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Prime ideals of skew polynomial rings (**)

1 - Preliminaries

All the rings considered in this paper are with identities.

Let R be a ring and let f be an endomorphism of R, then we recall that a map $d: R \to R$, such that $d(a+b) = d(a) + d(b)$ and $d(ab) = ad(b) + d(a)f(b)$ for all a, b in R is called a *f-derivation* of R; when f is the identity then d is a derivation of R.

Now let $H = \{f_1, \ldots, f_n\}$ be a finite set of automorphisms of R, then an ideal I of R is called a *H-ideal* if $f_i(I) = I$, for all f_i in H. Notice that when R is right Noetherian and $f_i(I) \subseteq I$ for some f_i in H, we have the ascending chain of ideals $I \subseteq f_i^{-1}(I) \subseteq f_i^{-2}(I) \subseteq \ldots$, which becomes stable after a finite number of steps, say n. Then $f_i^{-n}(I) = f_i^{-n-1}(I)$ and therefore $f_i(I) = I$.

Next let $D = \{d_1, \ldots, d_n\}$ be a finite set of mappings from R to R, such that d_i is a f_i-derivation of R, for all $i = 1, \ldots, n$. Then an ideal I of R is called a *D-ideal* if $d_i(I) \subseteq I$ for all d_i in D.

An ideal I of R which is both a H-ideal and a D-ideal is called for brevity in this paper a *(H, D)-ideal* of R. In the special case where $H = \{f\}$ and $D = \{d\}$ I is called a *(f, d)-ideal* of R.

Furthermore a (H, D)-ideal I of R is called a *(H, D)-prime ideal* if, given any two (H, D)-ideals A and B of R such that $AB \subseteq I$, is either $A \subseteq I$, or $B \subseteq I$ and R is called a *(H, D)-prime ring* if (0) is a (H, D)-prime ideal of R. The notions of a H-prime and of a D-prime ideal of R can be also defined in the obvious way.

Assume next that $d_i \circ d_j - d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $d_i \circ f_j = f_j \circ d_i$ for all i, $j = 1$, $\ldots$, n and consider the set S_n of all polynomials in n variables, say $x_1, \ldots, x_n$ over

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R. Define in S_n addition in the usual way and multiplication by the relations: $x_i r = f_i(r) x_i + d_i(r)$ and $x_i x_j = x_j x_i$, for all r in R and all $i, j = 1, \ldots, n$. Then S_i is an Ore extension over S_{i-1} (cfr. [6]) for all $i = 1, \ldots, n$, where $S_0 = R$ (cfr. Theorem 2.4 of $[8]_2$). We call the *ring* S_n a skew polynomial ring in n-variables over R and we denote it by $S_n = [x_1, f_1, d_1] \ldots [x_n, f_n, d_n]$.

Notice that under these conditions one can extend f_i to an automorphism and d_i to a f_i-derivation of S_n, by putting $f_i(x_j) = x_j$ and $d_i(x_j) = 0$, for all $i, j = 1, \ldots, n$ (cf. Theorems 2.2 and 2.3 of $[8]_2$).

For reasons of brevity we write $x^{(a)}$ instead of $x_1^{a_1} \ldots x_n^{a_n}$ for any non negative integers $a_1, \ldots, a_n$, therefore the typical element of S_n is a finite sum of the form $\sum\limits_{(a)} r_{(a)} x^{(a)}$, with $r_{(a)}$ in R, for each $(a) = (a_1, \ldots, a_n)$.

In the special case where f_i is the identity for all f_i in H we get the skew polynomial ring $S_n^* = R[x_1, d_1] \ldots [x_n, d_n]$, while if $d_i = 0$ for all α_i in D we get the skew polynomial ring $S'_n = R[x_1, f_1] \ldots [x_n, f_n]$.

When R is right Noetherian the usual proof of the Hilbert's Basis Theorem adapts easily to show (together with induction on n) that S_n is a right Noetherian ring too (this is not true if we take H to be any set of monomorphisms of R, cf. [5]).

2 - Relations among the prime ideals of S_n and those of R

In the next of this paper we deal with the skew polynomial rings S_n, S_n^* and S'_n defined in **1**. We need first the following

Lemma 2.1. (i) *If I is a H-ideal of S_n, then $I \cap R$ is a (H, D)-ideal of R.* (ii) *If A is a (H, D)-ideal of R, then AS_n is a (H, D)-ideal of S_n.*

Proof. (i) For all r in $I \cap R$ $d_i(r) = x_i r - f_i(r) x_i$ is in $I \cap R$, for each $i = 1, \ldots, n$. (ii) $x_i A \subseteq f_i(A) x_i + d_i(A) \subseteq A_{x_i} + A \subseteq AS_n$, therefore AS_n is an ideal of S_n. The rest of the proof is obvious.

We now prove the following

Theorem 2.2. *Let P be a H-prime ideal of S_n, then $P \cap R$ is a (H, D)-prime ideal of R.*

Proof. By the previous lemma $P \cap R$ is a (H, D)-ideal of R. Let A and B be any (H, D)-ideals of R such that $AB \subseteq R \cap P$. Then AS_n and BS_n are H-ideals of S_n and $(AS_n)(BS_n) = A(S_n BS_n) \subseteq ABS_n \subseteq (P \cap R) S_n \subseteq P$, therefore $AS_n \subseteq P$, or $BS_n \subseteq P$. Hence $A \subseteq AS_n \cap R \subseteq P \cap R$, or $B \subseteq P \cap R$.

The theorem above has the following two corollaries.

Corollary 2.3. *Let P be a prime ideal of S_n^*, then $P \cap R$ is a D-prime ideal of R.*

The proof is obvious.

Corollary 2.4. (i) *If P is a H-prime ideal of S_n' then $P \cap R$ is a H-prime ideal of R.* (ii) *If P is a prime ideal of S_n', such that x_i is not in P for each $i = 1, \dots, n$, then $P \cap R$ is a H-prime ideal of R.*

Proof. (i) Obvious. (ii) It suffices to show that P is a H-ideal of S_n'. For this, given g in P, $f_i(g)\,x_i = x_i g$ is also in P, therefore $f_i(g)\,S_n' x_i = f_i(g)\,x_i S_n' \subseteq P$ and so $f_i(g)$ is in P.

Conversely, if $f_i(g)$ is in P, then $x_i g$ is also in P, therefore $x_i S_n' g = S_n' x_i g \subseteq P$ and so g is in P.

Next we need the following

Lemma 2.5. *Let A be an H-ideal of S_n and let $T(A)$ be the set of all the leading coefficients of the elements of A, written as polynomials in x_n with coefficients in S_{n-1}. Put $T_i(A) = T(T_{i+1}(A))$ in S_i, for each $i = 0, 1, \dots, n-1$, where $S_0 = R$ and $T_n(A) = A$. Then:* (i) *$T_0(A)$ is a (H, D)-ideal of R.* (ii) *If B is another H-ideal of S_n, then $T_0(A)\,T_0(B) \subseteq T_0(AB)$.*

Proof. (i) Let a and b be any elements of $T(A)$, then there exist elements g an h of A of degrees k and m with respect to x_n and leading coefficients a and b respectively.

Without the loss of generality we may assume that $k \geqslant m$. Then , for all s in S_{n-1}, $a \pm b$, sa and as are all in $T(A)$, being either zero or the leading coefficients of $g \pm hx_n^{k-m}$, sg and $gf_n^{-m}(s)$ respectively. Thus $T(A)$ is an ideal of S_{n-1}.

Similarly $T_{n-2}(A)$ is an ideal of S_{n-2} and so on, so that $T_0(A)$ is an ideal of R. Furthermore $T(A)$ is a (f_n, d_n)-ideal of S_{n-1}, because $f_n(a)$, $f_n^{-1}(a)$ and $d_n(a)$ are the leading coefficients of $f_n(g)$, $f_n^{-1}(g)$ and $x_n g - f_n(g)\,x_n$ respectively.

In the same way $T_{n-2}(A)$ is a (f_{n-1}, d_{n-1})-ideal of S_{n-2}. Now let $r=r(x_1, \dots, x_{n-2})$ be an element of $T_{n-2}(A)$, then there exists s in $T(A)$ with leading coefficient r with respect to x_{n-1}. Hence $f_n(s)$, $f_n^{-1}(s)$ and $d_n(s)$ are all in $T(A)$, while $f_n(x_{n-1})=x_{n-1}$ and $d_n(x_{n-1})=0$, therefore $f_n(r)$, $f_n^{-1}(r)$ and $d_n(r)$ are all in $T_{n-2}(A)$. Thus $T_{n-2}(A)$ is also a (f_n, d_n)-ideal of S_{n-2}.

We keep going in the same way, until we finally find that $T_0(A)$ is a (H, D)-ideal of R_1 as we wish.

(ii) Given g in A and h in B with leading coefficients a and b and degrees m and m' with respect to x_n respectively, it is easy to check that ab is either zero or the leading coefficients of $gf_n^{-m}(h)$. Thus $T(A)\,T(B) \subseteq T(AB)$.

Next, applying induction on k, assume that $T_{n-k}(A)\,T_{n-k}(B) \subseteq T_{n-k}(AB)$. Then $T_{n-k-1}(A)\,T_{n-k-1}(B) = T(T_{n-k}(A))\,T(T_{n-k}(B)) \subseteq T(T_{n-k}(A)\,T_{n-k}(B)) \subseteq T(T_{n-k}(AB)) = T_{n-k-1}(AB)$ and we are through.

We now prove the following

Theorem 2.6. *If I is a (H, D)-prime ideal of R, then IS_n is a H-prime ideal of S_n. Therefore S_n is a H-prime ring if, and only if, R is a (H, D)-prime ring.*

Proof. Let A and B be any H-ideals of S_n such that $AB \subseteq IS_n$. Without loss of the generality we may assume that $A \supseteq IS_n$ and $B \supseteq IS_n$, otherwise we work with $A+IS_n$ and $B+IS_n$ respectively.

Then, by Lemma 2.5, $T_0(A)$ and $T_0(B)$ are (H, D)-ideals of R and $T_0(A)\,T_0(B) \subseteq T_0(AB) \subseteq T_0(IS_n)=I$, therefore $T_0(A) \subseteq I$ or $T_0(B) \subseteq I$.

Assume that $T_0(A) \subseteq I$ and let $g=\sum_{i=0}^{m_1} a_i x_n^i$ be a polynomial in A with coefficients in S_{n-1}; then a_{m_1} is in $T_{n-1}(A)$.

We write $a_{m_1}=\sum_{i=0}^{m_2} b_i^i x_{n-1}^i$; then b_{m_2} is in $T_{n-2}(A)$ and $a_{m_1}x_n^{m_1}=b_{m_2}x_n^{m_1}x_{n-1}^{m_2} + \sum_{i=0}^{m_2-1} b_i x_{n-1}^i x_n^{m_1}$.

We keep going in the same way, until we find some r in $T_0(A)$, such that $h=rx_n^{m_1}x_{n-1}^{m_2} \dots x_1^{m_n}$ is a term of $a_{m_1}x_n^{m_1}$. Then r is in I, therefore h is in $IS_n \subseteq A$. Thus $g-h$ is in A.

Repeating the same process for $g-h$ and keep going in the same way we eventually find that $a_{m_1}x_n^{m_1}$ is in $IS_n \subseteq A$. Thus $\tilde{g}=g-a_{m_1}x_n^{m_1}=\sum_{i=0}^{m_1-1} a_i x_n^i$ is in A.

Applying the same argument for $\bar{g}$ and keep going in the same way we finally find that g is in IS_n, as we wish.

The last part of the theorem is a straightforward consequence of the first part and of the Theorem 2.2.

The preceding theorem has the following two straightforward corollaries.

Corollary 2.7. *If I is a D-prime ideal of R, then IS_n^* is a prime ideal of S_n^*, therefore S_n^* is a prime ring if, and only if, R is a D-prime ring.*

Corollary 2.8. *If I is a H-prime ideal of R, then IS_n' is a H-prime ideal of S_n', therefore S_n' is a H-prime ring if, and only if, R is so.*

Next, with the additional assumption that R is right Noetherian, we prove the following result, stronger than Corollary 2.8.

Theorem 2.9. *Let R be a right Noetherian ring. Then, if I is a H-prime ideal of R, IS_n' is a prime ideal of S_n'; therefore S_n' is a prime ring if, and only if, R is a H-prime ring.*

Proof. Since I is a H-ideal of R, for all f_i in H, f_j induces an automorphism $\bar{f}_i$ of $R/I=\bar{R}$, by $\bar{f}_i(r+I)=f_i(r)+I$, for all r in R.

Then it is easy to check that the map t from $\bar{R}[x_1, \bar{f}_1] \dots [x_n, \bar{f}_n]$ to S_n'/IS_n', defined by $t(\sum_{(a)} \bar{r}_{(a)} x^{(a)}) = \sum_{(a)} r_{(a)} x^{(a)} + IS_n'$ is a ring isomorphism, therefore it suffices to show only the last part of the theorem.

For this, let A and B be any ideals of S_n' such that $AB=0$. Then, if B_1 is the right annihilator of A and A_1 is the best annihilator of B_1 it is clear that $A \subseteq A_1$, $B \subseteq B_1$ and that $A_1 B_1 = 0$.

From the other hand since $A_1 x_i \subseteq A_1$, is $A_1 x_i B_1 = A_1 f_i(B_1) x_i = 0$. But x_i is regular in S_n', therefore $A_1 f_i(B_1)=0$ and $f_i^{-1}(A_1) B_1 = 0$. Thus $f_i(B_1) \subseteq B_1$ and $f_i^{-1}(A_1) \subseteq A_1$, for each $i=1, \dots, n$.

Hence, since S_n' is right Noetherian, A_1 and B_1 are H-ideals of S_n'. Then, by Lemma 2.5, $T_0(A_1)$ and $T_0(B_1)$ are H-ideals of R and $T_0(A_1)\,T_0(B_1) \subseteq T_0(A_1 B_1) = 0$, therefore either $T_0(A_1)=0$ or $T_0(B_1)=0$, fact which shows that either $A=0$ or $B=0$ and this completes the proof.

3 - Remarks - Examples

(1) The statement of Theorem 2.2 remains true even if we take P to be a $(H\ D)$-prime ideal of S_n, because, by Lemma 2.1, AS_n and BS_n are in fact $(H,\ D)$-ideals of S_n (the rest of the proof remains unchanged). Therefore, an analogue to Corollary 2.3 gives that, if P is a D-prime ideal of S_n^*, then $P \cap R$ is a D-prime ideal of R.

(2) A $(H,\ D)$-ideal I of R is said to be a $(H,\ D)$-*semiprime ideal*, if, for all $(H,\ D)$-ideals A of R, such that $A^k \subseteq I$ for some non negative integer k, we have that $A \subseteq I$ and R is said to be a $(H,\ D)$-*semiprime ring*, if (0) is a $(H,\ D)$-semiprime ideal of R.

The statements of Theorems 2.2, 2.6, 2.9 and of their corollaries remain true if we replace the word «prime», whenever it appears, with the word «semiprime» (see also Remark 3). In fact the only modification, which is needed in the proofs, is to put $A^{k-1} = B$.

(3) Especially the statement of Corollary 2.4(ii) must be restated as follows: «*If R is right Noetherian and P is a semiprime ideal of S_n', none of whose minimal primes contains x_i for each $i = 1, \dots, n$, then $P \cap R$ is a H-semiprime ideal of R*».

For this notice that, since S_n' is right Noetherian, there exist finitely many prime ideals of S_n', say $P_1, \dots, P_k$, such that $P_1, \dots, P_k \subseteq P$ and $P_1, \dots, P_k \supseteq P$. Then $(P_1 \cap \dots \cap P_k)^k \subseteq P$ and, since P is a semiprime ideal, we get that $P = P_1 \cap \dots \cap P_k$. Therefore, in the proof of Corollary 2.4(ii), P can be replaced with one of the P_i's, while the rest of the proof remains unchanged.

(4) For $n = 1$, Corollaries 2.4 and 2.8 are due to Goldie and Michler [**1**], while Theorem 2.9 is essentially due to Jategaonkar [3], but in its final form can be also found in [**1**].

Also Corollaries 2.3 and 2.7, for $n = 1$, are due to Jordan $[4]_1$.

(5) The hypothesis that x_i is not in P for each i, appearing in the statement of Corollary 2.4(ii) is not superflous (cf. [**1**], Example 3, p. 338). The same counter example can be also used to show that, if P is a H-prime ideal of S_n', then $P \cap R$ need not be a prime ideal or R.

Also the assumption that R is right Noetherian, appearing in the statement of Theorem 2.9, is necessary (cf. $[4]_2$, Example 3.1.14, p. 71).

(6) The following example illustrates Theorems 2.6 and 2.9.

Let K be a field and let $R = K[y_1, y_2]$ be a polynomial ring over K. Define a K-automorphism of R by $f(y_1) = y_1$, $f(y_2) = y_2 + 1$ and let d be the f-derivation of R defined by $d(K) = 0$, $d(y_1) = 0$ and $d(y_2) = 1$.

Then it is easy to check that $f(d(y_1^n y_2^m)) = d(f(y_1^n y_2^m))$, for any non negative integers n and m, therefore $f \circ d = d \circ f$. Moreover it is clear that $y_1 R$ is a (f, d)-prime ideal of R, therefore, by Theorem 2.6, $y_1 S$ is a f-prime ideal of $S = R[x, f, d]$.

Also, since R is a Noetherian ring, Theorem 2.9 shows that $y_1 S'$ is a prime ideal of $S' = R[x, f]$.

(7) We denote by $J(R)$ the Jacobson radical of R. Assume that $J(R) \neq R$ and that R is a D-simple ring (i.e. it has not non zero, proper D-ideals). Then S_n^* has no non zero nil ideals.

For this, let A be a nil ideal of S_n^*, then, by Lemma 2.5, $T_0(A)$ is a D-ideal of R, therefore $T_0(A) = 0$ or $T_0(A) = R$. But it is easy to check that $T_0(A)$ is a nil ideal of R, therefore $T_0(A) \subseteq J(R)$, so $T_0(A) = 0$.

Notice that, when R is a D-simple, then, under some additional assumptions, S_n^* is a simple ring (cfr. Theorems 3.4 and 3.5 of $[8]_1$).

(8) It is well known (cfr. Corollary 2.5 of $[4]_1$) that, if R is right Noetherian and d-prime, then S_1^* is semiprimitive (i.e. $J(S_1) = 0$).

An analogue of this for S_n^* is as follows: *If R is right Noetherian and D-prime, then S_n^* is semiprimitive.* For this, by Corollary 2.7, S_n^* is a prime ring. Write $S_n^* = S_{n-1}^*[x_n, d_n]$; then, by Corollary 2.7 again, S_{n-1}^* is a d_n-prime ring and the result follows.

(9) Assume that, for all f_i in H, there exists a non negative integer m_i and a regular element t_i of R, such that $f_j(t_i) = t_i$ and $t_i r = f_i^{m_i}(r)\, t_i$, for all r in R and each $j = 1, \ldots, n$. Then, if R has non zero nil ideals, S_n' is semiprimitive.

The result above for $n = 1$, is essentially due to C. R. Jordan, but in its final form can be found in $[4]_2$ (Theorem 3.1.11, p. 69).

Next, applying induction on n, assume that S_{n-1}' is semiprimitive, then S_{n-1}' has no nonzero nil ideals. Write $S_n' = S_{n-1}'[x_n, f_n]$, then it is easy to check that $t_n g = f_n^{m_n}(g)\, t_n$, for all g in S_{n-1}' and the result follows.

Notice that the hypothesis $f_1(t_1) = t_1$ is not needed to show the result for $n = 1$.

(10) Given an ideal I of R we denote by $J(I)$ the ideal of R such that $J(I)/I$ is the Jacobson radical of R/I.

We recall that R is said to be a *Jacobson ring* if $J(P) = P$, for all prime ideals P of R.

It is well known that, if R is right Noetherian and Jacobson ring, then the rings S_1^*, S_1' and the skew Laurent polynomial ring T_1, obtained from S_1' by localizing at the powers of x_1 are Jacobson rings (see [4]$_1$ and [1]). Therefore, a straightforward induction shows, that the skew polynomial rings S_n^*, S_n' and the quotient ring T_n of S_n' with respect to the set of all powers $x^{(a)}$ where (a) are n-tuples of non negative integers (see [8]$_3$) are Jacobson rings.

The assumption that R is right Noetherian is not superflous, as an example of Pearson and Stephenson [7] shows.

Also notice that, since $S_1'/x_1S_1' \cong R$ and every homomorphic image of a Jacobson ring is a Jacobson ring, the converse is also true for S_n', but it is not true for S_n^* (see section 4 of [4]$_1$, where $n = 1$).

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Abstract

Results on the prime ideal structure of a skew polynomial ring $R[x, f, d]$ *over a ring* R *has been obtained only if additional assumptions are made; Goldie and Michler assume that* R *is right Noetherian and* $d = 0$, *Jordan assumes that* R *is right Noetherian and* f *is the identity map of* R, *while Irving assumes that* R *is a commutative ring.*

In the present paper we study what happens when f *is a non trivial automorphism of* R *and* d *is a non zero* f*-derivation of* R *and we give analogous results for skew polynomial rings in finitely many variables over* R.

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