{\gamma,\delta} \{ \rho_\gamma^+ (\frac{\partial l_\beta}{\partial \rho_\gamma})^+ \delta_{\delta\gamma} + (\frac{\partial l_\beta}{\partial \vartheta_\gamma}) Y_{\delta\gamma} + (\frac{\partial l_\beta}{\partial \boldsymbol{v}_\gamma})^+ \cdot \boldsymbol{n} u_n \delta_{\gamma\delta} + \rho_\gamma^+ r_\gamma \delta_{\beta\gamma} \} s_\gamma$$

$$- \frac{\rho_\beta^+}{u_n} \sum_{\gamma,\delta} \{ \frac{\rho_\delta^+}{\rho_\gamma^+} (\frac{\partial}{\partial \rho_\delta} (\rho_\gamma^2 \frac{\partial \varepsilon_\beta}{\partial \rho_\gamma}))^+ + \sum_\sigma (2\rho_\gamma^+ (\frac{\partial^2 \varepsilon_\beta}{\partial \rho_\gamma \partial \vartheta_\sigma})^+ + (\frac{\partial p_\beta}{\partial \vartheta_\sigma})^+ \delta_{\beta\gamma}) Y_{\sigma\delta} + \sum_{\sigma,\omega} (\frac{\partial^2 \varepsilon_\beta}{\partial \vartheta_\sigma \partial \vartheta_\omega})^+ Y_{\sigma\delta} Y_{\omega\gamma}$$

$$+ \rho_\gamma^+ (\frac{\partial p_\beta}{\partial \rho_\gamma})^+ \delta_{\beta\delta} \} s_\gamma s_\delta - u_n^2 \sum_\gamma (\rho_\beta^+ \rho_\gamma^+ (\frac{\partial \varepsilon_\beta}{\partial \rho_\gamma})^+ - p_\gamma^+ \delta_{\beta\gamma}) [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}_\gamma)]$$

where $\qquad\qquad\qquad Y_{\alpha\beta} = \rho_\beta^+ \sum_\gamma C_{\beta\gamma}^{-1} X_{\alpha\beta}.$

The jumps $[\boldsymbol{\nabla}(\partial p_\alpha / \partial t)]$ may now be evaluated making use of (3.15) and (4.12); we omit for brevity the resulting expression and note that after substitution into (4.2) and in turn, into (4.1), we finally have

$$(4.13) \qquad \rho_\alpha^+ \frac{\delta s_\alpha}{\delta t} = \frac{1}{2} \rho_\alpha^+ u \bar{K} s_\alpha + \vartheta_\alpha^+ \{ \sum_\beta (\Lambda_{\alpha\beta}^{(m)} + \Lambda_{\alpha\beta}^{(l)}) s_\beta + \sum_{\beta,\gamma} \bar{\Gamma}_{\alpha\beta\gamma} s_\beta s_\gamma \} \qquad \text{where}$$

$$(4.14) \qquad 2\Lambda_{\alpha\beta}^{(m)} = \frac{\rho_\beta^+}{u_n} (\frac{\partial(\boldsymbol{m}_\alpha / \vartheta_\alpha)}{\partial \rho_\beta})^+ \cdot \boldsymbol{n} + ((\frac{\partial(\boldsymbol{m}_\alpha / \vartheta_\alpha)}{\partial \boldsymbol{v}_\beta})^+ \cdot \boldsymbol{n}) \cdot \boldsymbol{n}$$

$$+ \frac{1}{\vartheta_\alpha^+ u_n} \sum_\gamma ((\frac{\partial \boldsymbol{m}_\alpha}{\partial \vartheta_\gamma})^+ \cdot \boldsymbol{n}) Y_{\gamma\beta}$$

$$(4.15) \qquad 2\Lambda_{\alpha\beta}^{(l)} = \frac{1}{u_n^2} \sum_{\gamma,\delta} \frac{C_{\delta\gamma}^{-1}}{\vartheta_\alpha^+ \vartheta_\gamma^{+2}} (\frac{\partial p_\alpha}{\partial \vartheta_\delta})^+ \{ (\frac{\partial l_\gamma}{\partial \vartheta_\beta})^+ \sum_\sigma Y_{\sigma\beta}$$

$$+ \rho_\beta^+ (\frac{\partial l_\gamma}{\partial \rho_\beta})^+ + u_n (\frac{\partial l_\gamma}{\partial \boldsymbol{v}_\beta}) \cdot \boldsymbol{n} + \rho_\gamma^+ r_\gamma \delta_{\gamma\beta} \}$$

(4.16) $2\bar{\Gamma}_{\alpha\beta\gamma}$

$$= \frac{1}{u_n^3} \{ \frac{\rho_\gamma^+}{\rho_\beta^+} (\frac{\partial}{\partial \rho_\gamma} (\rho_\alpha \rho_\beta^2 \frac{\partial (\mu_\alpha/\vartheta_\alpha)}{\partial \rho_\beta}))^+ + 2 \frac{\rho_\beta^+}{\vartheta_\alpha^+} \sum_\sigma (\frac{\partial^2 p_\alpha}{\partial \rho_\beta \partial \vartheta_\sigma})^+ Y_{\sigma\gamma} + \frac{1}{\vartheta_\alpha^+} \sum_{\sigma,\delta} (\frac{\partial^2 p_\alpha}{\partial \vartheta_\delta \partial \vartheta_\sigma})^+ Y_{\delta\beta} Y_{\sigma\gamma}$$

$$- \sum_{\sigma,\delta} \frac{\rho_\sigma^+}{\vartheta_\alpha^+ \vartheta_\sigma^{+2}} (\frac{\partial p_\alpha}{\partial \vartheta_\delta})^+ C_{\delta\sigma}^{-1} \{ \frac{\rho_\gamma^+}{\rho_\beta^+} (\frac{\partial}{\partial \rho_\gamma} (\rho_\beta^2 \frac{\partial \varepsilon_\sigma}{\partial \rho_\beta}))^+ + \sum_\omega (2\rho_\beta^+ (\frac{\partial^2 \varepsilon_\sigma}{\partial \rho_\beta \partial \vartheta_\omega})^+ + (\frac{\partial p_\sigma}{\partial \vartheta_\omega})^+ \delta_{\sigma\beta}) Y_{\omega\delta}$$

$$+ \rho_\beta^+ (\frac{\partial p_\sigma}{\partial \rho_\beta})^+ \delta_{\sigma\gamma} + \sum_{\omega,\tau} (\frac{\partial^2 \varepsilon_\sigma}{\partial \vartheta_\omega \partial \vartheta_\tau})^+ Y_{\omega\gamma} Y_{\tau\beta} \}\}$$

and where we have used the condition (3.17).

As in the previous case we multiply (4.13) by $z_\alpha/\vartheta_\alpha^+$ and, owing to (3.19), (3.20), after summation over α we get

(4.17) $\frac{\delta s}{\delta t} = (\bar{K}u_n + \Lambda(m) + \Lambda^{(l)}) s + \bar{\Gamma} s^2$

where $\Lambda^{(m)} = \sum_{\alpha,\beta} \Lambda_{\alpha\beta}^{(m)} z_\alpha z_\beta$ $\Lambda^{(l)} = \sum_{\alpha,\beta} \Lambda_{\alpha\beta}^{(l)} z_\alpha z_\beta$ $\bar{\Gamma} = \sum_{\alpha,\beta,\gamma} \bar{\Gamma}_{\alpha\beta\gamma} z_\alpha z_\beta z_\gamma$.

Before discussing eqs. (4.11) and (4.17) we observe that owing to (2.4) and (2.5), the quantities Γ, $\bar{\Gamma}$, $\Lambda^{(m)}$ and $\Lambda^{(l)}$ are functions of σ^+ only. Instead, def. (4.6) implies that Λ depends upon σ^-, and, consequently, on $s(t)$. It follows that (4.17) is a differential equation with constant coefficients, while, in (4.11), $\Lambda = \Lambda(s)$.

5 - Thermodynamic restrictions

We introduce a thermal equilibrium state σ_0 in which all the constituent of the mixture are at rest and have the same uniform temperature ϑ, i.e.

(5.1) $\sigma_0 = (\rho_{0\beta}, \ \vartheta, \ 0, \ 0, \ 0)$.

We let ϕ_1 to denote the left hand side of (2.11). We have $\phi_1 = \phi_1(\rho_\beta, \ \vartheta_\beta, \ v_\beta)$ and, owing to (5.1) and the condition $\sum_\alpha l_\alpha = 0$, we get $\phi_1(\sigma_0) = 0$.

Computing the partial derivative of ϕ_1 with respect to ϑ_γ at $\sigma = \sigma_0$ we have

(5.2) $\frac{\partial \phi_1}{\partial \vartheta_\gamma}|_{\sigma_0} = - \frac{1}{\vartheta^2} l_\gamma|_{\sigma_0}$.

By the other hand, (2.3) yields

(5.3) $l_\gamma|_{\sigma_0} = -\rho_{0\gamma}\, r_\gamma$.

Eqs. (5.2) and (5.3) show that if $r_\gamma \neq 0$ $(\gamma = 1, \ldots, \nu)$, $\dfrac{\partial \phi_1}{\partial \vartheta_\gamma}\Big|_{\sigma_0} \neq 0$ $(\gamma = 1, \ldots, \nu)$.

This enables us to apply the implicit function theorem to ϕ_1 obtaining

(5.4) $\phi_1(\rho_\beta,\ \vartheta_\beta,\ 0) = 0$

for any $(\rho_\beta,\ \vartheta_\beta,\ 0)$ in a neighborhood N of σ_0. If we assume that $\sigma^+ \in N$, from (5.4) we get

(5.5) $\left(\dfrac{\partial l_\alpha}{\partial \rho_\gamma}\right)^+ = 0$ $\left(\dfrac{\partial l_\alpha}{\partial v_\gamma}\right)^+ = 0$ $\sum_\alpha \dfrac{1}{\vartheta_\alpha^+}\left(\dfrac{\partial l_\alpha}{\partial \vartheta_\gamma}\right)^+ = -\dfrac{\rho_\gamma^+\, r_\gamma}{\vartheta_\gamma^{+2}}$

(5.6) $\left(\dfrac{\partial m_\alpha}{\partial \rho_\gamma}\right)^+ = 0$ $\left(\dfrac{\partial m_\alpha}{\partial \vartheta_\gamma}\right)^+ = 0$

where we have used (5.3), which holds also for $\sigma = \sigma^+$. To complete the restrictions on l_α and m_α we observe that inequality (2.11) also implies $\dfrac{d^2\phi_1}{d\sigma^2}\Big|_{\sigma^+} \geq 0$ which, in particular, requires that the matrices

(5.7)
$$\sum_\alpha \frac{1}{\vartheta_\alpha^+}\left(\frac{\partial^2 l_\alpha}{\partial v_\beta \partial v_\gamma}\right)^+ - \frac{1}{\vartheta_\gamma^+}\left(\frac{\partial m_\gamma}{\partial v_\beta}\right)^+ - \frac{1}{\vartheta_\beta^+}\left(\frac{\partial m_\beta}{\partial v_\gamma}\right)^+$$

$$\sum_\alpha \frac{1}{\vartheta_\alpha^+}\left(\frac{\partial^2 l_\alpha}{\partial \vartheta_\gamma \partial \vartheta_\beta}\right)^+ - \frac{1}{\vartheta_\gamma^{+2}}\left(\frac{\partial l_\gamma}{\partial \vartheta_\beta}\right)^+ - \frac{1}{\vartheta_\beta^{+2}}\left(\frac{\partial l_\beta}{\partial \vartheta_\gamma}\right)^+ + \frac{2}{\vartheta_\gamma^{+3}} l_\gamma|_{\sigma^+}\, \delta_{\beta\gamma}$$

be semidefinite positive. Henceforth we let the l_α's to depend linearly from the velocities and the themperatures [1], and replace the conditions on (5.7) with

(5.8) $\left(\dfrac{\partial(m_\gamma/\vartheta_\gamma)}{\partial v_\beta}\right)^+$ $\dfrac{1}{\vartheta_\gamma^{+2}}\left(\dfrac{\partial l_\gamma}{\partial \vartheta_\beta}\right)^+ - \dfrac{l_\gamma^+}{\vartheta_\gamma^{+3}}\delta_{\gamma\beta}$ negative semidefinite.

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[1] This restriction is unecessary if all the constituents have the same temperature $\vartheta_\alpha = \vartheta^+$ in front of the wave.

In connection with (2.10) we denote the left hand side with ϕ_2 and observe that $\phi_2(\sigma^+) = \phi_2(\sigma^-) = 0$.

Since ϕ_2 must be minimum at $\sigma = \sigma^+$, σ^- we have

(5.9)
$$\frac{d\phi_2}{d\sigma}\Big|_{\sigma^+,\sigma^-} = 0 \qquad \frac{d^2\phi_2}{d\sigma^2}\Big|_{\sigma^+,\sigma^-} \leq 0.$$

From the first of (5.9) we obtain $\boldsymbol{h}_\alpha^+ = \boldsymbol{h}_\alpha^- = \boldsymbol{0}$. With a procedure similar to that adopted for ϕ_1, it is possible to show that, owing to the non-singularity of the matrix $H_{\alpha\beta}$ defined in (4.6),

(5.10)
$$\frac{\partial \boldsymbol{h}_\alpha}{\partial \rho_\beta} = \boldsymbol{0}$$

for all σ in a neighborhood of σ^-. In the following we shall also assume for simplicity

(5.11)
$$\left(\frac{\partial \boldsymbol{h}_\alpha}{\partial \boldsymbol{v}_\beta}\right)^- = \boldsymbol{0}.$$

This hypothesis is coherent with some previous results [4]$_2$ and, at least within the assumption of linear constitutive relations, it is a direct consequence of the galileian invariance of $\boldsymbol{h}_\alpha$.

The second of (5.9) implies that the matrix $H_{i\alpha j\beta}$ be negative semidefinite with respect to the pairs $i\alpha$ and $j\beta$ at $\sigma = \sigma^+$ as well as just behind the wave. It follows that, in view of the definition (4.6),

(5.12) $H_{\alpha\beta}$ is negative definite.

We finally apply the present result to evaluate the coefficients Λ, $\Lambda^{(m)}$, and $\Lambda^{(l)}$. Making use of (2.9), the definition of ε_α and the results (5.6), (5.10) and (5.11), eq. (4.9) may be rearrenged to obtain

$$2\Lambda = \sum_{\alpha,\beta} \left\{ \boldsymbol{n} \cdot \left(\left(\frac{\partial(\boldsymbol{m}_\alpha/\vartheta_\alpha)}{\partial \boldsymbol{v}_\beta} \right)^+ \cdot \boldsymbol{n} \right) + \sum_{\gamma,\delta} H_{\gamma\delta}^{-1} \rho_\alpha X_{\gamma\alpha} \rho_\beta X_{\delta\beta} \right\} z_\alpha z_\beta.$$

Owing to (5.8) and (5.12) we realize that

(5.13) $\Lambda \leq 0.$

Analogously, from (4.14), (5.6) and (5.8)$_1$ we get

(5.14) $$\Lambda^{(m)} \leqq 0 \,.$$

Using (5.5), eq. (4.15) may be written more concisely as

(5.15) $$2\Lambda_{\alpha\beta}^{(l)} = \frac{1}{u_n^2} \sum_{\gamma,\delta} C_{\delta\gamma}^{-1} P_{\alpha\delta} L_{\gamma\beta} \qquad\qquad \text{where}$$

(5.16) $$P_{\alpha\delta} = \frac{1}{\vartheta_\alpha^+}\left(\frac{\partial p_\alpha}{\partial \vartheta_\delta}\right)^+ \qquad L_{\gamma\beta} = \frac{1}{\vartheta_\gamma^{+2}}\left\{\left(\frac{\partial l_\gamma}{\partial \vartheta_\beta}\right)^+ \sum_\sigma Y_{\sigma\beta} + \rho_\gamma^+ r_\gamma \delta_{\beta\gamma}\right\}\,.$$

The resulting sign of $\Lambda^{(l)}$ depends, in particular, on the external supplies r_α. We note here that for $r_\alpha = 0$ ($\alpha = 1, ..., \nu$), eq. (2.3) yields $l_\alpha^+ = 0$ ($\alpha = 1, ..., \nu$). This implies that no exchanges of energy occur among the different constituents so that the differences of temperature, if there exist, remain indefinitely for all time at $\sigma = \sigma^+$. In this case the constitutive assumptions (2.5) may be replaced by

(5.17) $$\psi_\alpha = \psi_\alpha(\rho_\alpha, \ \vartheta_\alpha) \qquad \varepsilon_\alpha = \varepsilon_\alpha(\rho_\alpha, \ \vartheta_\alpha)$$

which hold for a mixture of ideal fluids. On the basis of (5.17), the definitions (5.16) reduce to

(5.18) $$P_{\alpha\delta} = \frac{1}{\vartheta_\alpha^+}\left(\frac{\partial p_\alpha}{\partial \vartheta_\alpha}\right)^+ \delta_{\alpha\delta} \qquad L_{\gamma\beta} = \frac{1}{\vartheta_\beta^{+2}}\left(\frac{\partial l_\beta}{\partial \vartheta_\beta}\right)^+ Y_{\beta\beta}\, \delta_{\gamma\beta}\,.$$

Making use of (2.9), substitution of (5.18) into (5.15) gives

(5.19) $$2\Lambda_{\alpha\beta}^{(l)} = \frac{1}{u_n^2 \vartheta_\alpha^{+2}}\left(\frac{\partial l_\alpha}{\partial \beta}\right)^+ Y_{\alpha\alpha} Y_{\beta\beta}\,.$$

Then, owing to (5.8)$_2$, after multiplication of (5.19) by $\dfrac{z_\alpha z_\beta}{\vartheta_\alpha^+ \vartheta_\beta^+}$ and summation over α and β, we obtain

(5.20) $$\Lambda^{(l)} \leqq 0 \,.$$

This result corresponds to that obtained by Bowen and Rankin[5] in the case of ideal fluids at common temperatures in the state σ^+.

More generally, the result (5.15) shows that, if $r_\alpha \neq 0$ (at least for two different α) the sign of Λ depends on the positive or negative character of the matrices $C_{\alpha\beta}$, $P_{\alpha\beta}$ and $L_{\alpha\beta}$ under the only restriction $(5.5)_3$.

We summarize the results of this section. The coefficient Λ in (4.11) accounts for the rates of momentum exchanges $\boldsymbol{m}_\alpha$ and the heat fluxes $\boldsymbol{h}_\alpha$; it is a non-positive function of the wave amplitude s. The coefficient $\Lambda^{(m)}$ in (4.17) depends on the rates $\boldsymbol{m}_\alpha$ and is a non-positive constant. Finally, the coefficient $\Lambda^{(l)}$ in (4.17) is a constant which accounts for the rates of change of the energy exchanges l_α and the power densities r_α of the external sources; its sign depends on the relative weight of these two last quantities. In particular the power densities r_α could be intense enough to give $\Lambda^{(m)} + \Lambda^{(l)} > 0$.

6 - Growth and decay of waves into non-conducting mixtures

As it appears from (4.11) and (4.17), the wave evolution depends on the geometry of the initial perturbation and on the constitutive assumption for the mixture at hand. It has been shown [7] that the initial curvature of the waves affects indirectly their properties via the critical amplitudes, but it leaves unchanged the essential features of wave evolution. For this reason we shall consider the simple case of plane waves which amounts to setting $\bar{K} = 0$ in (4.11) and (4.17). In addition, since we are interested in the effects due to the dissipation rates l_α when different temperatures are present, we shall restrict our attention to eq. (4.17) which holds for mixtures of non-conducting fluids. The case of omothermal waves is quite similar to previous results $[4]_2$ in that the differences of temperatures do not affect the qualitative behaviour of the amplitude $s(t)$.

The solution of (4.17) is

$$(6.1) \qquad s(t) = \frac{s_0 \exp(\bar\Lambda t)}{1 - \dfrac{s_0 \bar\Gamma}{\bar\Lambda}(\exp(\bar\Lambda t) - 1)}$$

where $s_0 = s(0) \neq 0$ is the initial amplitude of the wave $\bar\Lambda = \Lambda^{(m)} + \Lambda^{(l)}$. Apart from the trivial case in which $\bar\Lambda = 0$ and $\bar\Gamma = 0$, we must distinguish the following two circumstances, depending on the sign of $s_0 \bar\Gamma$:

(1) $s_0 \bar\Gamma > 0$. This case corresponds to compressive waves when $\bar\Gamma > 0$ and

expansive waves when $\bar{\Gamma} < 0$. If $\bar{\Lambda} > 0$ eq. (6.1) yields

$$(6.2) \qquad \lim_{t \to T} s(t) = \infty \qquad \text{where} \qquad (6.3) \qquad T = \frac{1}{\bar{\Lambda}} \ln(1 + \frac{\bar{\Lambda}}{s_0 \bar{\Gamma}}) .$$

In this case the wave amplitude grows giving rise to a shock wave in a finite time T. If $\bar{\Lambda} < 0$, the evolution of $s(t)$ depends on the values of s_0. In particular, for $|s_0| > \bar{\Lambda}/\bar{\Gamma}|$ we obtain again (6.2) and (6.3), while for $|s_0| < |\bar{\Lambda}/\bar{\Gamma}|$ we get

$$(6.4) \qquad\qquad \lim_{t \to \infty} s(t) = 0 .$$

The quantity $s_c = |\bar{\Lambda}/\bar{\Gamma}|$ is a critical amplitude in that $s(t)$ grows without bound for $|s_0| > s_c$ and decays to zero for $|s_0| < s_c$.

(2) $s_0 \bar{\Gamma} < 0$. This case corresponds to expansive waves when $\bar{\Gamma} > 0$ and compressive waves when $\bar{\Gamma} < 0$. If $\bar{\Lambda} < 0$, from (6.1) we have

$$\lim_{t \to \infty} s(t) = 0$$

i.e. all compressive or expansive waves attenuate in time independently on the initial amplitude s_0. If $\bar{\Lambda} > 0$ we obtain

$$(6.5) \qquad\qquad \lim_{t \to \infty} s(t) = -\frac{\bar{\Lambda}}{\bar{\Gamma}} .$$

In this case compressive (expansive) waves tend to a positive (negative) finite amplitude of strength s_c.

7 - Final remarks

To sum up the main results we remark that propagation of acceleration waves in mixtures of non-ideal fluids with several temperatures is allowed if the matrices $\Omega_{\alpha\beta}$ and $A_{\alpha\beta}$ are positive definite. In particular, the matrix $\Omega_{\alpha\beta}$ may be obtained from the usual acoustic matrix for common temperatures substituting the partial chemical potentials μ_α with the quantities $\mu_\alpha/\vartheta_\alpha$. Differences of temperature among the mixture constituents ahead of the wave substantially

affect the wave evolution in mixtures of non-conducting fluids. This fact is illustrated by the matrix (5.16) whose sign depends on the balance between the power densities r_α and and the partial derivatives of the dissipation rates l_α.

In connection with the analysis of 6 we observe that in many circumstances $\bar{\Gamma} > 0$. Hence, having in mind (5.16)$_2$, the results (6.2) and (6.3) imply that, for sufficiently high power densities r_α, a compressive wave grows into a shock wave independently on its initial amplitude. Analogously, for $\bar{A} < 0$, an increase of power densities may cause the decreasing of the critical amplitude $|\bar{A}/\bar{\Gamma}|$, thus making the formation of a shock wave easier. Finally, (6.5) shows that an expansive wave cannot grow beyond a finite limiting amplitude, no matter how large the power densities of the external sources are.

As a final comment we note that, in view of (3.15) and (3.19) we may write

$$[\boldsymbol{\nabla} \vartheta_\beta] = -\frac{n}{u_n^2} (\sum_{\gamma, \delta} \rho_\delta^+ C_{\gamma\delta}^{-1} X_{\beta\gamma} z_\gamma) s$$

which shows that the differences of temperatures may be enhanced or reduced by wave propagation at a rate proportional to the wave amplitude.

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Sommario

Si studia l'evoluzione delle onde di accelerazione in una mistura non ideale di fluidi con diversa temperatura, in presenza di sorgenti esterne di energia. Si determinano le condizioni di propagazione nel caso di fluidi non conduttori di calore e nel caso delle onde omotermiche. Si scrivono poi le equazioni differenziali per l'ampiezza dell'onda nei due casi e si mette in evidenza la dipendenza della soluzione dai termini dissipativi e dalle sorgenti esterne nel caso particolare di fluidi non conduttori.

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