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Toeplitz operators associated with Bergman and Hardy spaces (**)

Introduction

Let $0<p<\infty$. The Hardy space H^p consists of all analytic functions f defined on the open unit disk $D=\{z\colon |z|<1\}$, for which $M_p(r, f)$ remains bounded as $r\to 1$, where

$$M_p(r, f)=\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\}^{1/p}.$$

The space H^∞ consists of all bounded analytic functions defined on D.

If a function f is in $H^p(1\leqslant p\leqslant\infty)$, then $\lim_{r\to1} f(re^{i\theta})$ exists almost everywhere on the unit circle ∂D. This of course, permits an association of H^p with a closed subspace of $L^p(\partial D)$. This subspace of $L^p(\partial D)$ (still denoted by H^p) consists of those functions in $L^p(\partial D)$ which has vanishing negative Fourier coefficients, that is

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, e^{int}\,dt=0 \qquad \text{for } n>0.$$

The above results appear in [7], [9].

The Bergman space $A^p(D)$ is the space of all analytic functions f defined on D

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for which $|f|^p$ is integrable with respect to the area measure $dA = \frac{1}{\pi} dy\, dx$. Obviously $A^p(D)$ is a Banach space where $p \geq 1$ (one can convince oneself that is the case by a proof in all similar to the classic one for H^p). In particular if $p = 2$, $A^2(D)$ is a Hilbert space. It can be easily checked from the definition of the Hardy space H^p, and the Bergman space $A^p(D)$, $0 < p < \infty$ that H^p is contained in $A^p(D)$. The inclusion $H^p \subset A^{2p}$ follows from a particular case of an inequality of Hardy and Littlewood (see [7], p. 87).

For $1 < p < \infty$ and $\psi \in L^\infty(D)$, the Toeplitz operator T_ψ is defined on the Bergman space $A^p(D)$ by $T_\psi(f) = P(\psi, f)$, where P is the projection of $L^p(D)$ onto $A^p(D)$. In particular P is defined as follows

$$(Pf)(z) = \langle f, k_z \rangle = \int_D f(\zeta)(1 - \bar{\xi} z)^{-2}\, dA(\xi)$$

where $k_z(\xi) = (1 - \bar{z}\xi)^{-2}$. Toeplitz operators on the Hardy spaces H^p are defined in a similar manner. The Hankel operator $H_\psi : A^2(D) \to (A^2(D))^\perp$ is defined by $H_\psi(f) = (I - P)\psi f$.

Toeplitz operators on the Hardy space H^2 have been the object of much study (see [3], [6]). In [3] the algebraic properties of Toeplitz operators were discussed. In [6], it was shown that the spectrum and essential spectrum of such operators were connected, and the only compact Toeplitz operator is the zero operator.

The primary goal of this paper is to study some of the above mentioned properties for Toeplitz operator defined on the Bergman space $A^2(D)$.

In 1 some algebraic properties of Toeplitz operators defined on $A^2(D)$ are proved. In particular, it is shown that if $T_\phi T_\psi = T_g$, and kernel $H_\psi \neq \{0\}$, then $g = \phi\psi$ almost everywhere. Also, it is proved that if $\phi \in C(\bar{D})$ and $|\phi|$ attains its maximum on the unit circle ∂D such that T_ϕ is an isometry, then ϕ is constant.

In 2 an example of a compact Toeplitz operator which is not Hilbert-Schmidt is constructed. In fact it is shown that if $\phi(r, \theta) = (1 - r)^\alpha$, where $0 < \alpha \leq 1/2$ then the Toeplitz operator T_ϕ defined on the Bergman space $A^2(D)$ is compact but nor Hilbert-Schmidt. It is also shown that the operator T_ϕ is actually a trace class only if $\alpha > 1$.

In 3 some spectral properties of Toeplitz operators defined on $A^2(D)$ are studied. In particular it is shown that $\sigma(T_\phi) = \overline{\phi(D)}$, where $\phi(z) = z^i|z|^n + \lambda\bar{z}^i|z|^n$, i and n are integers such that $i > 0$, $n \geq 0$, λ is real and $|\lambda| \leq 1$. Moreover, the spectrum of T_ψ is completely determined, where ψ is in a class of complex valued harmonic functions (not analytic).

1 - Algebraic properties

The purpose of this section is to study some algebraic properties of Toeplitz operators defined on the Bergman space $A^2(D)$ and to compare them with those on the Hardy space H^2.

The theory of Toeplitz operators is not easy, and this makes it difficult to predict all algebraic properties of T_ϕ by the behaviour of ϕ. One useful too in trying to get related good results, is the matrix associated with the Toeplitz operator.

Consider the orthonormal basis $\{e_n = e^{in\theta},\ n = 0,\ 1,\ 2,\ \dots\}$ for H^2, the matrix $\{a_{mn}\}$ of the Toeplitz operator T_ϕ, $\phi \in L^\infty(\partial D)$ associated with it is

$$a_{mn} = \langle T_\phi e_n,\ e_m \rangle = \langle T_\phi e_{n+1},\ e_{m+1} \rangle = a_{m+1,n+1} = \hat{\phi}(m-n)$$

where $\hat{\phi}(n)$ are the Fourier coefficients of ϕ. Thus, the matrix for T_ϕ is constant on diagonals, such a matrix is called a Toeplitz matrix. This special form of the Toeplitz matrix made the algebraic properties relatively easily derivable. An important result of this special form was proved in [3]. It was shown that if ϕ and ψ were in $L^\infty(\partial D)$ then $T_\phi T_\psi = T_{\phi\psi}$ if and only if ψ is analytic or $\bar{\phi}$ is analytic. As a consequence, it was proved that for $\phi \in L^\infty(\partial D)$ T_ϕ is an isometry if and only if ϕ is analytic and $|\phi| = 1$.

It would be interesting to have similar characterization for Toeplitz operators defined on the Bergman space $A^2(D)$. Since the Toeplitz matrix on $A^2(D)$ does not have a special form as that on H^2, different tools should be used in studying their algebraic properties. However, it can be easily established that if $\phi \in L^\infty(D)$ and $T_{\bar{\phi}} T_\phi = T_{|\phi|^2}$, then $\phi \in H^\infty(D)$. In what follows we prove the following.

Proposition 1.1. *Let ϕ, ψ and g be in $L^\infty(D)$. Assume that $T_\phi T_\psi = T_g$, and kernel $H_\psi \neq \{0\}$. Then $g = \phi\psi$.*

Proof. Let $f \in A^2(D)$ such that $H_\psi f = 0$. From this it follows that $\psi f \in A^2(D)$, and thus $\psi z^n f \in A^2(D)$ for all $n \geq 0$. Now

$$\langle T_\phi T_\psi z^n f,\ z^m \rangle = \langle T_g z^n f,\ z^m \rangle \qquad n,\ m \geq 0$$

implies that
$$\langle \phi\psi z^n f,\ z^m \rangle = \langle g z^n f,\ z^m \rangle .$$

Therefore
$$\langle \phi\psi f,\ \bar{z}^n z^m \rangle = \langle g f,\ \bar{z}^n z^m \rangle .$$

This show that $\langle(\phi\psi-g)f,\ \bar{z}^n z^m\rangle=0$ for all $n,\ m\geq 0$.

But the span of $\{z^m\bar{z}^n\}_{m,n,\geq 0}$ is dense in $L^2(D)$, so $(\phi\psi-g)f=0$. Consequently, $\phi\psi=g$ since f cannot vanish on a set of positive measure, and this ends the proof.

Remark 1.1. A question related to Proposition 1.1 is to characterize those functions in $L^\infty(D)$ such that kernel of $H_\phi\neq 0$.

Proposition 1.2. *Let $\phi\in C(\bar{D})$ such that $|\phi|$ attains its maximum on the unit circle ∂D. If $T_{\bar{\phi}}T_\phi=I$, then ϕ is constant.*

Proof. It can be easily checked that

$$H_\phi^* H_\phi = T_{|\phi|^2} - T_{\bar{\phi}} T_\phi = T_{|\phi|^2-1}.$$

Since $\phi\in C(\bar{D})$, then it follows by **[11]** that $H_\phi^* H_\phi$ is compact, and this will imply that $|\phi|=1$ on the unit circle ∂D (see [4]). Therefore, $|\phi(z)|\leq 1$ for all $z\in\bar{D}$, and hence $\|\phi\|_2\leq 1$. Let $\phi=\phi_1+\phi_2$, where $\phi_1\in A^2(D)$ and $\phi_2\in(A^2(D))^\perp$. Using the fact that T_ϕ is an isometry, it follows that $\|T_\phi 1\|=\|\phi_1\|_2$. Thus, $\phi_2=0$. Therefore, $\phi\in H^\infty(D)$, and hence $T_{\bar{\phi}}T_\phi=T_{|\phi|^2}=T_1$. From this it follows that $T_{|\phi|^2-1}\equiv 0$, and consequently ϕ is constant.

One final note regarding the algebraic properties of Toeplitz operators is that in [3] it was shown that the Toeplitz operator T_ϕ defined on the Hardy space H^2 is normal if and only if $\phi=\alpha+\beta f$, where α, β are complex numbers, and f is a real valued function. However, it can be easily checked that if $\phi(r,\ \theta)=\psi(r)$ (radial), then the matrix associated with the Toeplitz operator defined on the Bergman space $A^2(D)$ is diagonal. Thus, T_ϕ is a normal operator.

2 - Compactness

The theory of compact Toeplitz operators defined on the Hardy space H^p, $1<p<\infty$, is relatively easy. In fact, and in this case, the only compact Toeplitz operator is the zero operator. To see this, Widom **[15]** showed that if $\phi\in L^\infty(\partial D)$ where ∂D is the unit circle, $\phi=\phi_1+i\phi_2$, then spectrum of T_ϕ is connected. Moreover, using the fact that T_ϕ^* is compact and that the spectrum of a compact operator is countable with zero in the spectrum, it follows that $\sigma(T_{\phi_1})=\{0\}$ and

$\sigma(T_{\phi_2}) = \{0\}$. But T_{ϕ_1} and T_{ϕ_2} are self-adjoint and hence normal operators, so it follows by [8] that $T_{\phi_1} = T_{\phi_2} = 0$, and hence $T_\phi \equiv 0$.

The theory of compact Toeplitz operators on the Bergman space $A^2(D)$ is rather deep. In fact Coburn [4] showed that if $\phi \in C(\bar{D})$, then T_ϕ is compact if and only if $\phi|\partial D \equiv 0$. Moreover, compact Toeplitz operators were studied in $[10]_2$. Our goal now is to find a class of Toeplitz operators which is of Hilbert-Schmidt but not of trace class. First, the following is needed.

Def. 2.1. Let H be a Hilbert space, and $\{e_n\}$ be an orthonormal basis for H. *The operator* $T \in L(H)$ is said to be *Hilbert-Schmidt* if

$$\|T\|_2 = \left(\sum_{n=0}^{\infty} \|Te_n\|^2\right)^{\frac{1}{2}}$$

is finite. The set of all Hilbert-Schmidt operators will be denoted by σ_c.

Note that for each n, $\|Te_n\|^2 = \sum_{m=0}^{\infty} |\langle te_n, e_m \rangle|^2$. Thus, to show an operator $T \in L(H)$ is Hilbert-Schmidt, it suffices to show that $\sum_{m,n=0}^{\infty} |\langle Te_n, e_m \rangle|^2$ is finite. Also, it is well-known that every operator T in σ_c is necessarily compact.

Def. 2.2. The products of two operators in σ_c form the trace class τ_c. If $\{e_i\}$ is a given basis, then for $T \in \tau_c$ the finite number

$$t(T) = \sum_i \langle te_i, e_i \rangle$$

defines the *trace* of T.

As a consequence of Def. 2.2, every operator in τ_c is in σ_c, and hence is compact. Also, it is well-known that $T \in L(H)$ is a trace class operator if and only if $\operatorname{tr}(T^*T)^{\frac{1}{2}}$ is finite. For detailed and further more information about these operators (see [5], [14]).

Lemma 2.1. *For* $\alpha > -1$ *and* $n = 0, 1, 2,$ *we have*

$$\int_0^1 r^n (1-r)^\alpha \, dr = n!/(\alpha+1)(\alpha+2)\dots(\alpha+n+1).$$

Lemma 2.1 is the well-known Beta integral.

Proposition 2.1. *Let* $\phi(r, \theta) = (1-r)^{\alpha}$ *where* $\alpha > 0$, *and let* T_{ϕ} *be the Toeplitz operator defined on the Bergman space* $A^2(D)$. *Then:*

(i) T_{ϕ} *is Hilbert-Schmidt only if* $\alpha > 1/2$.

(ii) T_{ϕ} *is of trace class only if* $\alpha > 1$.

Proof. (i) If $\{a_{mn}\}$ is the matrix associated with T_{ϕ}, then $\{a_{mn}\}$ is a diagonal matrix, and

$$a_{nn} = (n+1)\langle (1-r)^{\alpha} z^n z^n \rangle = 2(n+1)\int_0^1 r^{2n+1}(1-r)^{\alpha}\, dr.$$

Using Lemma 2.1

$$a_{nn} = (2n+2)!\,/\,(\alpha+1)(\alpha+2)\ldots(\alpha+2n+2). \tag{1}$$

From previous discussion, it suffices to show that

$$\sum_{n=0}^{\infty} [(2n+2)!\,/\,(\alpha+1)(\alpha+2)\ldots(\alpha+2n+2)]^2 \tag{2}$$

diverges for $\alpha \leq 1/2$. The convergence of (2) can be checked very easily using Stirling formulas. Thus (i) is achieved.

To prove the second part of the theorem, consider equation (1) and apply Raab's test to the series

$$\sum_{n=0}^{\infty} (2n+2)!/(\alpha+1)(\alpha+2)\ldots(\alpha+2n+2).$$

If u_n denote the n^{th} term of the series, then

$$\lim_{n\to\infty} n\left(1 - \frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} n(1 - (2n+4)(2n+3)/(\alpha+2n+3)(\alpha+2n+4)) = \alpha.$$

Thus, T_{ϕ} is a trace class operator if $\alpha > 1$. Moreover, if $\alpha < 1$, T_{ϕ} is not of trace class. For the case $\alpha = 1$,

$$\sum_{n=0}^{\infty} a_{nn} = \sum_{n=0}^{\infty} (2n+2)!/2.3.4.\ \ldots\ (2n+3) = \sum_{n=0}^{\infty} \frac{1}{2n+3}$$

which is divergent and hence T_{ϕ} is not of trace class, completing the proof.

Remark 2.1. In Proposition 2.1 and for $\alpha \leqslant 1/2$, a class of compact Toeplitz operators on the Bergman space $A^2(D)$ which is not Hilbert-Schmidt was given. Also, for $1/2 < \alpha \leqslant 1$, T_ϕ is a Hilbert-Schmidt operator which is not of trace class. Finally, Luecking [10]$_2$ showed that the only finite rank Toeplitz operator T_ϕ defined on the Bergman space $A^2(D)$ is the zero operator.

3 - Spectral properties

In this section various results on the spectrum and essential spectrum of certain classes of Toeplitz operators defined on the Bergman space $A^2(D)$ are obtained.

Let $\phi \in L^\infty(\partial D)$ be a real-valued function. It was shown in [6], that the spectrum of T_ϕ, $\sigma(T_\phi) = [\text{ess inf}\,\phi,\ \text{ess sup}\,\phi]$, where T_ϕ is the Toeplitz operator defined on H^2. In fact, $\sigma(T_\phi) = \sigma_e(T_\phi)$, where $\sigma_e(T_\phi)$ denotes the essential spectrum of T_ϕ, and this is due to the fact that $\ker T_\phi = \{0\}$. For Toeplitz operators defined on the Bergman space $A^2(D)$, it is no longer true that $\sigma(T_\phi) = [\text{ess inf}\,\phi,\ \text{ess sup}\,\phi]$.

For example, if $\phi(z) = 1 - |z|^2$, then $\sigma(T_\phi)$ is countable, since T_ϕ is compact, while $[\text{ess inf}\,\phi,\ \text{ess sup}\,\phi] = [0,\ 1]$, and $\sigma_e(T_\phi) = \{0\}$; since $\phi \in C(\bar{D})$ and $\phi|\partial D \equiv 0$.

It was shown in [12] that if ϕ is a real-valued bounded harmonic function in the unit disk D, then $\sigma_e(T_\phi) = \sigma(T_\phi) = [\inf \phi,\ \sup \phi]$. However, the previous result is no longer true if ϕ is a complex bounded harmonic function in D. For example, let $\phi(z) = z$, $z \in D$. Then, $\sigma(T_\phi) = \bar{D}$, since $\phi \in H^\infty(D)$, while $\sigma_e(T_\phi) = \{z\colon\ |z| = 1\}$, since $\phi \in C(\bar{D})$. However, if $\phi \in H^\infty(B)$, where B is the open unit ball in the complex plane C^n $(n \geqslant 2)$ and T_ϕ is the Toeplitz operator defined on the Bergman space $A^2(B)$, then $\sigma_e(T_\phi) = \sigma(T_\phi) = \overline{\phi(B)}$. This follows from the facts that $\sigma(T_\phi) = \overline{\phi(B)}$ and index $T_\phi = \{0\}$ (see [11]).

In what follows, it will be shown that it is not necessarily true that $\sigma_e(T_\phi) = [\inf \phi,\ \sup \phi]$ whenever ϕ is a subharmonic function. Moreover, the spectrum of a class of complex bounded harmonic functions (not analytic) will be determined.

Example 3.1. Let $\phi(z) = |z|^2$, $z \in D$. The function ϕ is subharmonic, $[\inf \phi,\ \sup \phi] = [0,\ 1]$ and $\sigma_e(T_\phi) = \{1\}$. Also, note that, $T_{|z|^2-1}$ is a compact Toeplitz operator, and hence $\sigma(T_{|z|^2-1})$ is countable. Thus, $\sigma(T_\phi)$ is countable, and this ends the example.

Example 3.2. Let $\phi(x,\ y) = y^2$. Then ϕ is a subharmonic function in the open unit disk D and $\sigma_e(T_\phi)=[0,\ 1]$. Using the facts that $\sigma(T_\phi)\subset[\inf\phi,\ \sup\phi]$ and $\sigma_e(T_\phi)\subset\sigma(T_\phi)$, it follows that

$$[0,\ 1]=\sigma_e(T_\phi)\subset\sigma(T_\phi)\subset[\inf\phi,\ \sup\phi]=[0,\ 1].$$

Thus
$$\sigma_e(T_\phi)=\sigma(T_\phi)=[\inf\phi,\ \sup\phi].$$

Def. 3.1. Let T be an operator on a Hilbert space H. T is said to be *hyponormal* if and only if $T^*T-TT^*\geqslant 0$. T is said to be *pure* if the only subspace of H reducing T on which T is normal is the zero subspace.

Lemma 3.1. *Let $T=X+iY$ be hyponormal (X, Y are self-adjoint, $\lambda>0$. Then*

$$\sigma(M_\lambda(T))=M_\lambda(\sigma(T)).$$

Here, $M_\lambda(T)=\lambda X+iY$ and $M_\lambda(z)=\lambda x+iy$.

Proof. See [16].

Lemma 3.3. Let $\phi(x,\ y)=\lambda x+iy$, where $x^2+y^2<1$, and $\lambda=0$. Direct computations show that

$$T_{\bar\phi}T_\phi-T_\phi T_{\bar\phi}=2i\lambda(T_xT_y-T_yT_x)=\lambda(T_{\bar z}T_z-T_zT_{\bar z}).$$

Using the facts that T_z is hyponormal and $\lambda>0$, then it follows that T_ϕ is hyponormal. Moreover, $\sigma(T_z)=\{(x,\ y):x^2+y^2\leqslant 1\}$. Thus, by Lemma 3.1, $\sigma(T_\phi)=\{(\lambda x,\ y):x^2+y^2\leqslant 1\}$. Therefore, $\sigma(T_\phi)$ is connected.

Theorem 3.1. *Let i and n be integers such that $i>0$, $n\geqslant 0$ and $\phi(z)=z^i|z|^n$. Then:*

(i) *T_ϕ is a pure hyponormal operator. Moreover the self-commutator of T_ϕ, $T_{\bar\phi}T_\phi-T_\phi T_{\bar\phi}$ is Hilbert-Schmidt.*

(ii) *$\sigma(T_\phi)=\bar D$. The approximate point spectrum, $\sigma_{ap}(T_\phi)=\partial D=\sigma_e(T_\phi)$, and for every $\alpha\in D$, $T_{\phi-\phi(\alpha)}$ is Fredholm, with non-zero index.*

(iii) *$\sigma(T_{Re(\phi)})=[-1,\ 1]$, and the point spectrum $\sigma_p(T_{Re(\phi)})$ is empty.*

(iv) *For $i=1$, $n\geqslant 0$, T_ϕ is a unilateral weighted shift with trace class self-commutator.*

Proof. (i) It can be easily checked that

$$z^i|z|^n = \frac{2i+2}{n+2i+2} z^i \oplus z^i|z|^n - \frac{2i+2}{n+2i+2} z^i. \tag{3}$$

Thus, using (3) and for $m \geqslant 0$, we have

$$T_{\bar{\phi}} T_{\phi} z^m = 4 \frac{(i+m+1)(m+1)}{(n+2i+2m+2)^2} z^m. \tag{4}$$

However, for $0 \leqslant m < i$, $T_{\phi} T_{\bar{\phi}} z^m = 0$ and for $m \geqslant i$

$$T_{\phi} T_{\bar{\phi}} z^m = 4 \frac{(m-i+1)(m+1)}{(n+2m+2)^2} z^m. \tag{5}$$

Thus, it follows from (4) and (5) that T_ϕ is hyponormal. Moreover, if M is the smallest reducing subspace of T_ϕ that contains the range of the self-commutator of T_ϕ, then from (4) and (5), it can be concluded that $M = A^2(D)$. Consequently, T_ϕ is a pure hyponormal operator. Moreover, by direct computation, it can be checked that

$$\sum_{n=0}^{\infty} \|T_{\bar{\phi}} T_{\phi} - T_{\phi} T_{\bar{\phi}}) e_n\|^2 < \infty \qquad \text{where } e_n = \sqrt{n+1}\, z^n.$$

Thus, the self-commutator of T_ϕ is Hilbert-Schmidt.

(ii) Since ϕ is continuous on $\bar{D}$, then $\sigma_e(T_\phi) = \phi(\partial D) = \partial D$ (see [**11**]). Also, for any $\alpha \in D$, we can find a neighborhood N_α of ∂D such that $|\phi(z) - \phi(\alpha)| > \varepsilon$ for all $z \in N_\alpha$. Thus, it follows by [**11**] that $T_{\phi - \phi(\alpha)}$ is Fredholm, consequently $\phi(\alpha) \notin \sigma_{ap}(T_\phi)$.

However, it can be easily checked that $\ker T_\phi$ is empty and $T_{\bar{\phi}} z^m = 0$ for $0 \leqslant m < i$. Thus, index of T_ϕ is different from zero and hence index $(T_{\phi-\phi(\alpha)})$ is different from zero for all $\alpha \in D$, and this will imply that $\phi(D) = D \subset \sigma(T_\phi)$. Moreover, it is well-known that $\sigma(T_\phi) \subset \overline{co}(R(\phi)) = \bar{D}$. Therefore, $\sigma(T_\phi) = \bar{D}$. Finally, using the fact that $\partial\sigma(T_\phi) \subset \sigma_{ap}(T_\phi)$, it follows that $\sigma_{ap}(T_\phi) = \partial D$.

(iii) Since T_ϕ is hyponormal, then it follows by [**16**] that $\sigma(\mathrm{Re}(T_\phi)) = \mathrm{Re}(\sigma(T_\phi))$.

Thus, using (ii), we have $\sigma(\mathrm{Re}(T_\phi)) = [-1, 1]$. Moreover, if $N_\alpha = \{f \in A^2(D): T_{\mathrm{Re}\phi} f = \alpha f\}$ is not empty, then N_α reduces T_ϕ. Using the fact that T_ϕ is hyponormal, it follows that $T_\phi | N_\alpha$ is normal, and this contradicts the fact that T_ϕ is pure. Thus, $\sigma_p(T_{\mathrm{Re}\phi})$ is empty.

(iv) Using (3) and for $i \geq 0$ it can be checked easily that $T_{z|z|^n} e_i = \alpha_i e_{i+1}$, where $\alpha_i = \dfrac{2\sqrt{i+1}\sqrt{i+2}}{n+2i+4}$. Thus, $T_{z|z|^n}$ is a pure hyponormal unilateral weighted shift. Also, note that 1 is a cyclic vector for $T_{z|z|^n}$. Thus by a result of Berger and Shaw [2], the self-commutator of $T_{z|z|^n}$ is trace class.

Corollary 3.1. *Let i, n be integers such that $i > 0$, $n \geq 0$, and $\psi(z) = z^i|z|^n + \lambda \bar{z}^i |z|^n$, $\phi(z) = z^i|z|^n$. Then:*

(i) *If $|\lambda| < 1$, T_ψ is a pure hyponormal operator with Hilbert-Schmidt self-commutator.*

(ii) *If λ is real, and $|\lambda|$ is different from one, then $\sigma(T_\psi) = \overline{\psi(D)}$.*

Proof. (i) It follows from Theorem 3.1 and the fact that

$$T_{\bar\psi} T_\psi - T_\psi T_{\bar\psi} = (1 - |\lambda|^2)(T_{\bar\phi} T_\phi - T_\phi T_{\bar\phi}).$$

(ii) Applying the same argument in (ii) of Theorem 3.1 one gets that

$$\sigma(T_\psi) \subset \overline{co}(\psi(D)) = \overline{\psi(D)} \qquad \sigma_e(T_\psi) = \psi(\partial D)$$

and $T_{\psi - \psi(\alpha)}$ is Fredholm for every $\alpha \in D$. Moreover, it can be easily checked that range of T_ψ si not $A^2(D)$. Thus, T_ψ is not invertible. However, since T_ψ is a pure hyponormal operator and $0 \in \sigma(T_\psi) - \sigma_e(T_\psi)$, then index $(T_\psi) \leq 1$, (see [5]). Thus, index $(T_{\psi-\psi(\alpha)}) \leq -1$ for every $\alpha \in D$, hence $T_{\psi-\psi(\alpha)}$ is not invertible for every $\alpha \in D$. Consequently, $\psi(D) \subseteq \sigma(T_\psi)$. Therefore, $\sigma(T_\psi) = \overline{\psi(D)}$.

Remark 3.1. For $|\lambda| = 1$, see (iii) in Theorem 3.1.

It is known that if $\phi \in L^\infty(\partial D)$, then either $\ker T_\phi = \{0\}$ or $\ker T_{\bar\phi} = \{0\}$. As a corollary of this result, it follows that if $\phi \in L^\infty(\partial D)$ such that the Toeplitz operator defined on the Hardy space H^2, is Fredholm, then T_ϕ is invertible if and only if index $T_\phi = \{0\}$. These results are no longer true for the case of Toeplitz operators defined on the Bergman space $A^2(D)$.

Example 3.4. Let $\phi(z) = \ln 2 - 1/(1+|z|^2)$. Note that $\phi \in C(\bar{D})$, and $\sigma_e(T_\phi) = \ln 2 - \frac{1}{2}$. Therefore, T_ϕ is Fredholm, and hence index $T_\phi = \{0\}$. Yet it can be easily checked that $1 \in \ker T_\phi$ and thus T_ϕ is not invertible.

One of the basic results of Toeplitz operators defined on the Hardy space H^2 is that if $\phi \in L^\infty(\partial D)$ and T_ϕ is invertible, then ϕ^{-1} exist in $L^\infty(\partial D)$. This result is no longer true for Toeplitz operators defined on $A^2(D)$. To see this, let $\phi(z) = |z|^2$. Note that range of T_ϕ is closed, since T_ϕ is Fredholm, moreover, using the fact that $\phi \geqslant 0$, it can be easily established that T_ϕ is $1-1$. Thus T_ϕ is invertible yet ϕ^{-1} does not exist in $L^\infty(D)$.

The following proposition furnishes us with a class of functions $\psi \in L^\infty(D)$ such that T_ψ is invertible.

Proposition 3.1. *Let $\psi = \psi_1 + i\psi_2$ be in $L^\infty(D)$. Suppose that $\psi_1 \geqslant 0$ (ψ_1 is not the zero function) and the range of T_{ψ_1} is closed, then T_ψ is invertible.*

Proof. It can be easily proved that T_{ψ_1} is invertible. Also, note that $\sigma(T_{\psi_1})$ is non-negative. Moreover, it is known that $\sigma(T_{\psi_1}) \subseteq [m, M]$, where $m = \inf_{\|f\|=1} \langle T_{\psi_1} f, f\rangle$, $M = \sup_{\|f\|=1} \langle T_{\psi_1} f, f\rangle$ and $m, M \in \sigma(T_{\psi_1})$, (see [1]). Since $0 \notin \sigma(T_{\psi_1})$ it can be concluded that $\langle T_{\psi_1} f, f\rangle \geqslant m > 0$, $\|f\| 1$. Consequently, the numerical range $W(T_\psi)$ of T_ψ is contained in the half-plane $\{z \in \mathbb{C}\colon \operatorname{Re} z \geqslant m\}$. Since $\sigma(T_\psi) \subseteq \overline{W(T_\psi)}$, this will imply that T_ψ is invertible.

Remark 3.2. Sarason [13] showed that if the Toeplitz operator defined on H^2 is bounded below then ϕ^{-1} exist in $L^\infty(\partial D)$ moreover, and in this case, $T_{|\phi|}$ is invertible, and this is due to the fact that $\sigma(T_{|\phi|}) = [\operatorname{ess\,inf}|\phi|, \operatorname{ess\,sup}|\phi|]$ and $|\phi| \geqslant \varepsilon > 0$. However, by using different techniques it can be shown from Luecking paper **[10]**$_1$ that if $\phi \in L^\infty(D)$ and the Toeplitz operator T_ϕ defined on $A^2(D)$ is bounded below, then $T_{|\phi|}$ is invertible. Thus, from this we conclude that if T_ϕ is bounded below then either ϕ^{-1} exist in $L^\infty(D)$ or $\operatorname{ess\,inf}|\phi| \notin \sigma(T_{|\phi|})$.

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Abstract

In this paper some algebraic and spectral properties of Toeplitz operators on the Bergman space $A^2(D)$ are studied. Also, a class of Hilber-Schmidt Toeplitz operators which is not of trace class is obtained.

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