Riv. Mat. Univ. Parma (4) **15** (1989), 63-68

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Curvature on manifolds with almost contact 3-structure (**)

Introduction

The main purpose of this paper is to discuss some properties about curvature on manifolds with cosymplectic almost contact 3-structure and Sasakian 3-structure.

Among others results, we obtain that for a differentiable manifold M of dimension $\geqslant 11$ and with a φ_i-cosymplectic almost contact 3-structure, the curvature tensor vanishes identically if M has constant φ_i-sectional curvature.

1 - Preliminaries

Let M be a differentiable manifold with an *almost contact 3-structure* $(\varphi_i, \xi_i, \eta^i)$ $(i=1, 2, 3)$. For general references and notations, see $[5]_2$.

A Riemannian metric g on M is said to be *associated* to the almost contact 3-structure if it satisfies

$$g(\varphi_i X, \varphi_i Y) = g(X, Y) - \eta^i(X)\eta^i(Y) \qquad (i=1, 2, 3)$$

for any vector fields X, Y on M.

In a differentiable manifold with an almost contact 3-structure there always exists an associated metric g, and $(\varphi_i, \xi_i, \eta^i, g)$ $(i=1, 2, 3)$ is called an *almost contact metric* 3-structure.

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Let us consider the product manifold $M \times \mathbb{R}$. A *quaternion structure* on $M \times \mathbb{R}$ can be defined as follows

$$\psi_i(X, \ a\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi_i X - a\xi_i, \ \eta^i(X)\frac{\mathrm{d}}{\mathrm{d}t}) \qquad (i = 1, 2, 3) \tag{1.1}$$

for any vector field $(X, \ a\frac{\mathrm{d}}{\mathrm{d}t})$ on $M \times \mathbb{R}$, i.e. X is a differentiable vector field on M, a is a C^∞ function on $M \times \mathbb{R}$ and t is the usual coordinate on $\mathbb{R}$.

Let g be an associated metric for the almost contact 3-structure. Then, $\hat{g}$ given by

$$\hat{g}((X, \ a\frac{\mathrm{d}}{\mathrm{d}t}), \ (Y, \ b\frac{\mathrm{d}}{\mathrm{d}t})) = g(X, \ Y) + ab$$

is an *associated metric for the quaternionic structure* $(M \times \mathbb{R}, \ \psi_i)$, $(i = 1, 2, 3)$.

Let $(\varphi_i, \ \xi_i, \ \eta^i, \ g)$ $(i = 1, 2, 3)$, be an almost contact metric 3-structure on M. Then $(\varphi_i, \ \xi_i, \ \eta^i, \ g)$, $(i = 1, 2, 3)$, is called *Sasakian 3-structure* if the following relations for the structure tensor fields hold

$$(1) \quad \mathrm{d}\eta^i(X, \ Y) = g(X, \ \varphi_i Y) \qquad (2) \quad N_{\varphi_i} + 2\mathrm{d}\eta^i \otimes \xi_i = 0 \qquad (i = 1, 2, 3)$$

where X, Y are arbitrary vector fields on M, and N_{φ_i} is the Nijenhuis tensor of φ_i.

Note that the second condition implies that $(\mathcal{L}_{\xi_i}\varphi_i)Y = 0$ for any vector field Y such that $\eta^i(Y) = 0$, where $\mathcal{L}$ is the Lie differentiation.

We say that the almost contact metric 3-structure $(\varphi_i, \ \xi_i, \ \eta^i, \ g)$ $(i = 1, 2, 3)$, is *cosymplectic* if $(M \times \mathbb{R}, \ \psi_i)$ $(i = 1, 2, 3)$ is a quaternion Kaehler manifold, and we say that $(\varphi_i, \ \xi_i, \ \eta^i, \ g)$ $(i = 1, 2, 3)$, is *φ_i-cosymplectic*, if for any $i = 1, 2, 3$ the almost contact structure $(\varphi_i, \ \xi_i, \ \eta^i)$ is cosymplectic [6].

In the sequel, we denote by Ω the *fundamental* 4*-form of the almost contact* 3*-structure*, given by

$$\Omega = \sum_{i=1}^{3} F^i \wedge F^i$$

being F^i the fundamental 2-form given by $F^i(X, \ Y) = g(X, \ \varphi_i Y)$, and ∇ the Riemannian connection of g.

Then we have

Theorem. *$(\varphi_i, \xi_i, \eta^i, g)$ $(i=1, 2, 3)$ is cosymplectic if and only if the following identities hold*

$$(\nabla_X \Omega)(Y, Z, V, W) = 0 \qquad \sum_{i=1}^{3} (\nabla_X(\eta^i \wedge F^i))(Y, Z, V) = 0$$

for any vector fields X, Y, Z, V, W on M.

The proof requires long but not difficult calculations [5].

2 - Curvature properties

Let $(\varphi_i, \xi_i, \eta^i, g)$ $(i=1, 2, 3)$ be an almost contact metric 3-structure on M, and let $(\psi_i, \hat{g})$ the quaternionic structure associated on $M \times \mathbb{R}$ as in (1.1). Denote by ∇ and D the Riemannian connections on M and $M \times \mathbb{R}$ respectively.

The bracket product of two vectors fields on $M \times \mathbb{R}$ is given by

$$[(X, a\frac{d}{dt}), (Y, \frac{d}{dt})] = ([X, Y], (X(b) - Y(a) + a\frac{db}{dt} - b\frac{da}{dt})\frac{d}{dt}).$$

Denoting by $\hat{R}$ and R the *curvature tensors* on $M \times \mathbb{R}$ and M respectively, we get

$$\hat{R}((X, a\frac{d}{dt}), (Y, \frac{d}{dt}))(Z, \frac{d}{dt}) = (R(X, Y)Z, 0).$$

If we denote by $\hat{S}$ and S the *Ricci tensors* of $M \times \mathbb{R}$ and M repectively, then

$$\hat{S}((X, A\frac{d}{dt}), (Y, b\frac{d}{dt})) = S(X, Y) \circ \Pi$$

being $\Pi: M \times \mathbb{R} \to M$ the natural projection.

Theorem 2.1. *Let M be a differentiable manifold of dimension $\geqslant 7$ with a cosymplectic almost contact 3-structure. Then, the Ricci tensor vanishes identically.*

Proof. Since $M \times \mathbb{R}$ is a quaternion Kaehler manifold of dimension $\geqslant 8$ and $M \times \mathbb{R}$ is reducible, then the Ricci tensor of $M \times \mathbb{R}$ vanishes [2], and by consequence the Ricci tensor of M vanishes.

Corollary 2.1. *For any manifold of dimension $\geqslant 7$, with a cosymplectic almost contact 3-structure the scalar curvature vanishes identically.*

Theorem 2.2. *For any 3 dimensional manifold, with a φ_i-cosymplectic almost contact 3-structure the curvature tensor vanishes identically.*

Proof. For any $i=1,\ 2,\ 3$ the almost contact structure $(\varphi_i,\ \xi_i,\ \eta^i,\ g)$ is cosymplectic, i.e. $(\nabla_X \varphi_i) Y = 0$, for any vector fields X, Y on M. Then, $\nabla_X \xi_i = 0$, for any vector field X on M. Hence $R(\xi_i,\ \xi_j)\,\xi_k = 0$ and the curvature tensor vanishes.

Theorem 2.3. *If a differentiable manifold M with a cosymplectic almost contact 3-structure has nonvanishing constant curvature, then M has dimension* 3.

Proof. It follows from [2] taking into account that $M \times \mathbb{R}$ is a quaternion Kaehler manifold with nonvanishing constant curvature, and hence $M \times \mathbb{R}$ has dimension 4.

Theorem 2.4. *Let M be a differentiable manifold with Sasakian 3-structure $(\varphi_i,\ \xi_i,\ \eta^i,\ g)$ $(i=1,\ 2,\ 3)$. Then the sectional curvature verifies $K(\xi_i,\ X)=1$ for any vector field X on M such that $\eta^i(X)=0$ and $g(X,\ X)=1$.*

Proof. Since $(\varphi_i,\ \xi_i,\ \eta^i,\ g)$ $(i=1,\ 2,\ 3)$, is a Sasakian 3-structure, then $\nabla_X \xi_i = -\varphi_i X$ [1]. Hence

$$K(\xi_i,\ X) = g(R(X,\ \xi_i)\,\xi_i,\ X)$$

$$= g(\nabla_X \nabla_{\xi_i} \xi_i - \nabla_{\xi_i} \nabla_X \xi_i - \nabla_{[X,\xi_i]} \xi_i,\ X) = g(\nabla_{\xi_i} \varphi_i X + \varphi_i [X,\ \xi_i],\ X)\,.$$

On the other hand

$$0 = (\mathscr{L}_{\xi_i} \varphi_i) X = [\xi_i,\ \varphi_i X] - \varphi_i [\xi_i,\ X]\,.$$

Thus we have

$$g(\nabla_{\xi_i}\varphi_i X+\varphi_i[X,\ \xi_i],\ X)=-g(\varphi_i^2 X,\ X)=g(X,\ X)=1.$$

Next, let p be a point of a manifold M with an almost contact metric 3-structure $(\varphi_i,\ \xi_i,\ \eta^i,\ g)$ $(i=1, 2, 3)$, and let X be a tangent vector to M at p such that $\eta^i(X)=0$ $(i=1,\ 2,\ 3)$. Then, the 4-dimensional subspace $\mathcal{S}_{\varphi_i}(X)$ of the tangent space to M at p, $T_p(M)$, defined by

$$\mathcal{S}_{\varphi_i}(X)=\{Y|Y=aX+b\varphi_1 X+c\varphi_2 X+d\varphi_3 X\}$$

$a,\ b,\ c,\ d$ being arbitrary real numbers, is called the *φ_i-section* determined by X at p.

If the sectional curvature $K(Y,\ Z)$ for any $Y,\ Z\in\mathcal{S}_{\varphi_i}(X)$ is a constant $\rho(X)$, it will be called *φ_i-sectional curvature* with respect to X at p.

Now, if $(\varphi_i,\ \xi_i,\ \eta^i,\ g)$ $(i=1,\ 2,\ 3)$ is a φ_i-cosymplectic almost contact 3-structure on a manifold M of dimension $\geqq 11$, and if M has constant φ_i-sectional curvature $c(p)=\rho(X)$, then its curvature tensor has the nice form [3]

$$R(X,\ Y)Z=\frac{1}{4}c\{g(Y,\ Z)X-g(X,\ Z)Y \tag{2.1}$$
$$+\sum(g(\varphi_i Y,\ Z)\varphi_i X-g(\varphi_i X,\ Z)\varphi_i Y+g(X,\ \varphi_i Y)\varphi_i Z)$$
$$+\sum(\eta^i(X)\eta^i(Z)Y-\eta^i(Y)\eta^i(Z)X)+\sum(\eta^i\wedge\eta^j)(X,\ Z)\varphi_k Y$$
$$-(\eta^i\wedge\eta^j)(Y,\ Z)\varphi_k X-2(\eta^i\wedge\eta^j)(Y,\ Z)\varphi_k Z)+B(\xi_1,\ \xi_2,\ \xi_3,\ X,\ Y,\ Z)\}$$

where $i,\ j,\ k$ take the values 1, 2, 3, for any vector fields $X,\ Y,\ Z$, on M, being $B(\xi_1,\ \xi_2,\ \xi_3,\ X,\ Y,\ Z)$ a vector field belonging to the subspace of $T_p(M)$ spanned by $\xi_1,\ \xi_2,\ \xi_3$ [3].

Theorem 2.5. *Let M be a differentiable manifold of dimension $\geqq 11$ and with a φ_i-cosymplectic almost contact 3-structure. If M has constant φ_i-sectional curvature at each point, then the curvature tensor vanishes identically.*

Proof. From Theorem 2.1, the Ricci tensor vanishes. Consider the φ_i-basis: $\{X_i,\ \varphi_j X_i,\ \xi_i\}$ $(i=1,\ \dots,\ n,\ j=1,\ 2,\ 3)$.

Denote by S the Ricci tensor. Then we have

$$0 = S(X_1, X_1) = \sum g(R(X_i, X_1)X_1, X_i)$$

$$+ \sum g(R(\varphi_j X_i, X_1)X_1, \varphi_j X_i) + \sum g(R(\xi_j, X_1)X_1, \xi_j)$$

where i takes the values 1, 2, ..., n and j takes the values 1, 2, 3.

Using (2.1), we get

$$g(R(X_i, X_1)X_1, X_i) = \frac{n+1}{4}c$$

$$g(R(\varphi_j X_i, X_1)X_1, \varphi_j X_i) = 3c + \frac{3(n+1)}{4}c \qquad g(R(\xi_j, X_1)X_1, \xi_j) = 0.$$

Hence, $0 = S(X_1, X_1) = (n+2)c$ and, by consequence, the curvature tensor vanishes.

References

[1] D. BLAIR, *Contact manifolds in Riemannian Geometry*, Lecture Notes in Math. **509**, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[2] S. ISHIHARA, *Quaternion Kaehlerian manifolds*, J. Differential Geom. **9** (1974), 483-500.

[3] M. KOBAYASHI, *Differential Geometry of symmetric twofold CR submanifolds in manifolds with cosymplectic 3-structure*, Tensor N.S. **41** (1984), 69-89.

[4] Y. Y. KUO, *On almost contact 3-structure*, Tôhoku Math. J. **22** (1970), 325-332.

[5] D. MONAR: [•]$_1$ *3-estructuras casi contacto*, Doctoral Thesis. Sec. de Public. Univ. de La Laguna, 1987; [•]$_2$ *Affine connections on manifolds with almost contact 3-structure*, Riv. Mat. Univ. Parma (4) **14** (1988), 237-248.

[6] J. A. OUBIÑA, *A classification for almost contact structures*, Preprint.

Summary

See Introduction.

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