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M. SQUILLANTE, A. S. G. VENTRE and S. WEBER (*)

Closure and convergence properties for classes of decomposable measures (**)

1 - Introduction

The concept of t-conorm was introduced by Schweizer and Sklar [8] on the basis of the work of Menger [7] who defined generalized triangle inequalities by means of triangular norms (t-norms). Schweizer and Sklar used the concepts of t-norm and t-conorm in the theory of probabilistic metric spaces. Weber $[11]_1$ defined a special class of set functions by means of a t-conorm operator $\perp$, in order to state a general theory of non additive measures (called $\perp$-*decomposable measures*) and consequent integration, that reduces to the Lebesgue theory in the additive case. Comparisons with other non-additive theories (e.g. Choquet's and Sugeno's integrals) were illustrated in $[11]_1$.

A basic condition for many interesting developments is to consider Archimedean t-conorms.

In this framework it is worth mentioning some applications to measure of information and probability theory (see, e.g., [1], [10] and $[11]_2$), and mathematical economics (see, e.g., [4]).

Let us observe that the structure ([0, 1], $\perp$) is a semigroup. We just recall that semigroup valued measures are extensively treated in Sion's book [9].

Our present aim is to analyze the closure of some families of $\perp$-decomposable measures with respect to the operators «lim», and «t-conorm». Furthermore

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(*) Indirizzi degli AA.: M. SQUILLANTE e A. S. G. VENTRE, Istituto di Matematica, Facoltà di Architettura, Via Monteoliveto 3, I-80134 Napoli; S. WEBER, Johannes Gutenberg Universität, Fachbereich Mathematik, Mainz. Fed. Rep. of Germany.

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some convergence theorems for sequences of decomposable measures are proved (e.g. Nikodým and Phillips like theorems). We shall adopt [2] and [5] as reference books.

2 - Some basic results and definitions

Recall that a t-conorm $\perp$ is a binary operation on the interval $J=[0,\ 1]$ which is non decreasing in each argument, commutative, associative and has 0 as unit.

An *Archimedean t-conorm* $\perp$ is, by definition, continuous and such that $\perp(x,\ x)>x$, for every $x\in(0,\ 1)$. The following representation theorem, due to Ling [6], holds.

Theorem 1. A binary operation $\perp$ on J is an Archimedean t-conorm iff there exists an increasing and continuous function $g:J\to[0,\ \infty]$ with $g(0)=0$, such that

$$\perp(x,\ y)=g^{(-1)}(g(x)+g(y))$$

where $g^{(-1)}$ is the pseudo-inverse of g defined by

$$g^{(-1)}(x)=g^{-1}(\min(x,\ g(1)).$$

Moreover $\perp$ is strictly increasing iff $g(1)=\infty$.

Let $(X,\ A)$ be a measurable space. A function $\mu:A\to J$, with $\mu(\emptyset)=0$ and $\mu(X)=1$, is called a *$\perp$-decomposable measure* $[11]_1$ (or measure decomposable with respect to a t-conorm $\perp$) if $\mu(A\,\dot\cup\,B)=\mu(A)\perp\mu(B)$; *$\sigma$-$\perp$-decomposable measure* if $\mu(\dot{\bigcup}_{n=1}^{\infty}A_n)=\perp_{n=1}^{\infty}\mu(A_n)$; if condition $\mu(X)=1$ is replaced by $\mu(X)\leq 1$, then the set function μ is called *subnormed $\perp$-decomposable measure*; *σ-$\perp$-decomposable measure* if $\mu(\dot{\bigcup}_{n=1}^{\infty}A_n)=\perp_{n=1}^{\infty}\mu(A_n)$; *continuous* from below or above, resp., if $\lim_k\mu(A_k)=\mu(A)$ for $A_k\uparrow A$ or $A_k\downarrow A$, resp.

The following propositions are valid $[11]_1$.

(a) If μ is $\perp$-decomposable, then μ is monotone.
(b) μ is $\perp$-decomposable if and only if

$$\mu(A\cup B)\perp\mu(A\cap B)=\mu(A)\perp\mu(B).$$

(c) μ is σ-$\perp$-decomposable if and only if μ is $\perp$-decomposable and continuous from below.

An example of decomposable measure with respect to any Archimedean conorm $\perp$ is the Dirac measure, namely the measure concentrated in a point. For non trivial examples see, e.g., [11]$_1$.

Let us recall that given an Archimedean r-conorm $\perp$ with an additive generator g, the following classification for $\perp$-decomposable measures μ holds [11]$_1$:

(S): $\perp$ strict. Then $g \circ \mu : A \to [0, \infty]$ is an infinite (σ-)additive measure, whenever μ is a (σ-)$\perp$-decomposable one.

(NSA): $\perp$ non-strict Archimedean and $g \circ \mu : A \to [0, g(1)]$ a finite (σ-)additive measure with $(g \circ \mu)(X) = g(1)$.

(NSP): $\perp$ non-strict Archimedean and $g \circ \mu$ a finite measure with $(g \circ \mu)(X) = g(1)$, which is only pseudo (σ-)additive, i.e., it is possible that

$$(g \circ \mu)(\dot{\cup}_k A_k) = g(1) < \Sigma_k (g \circ \mu) A_k .$$

Furthermore, let μ be a $\perp$-decomposable measure, then the following propositions hold:

(i) If μ is continuous from below, then μ is continuous from above for all decreasing sequences $\{A_k\}$ in case (NSA), for all $\{A_k\}$, with $\mu(A_1) < 1$, in the other cases.

(ii) If μ is continuous from above for all $\{B_k\} \downarrow \emptyset$, then μ is continuous from below.

The following properties for decomposable measures can be easily checked.

2.2 - Let μ be a $\perp$-decomposable measure on (X, A). Then:

(a) If $\{A_i\}$ is a finite family of measurable subsets of X, then

$$\mathop{\perp}_{i=1}^{n} \mu(A_i) = \mu\Big(\bigcup_{i=1}^{n} A_i\Big) \perp \mu\Big(\bigcup_{i=1}^{n} \bigcup_{\substack{j=1 \\ i<j}}^{n} (A_i \cap A_j)\Big)$$

$$\perp \mu\Big(\bigcup_{i=1}^{n} \bigcup_{\substack{j=1 \\ i<j<k}}^{n} \bigcup_{k=1}^{n} (A_i \cap A_j \cap A_k)\Big) \perp \ldots \mu\Big(\bigcap_{i=1}^{n} A_i\Big).$$

(b) For every A, $B \in \mathbf{A}$, it is $\mu(A \cup B) \leq \mu(A) \perp \mu(B)$.

If $\{A_n\}$ is a sequence of measurable subsets of X, with $A_i \cap A_j = \emptyset$, $(i \neq j)$ for every $A \in \mathbf{A}$ such that $A \supseteq \bigcup_{n=1}^{\infty} A_n$, it is

$$\mathop{\perp}_{n=1}^{\infty} \mu(A_n) \leq \mu(A).$$

In particular

$$\mathop{\perp}_{n=1}^{\infty} \mu(A_n) \leq \mu(\bigcup_{n=1}^{\infty} A_n).$$

(c) If μ is σ-$\perp$-decomposable and $\{A_n\}$ is an arbitrary sequence of measurable subsets of $\mathbf{A}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \mathop{\perp}_{n=1}^{\infty} \mu(A_n).$$

3 - Closure properties

Let $M_\perp(M_{\sigma\text{-}\perp})$ denote the set of $\perp$-(σ-$\perp$)-decomposable measures on the measurable space $(X, \mathbf{A})$. The following Nikodým property, whose proof is omitted, holds.

3.1 - Let $\{\mu_n\}$ be a sequence in $M_\perp$, with $\perp$ continuous in $J \times J$, such that $\{\mu_n(A)\}$ converges for every $A \in \mathbf{A}$. Then the set function

$$\mu : A \in \mathbf{A} \to \lim_n \mu_n(A)$$

is a $\perp$-decomposable measure on $(X, \mathbf{A})$. Furthermore, if $\{\mu_n\}$ is in $M_{\sigma\text{-}\perp}$ and converges uniformly with respect to $A \in \mathbf{A}$, then μ is in $M_{\sigma\text{-}\perp}$.

3.2 - In **3.1** monotonicity can take the place of uniform convergence; indeed we can state

Theorem 2. *Let $\{\mu_n\}$ be an increasing sequence in $M_{\sigma\text{-}\perp}$, with $\perp$ continuous. The set function $\mu = \lim \mu_n$ is in $M_{\sigma\text{-}\perp}$.*

Proof. By **3.1** μ is $\perp$-decomposable. Let $\{A_k\}$ be a sequence of disjoint subsets in $\mathbf{A}$, $B_m := \dot{\bigcup}_{k=1}^{m} A_k$ and $a_{nm} := \mu_n(B_m)$. By monotonicity of μ_n the double sequence $\{a_{nm}\}$ converges. The partial sequences have the limits $\mu(B_m)$, for all m, and $\mu_n(\dot{\bigcup}_{k=1}^{\infty} A_k)$, for all n, respectively. Therefore, interchanging the limits on m and n, we obtain

$$\lim_{n,m\to\infty} a_{nm} = \mathop{\perp}_{k=1}^{\infty} \mu(A_k) = \mu(\bigcup_{k\in N} A_k) .$$

In virtue of Theorem 2, the behaviour of sequences in $M_\perp$ ($M_{\sigma\text{-}\perp}$) can be illustrated as follows.

3.3 - Let $\{\mu_n\}$ be a sequence of measures in $M_\perp$, with $\perp$ continuous. Then the set function

$$\mu : A \in \mathbf{A} \to \mathop{\perp}_{n=1}^{\infty} \mu_n(A)$$

is in $M_\perp$; if $\{\mu_n\}$ is in $M_{\sigma\text{-}\perp}$, then μ is in $M_{\sigma\text{-}\perp}$ too.

4 - Limit properties with variable subsets

It will be useful in the sequel the following definition and propositions **[11]**$_1$.

Def. For any t-conorm $\perp$ the operation $\dot{-}$ is defined as

$$b \dot{-} a = \inf\{y / a \perp y \geq b\} .$$

4.1 - For every strict Archimedean t-conorm with additive generator g,

$$b \dot{-} a = \begin{cases} g^{-1}(g(b) - g(a)) & \text{if } a \leq b,\ a < 1 \\ 0 & \text{otherwise.} \end{cases}$$

For every non-strict Archimedean t-conorm, and $a \leq b$

$$b \dot{-} a = g^{-1}(g(b) - g(a)) .$$

4.2 - Let $\perp$ be an Archimedean t-conorm and μ a $\perp$-decomposable measure. If $A \subseteq B$ and under the additional conditions for (S) with $\mu(A) < 1$ or (NSP) with $\mu(B) < 1$, resp., then $\mu(B \setminus A) = \mu(B) \dot{-} \mu(A)$.

The following theorems hold.

Theorem 4.3. *Let $\perp$ be an Archimedean t-conorm with additive generator g, μ a $\perp$-decomposable measure on $(X, \boldsymbol{A})$.*

For every sequence $\{A_i\} \subseteq \boldsymbol{A}$ of pairwise disjoint subsets, it is

$$\lim_i \mu(A_i) = 0 \tag{1}$$

in case (NSA). *Eq.* (1) *holds in case* (NSP) *under the additional condition* $\mu(\dot{\bigcup}_{i=1}^{k} A_i) < 1$, *for every k, and in the case* (S) *under the additional condition* $\mu(\dot{\bigcup}_{i=1}^{\infty} A_i) < 1$.

Proof. It is

$$\lim_i \mu(A_i) = \lim_i \mu(\dot{\bigcup}_{h=1}^{i} A_h) \dot{-} \lim_i \mu(\dot{\bigcup}_{h=1}^{i-1} A_h) = \mathop{\perp}_{h=1}^{\infty} \mu(A_h) \dot{-} \mathop{\perp}_{h=1}^{\infty} \mu(A_h) = 0\,.$$

This follows by **4.2** and the continuity of the operator $\dot{-}$, which is true only in $[0, 1) \times [0, 1)$ for the case (S) and therefore is required the additional condition.

Let us state a tie between monotone and uniform convergence for a sequence of σ-$\perp$-decomposable measures.

Let $\boldsymbol{B}$ denote a subfamily in the σ-algebra $\boldsymbol{A}$ of the measurable space (X, $\boldsymbol{A}$). Consider the property

(C) Every sequence of elements in $\boldsymbol{B}$ contains a convergent subsequence.

Theorem 4.4. *Let $(X, \boldsymbol{A})$ be a measurable space and $\{\mu_n\}$ an increasing sequence of elements of $M_{\sigma\text{-}\perp}$ with $\perp$ archimedean and every μ_n of type* (NSA). *If $\boldsymbol{B}$ is a subfamily of $\boldsymbol{A}$ enjoing property* (C), *then the equality*

$$\lim_n \mu_n(A) = \mu(A)$$

is valid uniformly with respect to $A \in \boldsymbol{B}$.

Proof. Let us prove the assumption by contradiction, and then assume that the convergence of $\{\mu_n\}$ is not uniform on $\boldsymbol{B}$. Precisely if $\lim \mu_n(A) = \mu(A)$, for every $A \in \boldsymbol{A}$, suppose that there exists $\varepsilon > 0$ such that there is a sequence $\{n_k\}$ of indices and a sequence $\{C_k\}$ of subsets of $\boldsymbol{B}$, for which

$$\mu_{n_k}(C_k) \leqslant \mu(C_k) - \varepsilon . \tag{2}$$

By property (C) there is a subsequence $\{C_{k_j}\}$ in $\{C_k\}$ that converges to a measurable subset C

$$\lim_j C_{k_j} = C .$$

Let $\bar{k}$ be a given positive integer; for $n_{k_j} \geqslant \bar{k}$ we get $\mu_{\bar{k}}(C_{k_j}) \leqslant \mu_{n_{k_j}}(C_{k_j})$ hence, by (2),

$$\mu_{\bar{k}}(C_{k_j}) \leqslant \mu_{n_{k_j}}(C_{k_j}) < \mu(C_{k_j}) - \varepsilon . \tag{3}$$

By Theorem 2, μ is in $M_{\sigma\text{-}\perp}$ and therefore $g \circ \mu$ is a finite σ-additive measure, in virtue of Nikodým Theorem. Thus condition (NSA) is satisfied for μ; therefore by **[11]**$_2$ μ is continuous. Then (3) implies

$$\mu_{\bar{k}}(C) = \lim_j \mu_{\bar{k}}(C_{k_j}) \leqslant \mu(C) - \varepsilon .$$

The inequality $\mu_{\bar{k}}(C) \leqslant \mu(C) - \varepsilon$, stated for an arbitrary $\bar{k}$, contradicts the hypothesis of monotonicity and, then, convergence of μ_n on every measurable subset. Thus the uniform convergence of the sequence $\{\mu_n\}$ on $\boldsymbol{B}$ follows.

Remarks (i) If $\{\mu_n\}$ is a sequence fulfilling the hypotheses of Theorem 4, then, in particular, $\lim_n \mu_n(B) = \mu(B)$, where B belongs to any convergent sequence of measurable subsets.

(ii) Theorem 4.4 holds, in particular, if $\{\mu_n\}$ is an increasing sequence of σ-additive measures (that are σ-$\perp$-decomposable with respect to the non strict t-conorm $a \perp b = \min(a + b,\ 1)$).

The remarks above allow to restate a classical result (see, e.g., [3], p. 275).

(iii) If $\{\mu_n\}$ is a monotone sequence of σ-additive measures and $\sup_n \mu_n$ is finite, then $\{\mu_n\}$ is uniformly convergent on every subfamily in $\boldsymbol{A}$, contained in a convergent sequence of measurable subsets.

(iv) Let (X, T) be a Hausdorff locally compact space, such that T contains all countable intersections of open subsets (e.g. T is the discrete topology). T fulfils condition (C).

Let N denote the set of positive integers and $P(N)$ the power set of N. A Phillips' Lemma analogue for $\perp$-decomposable measures holds.

Theorem 5. *Let $\{\mu_n\}$ be a sequence of subnormed $\perp$-decomposable measures on $(N, P(N))$ and $\perp$ continuous. Then the following propositions*

$$\text{(a)} \quad \lim_n \mu_n(A) = 0 \qquad \textit{for every} \qquad A \in P(N)$$

$$\text{(b)} \quad \lim_n S_n = 0 \qquad \text{(c)} \quad \lim_n \bar{S}_n = 0 \qquad \text{(d)} \quad \lim_n \mathop{\perp}_{k=1}^{\infty} \mu_n(k) = 0$$

where S_n, $\bar{S}_n$ are the suprema of the sets $\{\mu_n(A)$, for every finite $A \in N\}$, $\{\mu_n(A)$, for every $A \subseteq N\}$, respectively, are linked by the implications

$$\text{(a)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(d)}.$$

Proof. $(a) \Rightarrow (c)$. Indeed, by contradiction if (c) is false, there is $\eta > 0$, such that for every $m \in N$, there exists $p_m > m$ and $A_m \subseteq N$, such that $\mu_{p_m}(A_m) \geqslant \eta$, and setting $A = \bigcup_{m=1}^{\infty} A_m$, by monotonicity of every μ_{p_m}, $\mu_{p_m}(A) \geqslant \mu_{p_m}(A_m) \geqslant \eta$. Then the absurd.

(c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b) are evident.

Let us prove (b) implies (d). Indeed

$$0 \leqslant \lim_n{}'' \mathop{\perp}_{k=1}^{\infty} \mu_n(k) = \lim_n{}'' \lim_m \mathop{\perp}_{k=1}^{m} \mu_n(k) < \lim_n{}'' S_n = 0\,.$$

(d) $\Rightarrow$ (b). We just observe that for any finite sequence $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_{j_m}$. We have

$$0 < \lim{}'' S_n = \lim_n{}'' \operatorname*{Sup}_{j_m \in N} \{\mu_n(\{k_1, k_2 \ldots k_{j_m}\})\}$$

$$= \lim_n{}'' \operatorname*{Sup}_{j_m < N} \mathop{\perp}_{i=1}^{j_m} \mu_n(k_i) \leqslant \lim_n{}'' \operatorname*{Sup}_{j_n} \mathop{\perp}_{k=1}^{k_{j_m}} \mu_n(k) = \lim_n{}'' \mathop{\perp}_{k=1} \mu_n(k)\,.$$

As a concluding remark let us observe that if all measures μ_n are σ-$\perp$-decomposable, then

$$\lim_n \mathop{\perp}_{k=1}^{\infty} \mu_n(k) = \lim_n \mu_n\Big(\bigcup_{k=1}^{\infty} \{k\}\Big) = \lim_n \mu(N)$$

and all the propositions (a), (b), (c), (d) are equivalent between them.

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Sommario

Si dimostrano alcuni teoremi di convergenza per successioni di funzioni d'insieme decomponibili rispetto a conorme triangolari; in particolare si stabilisce una condizione sufficiente per la convergenza uniforme e l'analogo del classico teorema di Phillips.

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