Riv. Mat. Univ. Parma (4) **14*** (1988), 1-28

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Finsler metrics on almost complex manifolds

Dedicated to professor G. B. Rizza, who is the originator of the notion of Rizza manifolds.

Introduction

In the present paper, we are mainly concerned with Finsler metrics defined on almost complex manifolds.

The tangent space at any point of an almost complex manifold can be regarded as a complex vector space. On the other hand, the tangent space at any point of a Finsler manifold can be regarded as a normed linear space. Thereupon, with respect to an almost complex manifold endowed with a Finsler metric, first of all, we should consider the condition under which the complex structure and the norm are compatible as a complex Banach norm in each tangent space of the manifold.

Our main purpose is to study the manifolds satisfying the above condition, which will be called *Rizza manifolds*. Moreover we also deal with the case where the given metric is not a Finsler metric but is a generalized Finsler metric. The contents of the present paper are the following:

1 - C-Minkowski spaces **2** - Rizza manifolds **3** - The Rizza condition
4 - Kaehlerian Finsler structures **5** - The induced Moór metric
6 - $(f, \tilde{g}, N)$-structures **7** - A generalization of Yano-Westlake's theorem.

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1 - C-Minkowski spaces

Let R^{2n} be a $2n$-dimensional real vector space and $\{e_1, \ldots, e_{2n}\}$ be its normal basis where $e_a = {}^t(0, \ldots, \underset{(a)}{1}, \ldots, 0)$ (1). Now R^{2n} admits a complex structure J such that $J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$ where E_n is the unit matrix of order n. It is clear that $Je_\alpha = e_{\bar\alpha}$. So, any vector $\xi \in R^{2n}$ can be represented as $\xi = \xi^i e_i = \xi^\alpha e_\alpha + \xi^{\bar\alpha} Je_\alpha$. Hence for any $\xi \in R^{2n}$, we can define a complex vector $\tilde\xi$ as $\tilde\xi = (\xi^\alpha + i\xi^{\bar\alpha})\, e_\alpha$ and consider an n-dimensional complex vector space C^n which is the set of all the complex vectors $\tilde\xi$. Of course, the set $\{e_1, \ldots, e_n\}$ becomes a basis of C^n. The mapping $\xi \to \tilde\xi$ is bijective from R^{2n} to C^n and maps $x\xi + yJ\xi$ to $z\tilde\xi$ where $z = x + iy$.

Now we can introduce an inner product in R^{2n} such that $\langle e_a,\ e_b\rangle = \delta_{ab}$ and we can regard R^{2n} as an *Euclidean space* E^{2n}. For any $\xi = \xi^i e_i \in E^{2n}$, the norm of ξ is given by $\|\xi\|_{E^{2n}} = (\langle \xi,\ \xi\rangle)^{\frac12} = (\sum_{a=1}^{2n} (\xi^a)^2)^{\frac12}$. Corresponding to this, for any $\tilde\xi = (\xi^\alpha + i\xi^{\bar\alpha})\, e_\alpha$ and $\tilde\eta = (\eta^\alpha + i\eta^{\bar\alpha})\, e_\alpha$, which belong to C^n, the Hermitian inner product $h(\tilde\xi,\ \tilde\eta)$ is given by

$$h(\tilde\xi,\ \tilde\eta) = \sum_{\alpha=1}^{n} (\xi^\alpha + i\xi^{\bar\alpha})\, \overline{(\eta^\alpha + i\eta^{\bar\alpha})} = \sum_{\alpha=1}^{n} \{(\xi^\alpha \eta^\alpha + \xi^{\bar\alpha} \eta^{\bar\alpha}) + i(\xi^{\bar\alpha} \eta^\alpha - \xi^\alpha \eta^{\bar\alpha})\}.$$

Thus C^n can be regarded as a *unitary space*. In this unitary space, the norm of a vector $\tilde\xi = (\xi^\alpha + i\xi^{\bar\alpha})\, e_\alpha$ is given by $\|\tilde\xi\|_{C^n} = (h(\tilde\xi,\ \tilde\xi))^{\frac12}$. It is easy to see that $\|\tilde\xi\|^2_{C^n} = h(\tilde\xi,\ \tilde\xi) = \|\xi\|^2_{E^{2n}}$. Moreover it follows that

$$(1.1) \qquad (h(z\tilde\xi,\ z\tilde\xi))^{\frac12} = |z|\, \|\xi\|_{E^{2n}}.$$

Next, we introduce the concept of Minkowski norm. A vector space admitting the Minkowski norm is said to be a *Minkowski space* and is denoted by V hereafter. The Minkowski norm is the following:

(1) for any $\xi \in V$, the norm function $\|\xi\| = N(\xi^1, \ldots, \xi^{2n}) = N(\xi)$ is 3-times

(1) In this paper, the latin indices $a, b, c, \ldots, i, j, k, \ldots$ run over the range $1, \ldots, 2n$; and the Greek indices $\alpha, \beta, \ldots, \lambda, \mu, \ldots$ run over the range $1, \ldots, n$; and the indices $\bar\alpha, \bar\beta, \bar\gamma, \ldots$ stand for $\alpha + n,\ \beta + n,\ \gamma + n,\ \ldots$ respectively.

continuously differentiable at $\xi \neq 0$,

(2) $\|\xi\| \geqslant 0$, (3) $\|\xi\| = 0$ if and only if $\xi = 0$,

(4) $\|k\xi\| = k\|\xi\|$ for any $k > 0$, (5) $\dfrac{\partial^2 N^2(\xi)}{\partial \xi^i \partial \xi^j}\eta^i \eta^j$ is positive definite.

Remark. According to circumstances, the condition (5) can be replaced by the weaker condition $\|\xi_1 + \xi_2\| \leqslant \|\xi_1\| + \|\xi_2\|$ (Rund [23]).

Now, let V^{2n} be a Minkowski space. In the complex vector space C^n corresponding to V^{2n}, we can naturally introduce a norm such that

$$\|\xi\|_{C^n} = \|\xi\|_{V^{2n}} = N(\xi)\,. \tag{1.2}$$

Then we have

Theorem 1.1. *Let us consider a 2n-dimensional Minkowski space V^{2n}, identify the space V^{2n} with an n-dimensional complex vector space C^n, and define a norm in C^n by* (1.2). *The space C^n is a complex Banach space if and only if the relation*

$$\|x\xi + yJ\xi\|_{V^{2n}} = (x^2 + y^2)^{\frac{1}{2}} \|\xi\|_{V^{2n}} \tag{1.3}$$

holds for any real numbers x, y and for any $\xi \in V^{2n}$.

Proof. The definition shows us that

$$\|z\xi\|_{C^n} = \|x\xi + yJ\xi\|_{V^{2n}} = N(x\xi^a + yJ^a{}_b\,\xi^b)$$

and $|z|\,\|\xi\|_{C^n} = (x^2+y^2)^{\frac{1}{2}} N(\xi)$ for any $z = x + iy$. Hence (1.3) means that $\|z\xi\|_{C^n} = |z|\,\|\xi\|_{C^n}$. The rest of the condition for C^n to be a complex Banach space is evidently satisfied from the definition of the Minkowski norm (Taylor [25]).

In what follows, a $2n$-dimensional Minkowski space satisfying the condition (1.3) is said to be a *C-Minkowski space* and the norm is said to be a C-Minkowski norm.

Now we show some examples of C-Minkowski spaces.

Example 1. In an Euclidean space E^{2n}, the norm $\|\xi\| = (\sum_{i=1}^{n} (\xi^i)^2)^{\frac{1}{2}}$ is a C-Minkowski norm.

Example 2. In R^4, $\|\xi\| = (\sum_{i=1}^{n} (\xi^i)^4)^{\frac{1}{4}}$ is not a C-Minkowski norm.

Example 3. In R^4, $\|\xi\| = (\{(\xi^1)^2 + (\xi^3)^2\}^2 + \{(\xi^2)^2 + (\xi^4)^2\}^2)^{\frac{1}{4}}$ is a C-Minkowski norm. If we put $z^\alpha = \xi^\alpha + i\xi^{\bar{\alpha}}$, then we can see that $\tilde{\xi} = z^\alpha e_\alpha$ and $\|\xi\| = \|\tilde{\xi}\|_{C^2} = (|z^1|^4 + |z^2|^4)^{\frac{1}{4}}$.

Example 4. In R^{2n}, we introduce the following

$$\|\xi\| = (\sum_{a=1}^{2n} (\xi^a)^2)^{\frac{1}{2}} + ((\xi^1)^2 + (\xi^2)^2)^{\frac{1}{2}}.$$

This is not a Minkowski norm in the strict sense, because the condition (1) of a Minkowski norm is not satisfied in some region. However, the rest of the condition for a Minkowski norm and the condition (1.3) are all fulfilled. In such a case, we call it a C-Minkowski norm *in the wide sense.*

If we put $x = r\cos\theta$, $y = r\sin\theta$, the condition (4) of the Minkowski norm tells us that the condition (1.3) is equivalent to $\|(\cos\theta\, E + \sin\theta\, J)\,\xi\| = \|\xi\|$, where E is the identity mapping on V. By putting

$$P_\theta = \cos\theta\, E + \sin\theta\, J \qquad (P_\theta{}^i{}_j = \cos\theta\, \delta^i{}_j + \sin\theta\, J^i{}_j)\,, \tag{1.4}$$

we can rewrite the condition (1.3) as

$$\|P_\theta \xi\| = \|\xi\| \qquad \text{for any } \theta\,. \tag{1.5}$$

With respect to the Minkowski norm and C-Minkowski norm, Rizza [20] has shown

Theorem 1.2 (Rizza). *Let V be an even dimensional Minkowski space with a norm $\|\xi\|$, then the so-called Rizza norm defined by*

$$\|\xi\|_R = (\frac{1}{2\pi}\int_0^{2\pi} \|P_\theta \xi\|^2 \, d\theta)^{\frac{1}{2}} \tag{1.6}$$

gives V a C-Minkowski norm.

Proof. By direct calculation, we have $P_\theta P_\phi = P_{\theta+\phi}$. Hence we see

$$\|P_\phi \xi\|_R = \Big(\frac{1}{2\pi}\int_0^{2\pi} \|P_\theta P_\phi \xi\|^2 \, d\theta\Big)^{\frac{1}{2}} = \Big(\frac{1}{2\pi}\int_0^{2\pi} \|P_{\theta+\phi}\xi\|^2 \, d\theta\Big)^{\frac{1}{2}} = \|\xi\|_R .$$

Thus $\|\xi\|_R$ is a C-Minkowski norm.

Now, in a C-Minkowski space V, let $g_{ij}(\xi) = \frac{1}{2}\frac{\partial^2 N^2(\xi)}{\partial\xi^i \partial\xi^j}$, then $g_{ij}(\xi)$ gives V a Riemann metric and satisfies $g_{ij}(k\xi) = g_{ij}(\xi)$ for any non-zero real number k. Next, let $\tilde{g}_{ij}(\xi) = \frac{1}{2\pi}\int_0^{2\pi} g_{ij}(P_\theta \xi)\, d\theta$, then we have

$$\tilde{g}_{ij}(\xi) = \frac{1}{2}\big(g_{ij}(\xi) + g_{pq}(\xi)\, J^p{}_i J^q{}_j\big) . \tag{1.7}$$

It is easy to verify that $\tilde{g}_{pq}(\xi)\, J^p{}_i J^q{}_j = \tilde{g}_{ij}(\xi)$ and

$$\partial(\tilde{g}_{ir} J^r{}_j)/\partial\xi^k + \partial(\tilde{g}_{jr} J^r{}_k)/\partial\xi^i + \partial(\tilde{g}_{kr} J^r{}_i)/\partial\xi^j = 0 .$$

Thus we obtain

Theorem 1.3. *In a C-Minkowski space, the Riemann metric given by* (1.7) *together with the complex structure* $J^i{}_j$ *defines a Kaehler structure.*

2 - Rizza manifolds

Let M be an n-dimensional *Finsler manifold* whose fundamental function is given by $L(x, y)$ (²). The metric tensor $g_{ij}(x, y)$ is introduced by $g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2(x, y)$. Here we assume that $g_{ij}(x, y)\,\xi^i \xi^j$ is positive definite. As is well-known, the tangent space $T_p(M)$ at any point $p = (x^i) \in M$ can be regarded as an n-dimensional Minkowski space such that the Minkowski norm of any tangent vector $y = y^i(\partial/\partial x^i)_p$ is given by $\|y\| = L(x, y)$, which is sometimes represented by $L(p, y)$.

(²) Terminologies and notations in the present paper are referred to Matsumoto's monograph [16] unless otherwise stated.

In the present paper, we assume moreover that M admits an almost complex structure $f^i{}_j(x)$. Then n is even and $T_p(M)$ can be regarded as an $\frac{n}{2}$-dimensional complex vector space due to $f^i{}_j(x)$ [17]. That is, for any complex number $\tilde{c}=a+ib$ and for any tangent vector $y=y^i(\partial/\partial x^i)_p\in T_p(M)$, we define $\tilde{c}y=(ay^i+bf^i{}_m(x)y^m)(\partial/\partial x^i)_p$.

Now, we consider the case where the Minkowski norm given by $\|y\|=L(p,\ y)$ is *compatible*, in each $T_p(M)$, with the complex structure $f^i{}_j(p)$. That is, the norm in $T_p(M)$ is a kind of complex Banach norm with respect to the complex structure $f^i{}_j(p)$. Concretely speaking, the relation

$$L(x,\ \tilde{c}y)=|\tilde{c}|L(x,\ y) \tag{2.1}$$

holds good for any complex number $\tilde{c}$. Since $L(x,\ ky)=kL(x,\ y)$ for any $k>0$ holds, we can rewrite (2.1) as

$$L(x,\ \phi_\theta y)=L(x,\ y) \tag{2.2}$$

for any θ where we put

$$\phi_\theta{}^i{}_j=\cos\theta\,\delta^i{}_j+\sin\theta\,f^i{}_j(x)\,. \tag{2.3}$$

It is easy to verify that condition (2.2) is equivalent to

$$g_{pq}(x,\ \phi_\theta y)\,\phi_\theta{}^p{}_i\,\phi_\theta{}^q{}_j=g_{ij}(x,\ y)\,. \tag{2.4}$$

If a manifold admits a Finsler metric and an almost complex structure satisfying the condition (2.1) (or equivalently (2.2) or (2.4)), then the manifold is called an *almost Hermitian Finsler manifold* or simply a *Rizza manifold* and the structure is called an almost Hermitian Finsler structure or a Rizza structure. The notion of the Rizza structure was, for the first time, introduced by G. B. Rizza. It is known that the Rizza manifold has been studied by Rizza [20], [21], Rund [24], Kobayashi [14], Ichijyō [6], [9], Royden [22] and Fukui [2].

With respect to Rizza manifolds, we show first

Theorem 2.1. *The tangent space at any point of a Rizza manifold is a C-Minkowski space.*

Proof. The norm of any tangent vector $y=y^i\partial/\partial x^i$ at any point p in M is given by $\|y\|=L(p, y)$. Since M admits an almost complex structure f, the tangent space at any point of M admits a basis such that $\{X_a\}=\{X_\alpha, f(X_\alpha)\}$. Operating f, we see $\{f(X_a)\}=\{f(X_\alpha), -X_\alpha\}=\{X_{\bar\alpha}, -X_\alpha\}$. If we put $f(X_a)=Q^b{}_aX_b$, we have $f(X_\alpha)=Q^b{}_\alpha X_b=X_{\bar\alpha}$ and $f(X_{\bar\alpha})=Q^b{}_{\bar\alpha}X_b=-X_\alpha$. Hence we otain $Q^b{}_a=J^b{}_a$. Thus we have $\{X_a\}=\{X_\alpha, JX_\alpha\}$. And also we have that the components of f with respect to the basis $\{X_a\}$ coincide with the components of J. Thus $\phi_\theta=P_\theta$ holds with respect to $\{X_a\}$. Then the condition $L(x, y)=L(x, \phi_\theta y)$ leads us to $\|y\|=\|P_\theta y\|$ with respect to $\{X_a\}$, which implies that the tangent space is a C-Minkowski space.

Theorem 2.2. *A Rizza manifold is an almost Hermitian manifold with respect to the given structure if and only if the tensor* $C_{ijk}=\frac{1}{2}\partial g_{ij}/\partial y^k$ *vanishes.*

Proof. Suppose that $C_{ijk}=0$ holds in a Rizza manifold. Then, the Finsler metric is, of course, a Riemann metric and the condition $L(x, y)=L(x, \phi_\theta y)$ shows us that $g_{ij}(x)=g_{pq}(x)\phi_\theta{}^p{}_i\phi_\theta{}^q{}_j$. Hence we see

$$g_{ij}(x)=\cos^2\theta\, g_{ij}(x)+\sin^2\theta\, g_{pq}(x)f^p{}_i(x)f^q{}_j(x)$$

$$+\cos\theta\sin\theta\,(g_{ir}(x)f^r{}_j(x)+g_{jr}(x)f^r{}_i(x)).$$

Consider the case where $\theta=\frac{\pi}{2}$, we have $g_{ij}=g_{pq}f^p{}_if^q{}_j$. Consequently the manifold is an almost Hermitian one. Conversely, let us consider an almost Hermitian manifold. Then $C_{ijk}=0$ holds evidently. Moreover the condition $g_{ij}=g_{pq}f^p{}_if^q{}_j$ shows us that $g_{ir}f^r{}_j+g_{jr}f^r{}_i=0$. Hence we have that $g_{pq}\phi_\theta{}^p{}_i\phi_\theta{}^q{}_j=g_{ij}$ holds good.

Applying Theorem 1.2 to a Finsler space, we obtain directly

Theorem 2.3 (Rizza). *If a* $2n$*-dimensional Finsler manifold admits an almost complex structure* f*, the following fundamental function*

$$\tilde{L}(x, y)=\Big(\frac{1}{2\pi}\int_0^{2\pi}L^2(x, \phi_\theta y)\,d\theta\Big)^{\frac{1}{2}} \tag{2.5}$$

together with f *defines a Rizza structure.*

Next, we consider a concrete example. Suppose that M is an almost Hermitian manifold, that is, M admits a Riemann metric $a_{ij}(x)$ and an almost complex structure f satisfying $a_{ij}=a_{pq}f^p{}_i f^q{}_j$. Further, we suppose that M admits a vector field $b_i(x)$. Here, we put

$$L(x,\ y)=(a_{ij}(x)\,y^i y^j)^{\frac{1}{2}}+((b_i(x)\,y^i)^2+(b_m(x)f^m{}_j(x)\,y^j)^2)^{\frac{1}{2}} \tag{2.6}$$

Then, the function $L(x,\ y)$ gives M a Finsler metric (in the wide sense). In what follows, we call this metric an *$(a,\ b,\ f)$-metric*. Now we show

Theorem 2.4. *A manifold admitting an $(a,\ b,\ f)$-metric is a Rizza manifold (in the wide sense).*

Proof. From the definition of $L(x,\ y)$, it follows

$$L(x,\ \phi_0 y)=(a_{ij}\,\phi_0{}^i{}_p\,\phi_0{}^j{}_q\,y^p y^q)^{\frac{1}{2}}+((b_i\,\phi_0{}^i{}_r\,y^r)^2+(b_m\,f^m{}_i\,\phi_0{}^i{}_r\,y^r)^2)^{\frac{1}{2}}$$

Now it is easy to show that $a_{ij}\,\phi_0{}^i{}_p\,\phi_0{}^j{}_q=a_{pq}$ holds. And, for the second term of $L(x,\ \phi_0 y)$, we see

$$(b_i\,\phi_0{}^i{}_r\,y^r)^2+(b_m\,f^m{}_i\,\phi_0{}^i{}_r\,y^r)^2$$

$$=\cos^2\theta\ (b_r y^r)^2+2\ \sin\theta\ \cos\theta\ b_r y^r\,b_i f^i{}_m y^m+\sin^2\theta\ (b_i f^i{}_r y^r)^2$$

$$+\cos^2\theta\ (b_m f^m{}_j y^j)^2-2\ \sin\theta\ \cos\theta\ b_r y^r b_i f^i{}_m y^m+\sin^2\theta\ (b_r y^r)^2$$

$$=(b_i y^i)^2+(b_m f^m{}_i y^i)^2\,.$$

Consequently, we obtain $L(x,\ \phi_0 y)=L(x,\ y)$, that is, $L(x,\ y)$ is a Rizza metric.

3 - The Rizza condition

Let M be a Rizza manifold and $T(M)$ be the tangent bundle over M. $T(M)$ admits a local canonical coordinates system $(x^i,\ y^i)$.

Now we define a point transformation Φ_0 in $T(M)$ by

$$\Phi_0\colon\ (x^i,\ y^i)\to(x^i,\ \phi_0{}^i{}_m\,y^m)\,. \tag{3.1}$$

The straightforward calculation gives us $\phi_{\theta_1}{}^i{}_m \phi_{\theta_2}{}^m{}_j = \phi_{\theta_1+\theta_2}{}^i{}_j$. So, we have $\Phi_{\theta_1} \circ \Phi_{\theta_2} = \Phi_{\theta_1+\theta_2}$. Hence $\{\Phi_\theta;\ -\infty < \theta < \infty\}$ is a *group of* 1-*parameter transformations in* $T(M)$. Let ξ be the vector field induced by the group of 1-parameter transformations. We have

$$\xi_{(x,y)} = (\frac{\mathrm{d}}{\mathrm{d}\theta}\Phi_\theta((x^i,\ y^i)))_{\theta=0} = f^i{}_m(x)\, y^m (\partial/\partial y^i)_{(x,y)}$$

that is

$$\xi = f^i{}_m(x)\, y^m\, \partial/\partial y^i\,. \tag{3.2}$$

Since the fundamental function $L(x,\ y)$ is a scalar field in $T(M)$, the Rizza condition (2.2) implies that $L(x,\ y)$ is *invariant under the group of* 1-*parameter transformations* $\{\Phi_\theta;\ -\infty < \theta < \infty\}$. Thus the condition (2.2) is equivalent to $\mathscr{L}_\xi(\frac{1}{2}L^2) = 0$. Hence we can rewrite (2.2) as $f^i{}_m y^m \dot{\partial}_i(\frac{1}{2}L^2) = 0$. By the homogeneity property of $L(x,\ y)$ for y, this can be written as $f^i{}_m y^m y^j \dot{\partial}_j \dot{\partial}_i(\frac{1}{2}L^2) = 0$. Thus the condition (2.2) is rewritten as

$$g_{ij}(x,\ y) f^i{}_m(x)\, y^m y^j = 0\,. \tag{3.3}$$

Next, differentiating (3.3) with respect to y^k, we have

$$g_{im}(x,\ y)\, f^i{}_k(x)\, y^m + g_{ik}(x,\ y)\, f^i{}_m(x)\, y^m = 0\,. \tag{3.4}$$

Transvecting (3.4) with $f^k{}_h$, we get

$$(g_{hr}(x,\ y) - g_{pq}(x,\ y)\, f^p{}_h(x)\, f^q{}_r(x))\, y^r = 0\,. \tag{3.5}$$

Conversely, multiplying (3.5) by $f^h{}_t y^t$ and contracting, we get (3.3). Therefore the condition (3.3) and (3.5) are mutually equivalent.

Differentiating (3.4) with respect to y^h, we get

$$g_{im}(x,\ y)\, f^m{}_j(x) + g_{jm}(x,\ y)\, f^m{}_i(x) + 2C_{ijm}(x,\ y)\, f^m{}_r(x)\, y^r = 0\,. \tag{3.6}$$

Conversely, transvecting (3.6) with $y^i y^j$, we get (3.3). Therefore the condition (3.3) and (3.6) are also mutually equivalent. Consequently we have proved

Theorem 3.1. *Let M be a manifold admitting an almost complex structure $f^i{}_j(x)$ and a Finsler metric $g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2(x, y)$. The condition for the couple $(f^i{}_j(x), g_{ij}(x, y))$ to define a Rizza structure is given by $L(x, \phi_0 y) = L(x, y)$. This condition is equivalent to any one of the following*

(1) $g_{pq}(x, \phi_0 y)\phi_0{}^p{}_i \phi_0{}^q{}_j = g_{ij}(x, y)$

(2) $g_{ij}(x, y) f^i{}_m(x) y^m y^j = 0$

(3) $(g_{ij}(x, y) - g_{pq}(x, y) f^p{}_i(x) f^q{}_j(x)) y^i = 0$

(4) $g_{im}(x, y) f^m{}_j(x) + g_{jm}(x, y) f^m{}_i(x) + 2C_{ijm}(x, y) f^m{}_r(x) y^r = 0\,.$

Remark. The Rizza's condition (2.2) implies that Φ_0 is an *isometry*. Hence, the general theory of the V-transformations in a Finsler space obtained by Matsumoto [**16**] directly shows us that (2.2) is equivalent to (3.6).

From the condition (2.4), we have $g_{pq}(x, fy) f^p{}_i(x) f^q{}_j(x) = g_{ij}(x, y)$. On the contrary, Fukui [2] has proved

Theorem 3.2 (Fukui). *If a Finsler metric $g_{ij}(x, y)$ and an almost complex structure $f^i{}_j(x)$ satisfy the condition*

$$g_{pq}(x, y) f^p{}_i(x) f^q{}_j(x) = g_{ij}(x, y) \tag{3.7}$$

then g_{ij} is a Riemann metric, that is, (f, g) is an almost Hermitian structure.

Proof. Differentiating (3.7) with respect to y^k, we have

$$C_{pqk} f^p{}_i f^q{}_j = C_{ijk}\,.$$

Multiplying by $f^j{}_h f^k{}_m$ and contracting with j and k, we have

$$-C_{phk} f^p{}_i f^k{}_m = C_{ijk} f^j{}_h f^k{}_m\,.$$

Thus we have $-C_{him} = C_{ihm}$. Namely, we have $C_{ijk} = 0$. Therefore g_{ij} is a Riemann metric.

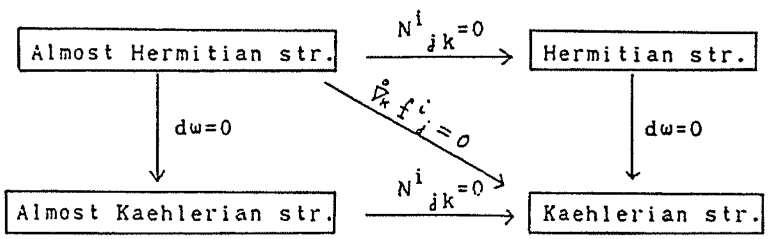
Remark. This result is a generalization of Heil's Theorem [**4**].

4 - Kaehlerian Finsler structures

Let $f^i{}_j(x)$ be an almost complex structure in a manifold M^{2n}. The *Nijenhuis tensor* of $f^i{}_j(x)$ is represented by

(4.1) $$N^i{}_{jk} = \partial_r f^i{}_j f^r{}_k - \partial_r f^i{}_k f^r{}_j + f^i{}_r \partial_j f^r{}_k - f^i{}_r \partial_k f^r{}_j .$$

We assume that a_{ij} is a Riemann metric constructing an almost Hermitian structure together with $f^i{}_j$. Putting $f_{ij} = a_{im} f^m{}_j$, we have a globally defined 2-form $\omega = f_{ij}\, dx^i \wedge dx^j$. With respect to this, the following diagram is well-known:



where $\overset{\circ}{\nabla}_k$ represents the covariant derivative with respect to the Levi-Civita's connection. By putting $f_{ijk} = \partial_i f_{jk} + \partial_j f_{ki} + \partial_k f_{ij}$, we know that the condition $d\omega = 0$ is equivalent to $f_{ijk} = 0$.

Now, in a Rizza manifold, let $(\overset{*}{\Gamma}{}^i{}_{jk}, G^i{}_j, C^i{}_{jk})$ be the Cartan's Finsler connection [16]. Let us represent by $\overset{*}{\nabla}_k$ the h-covariant derivative for the Cartan's Finsler connection. Then it is easy to see

(4.2) $$N^i{}_{jk} = \overset{*}{\nabla}_r f^i{}_j f^r{}_k - \overset{*}{\nabla}_r f^i{}_k f^r{}_j + f^i{}_r \overset{*}{\nabla}_j f^r{}_k - f^i{}_r \overset{*}{\nabla}_k f^r{}_j .$$

Hence we obtain

Theorem 4.1. *A Rizza manifold is a complex manifold if* $\overset{*}{\nabla}_k f^i{}_j = 0$ *holds good.*

Now a Rizza manifold satisfying the condition $\overset{*}{\nabla}_k f^i{}_j = 0$ is said to be a *Kaehlerian Finsler manifold*, and a Rizza manifold satisfying the condition $N^i{}_{jk} = 0$ is said to be an *Hermitian Finsler manifold.*

In a Rizza manifold, we put

$$f_{ij} = g_{im} f^m{}_j. \tag{4.3}$$

If $f_{ij} = -f_{ji}$ holds, we have $g_{pq} f^p{}_i f^q{}_j = g_{ij}$. Then Theorem 3.2 tells us that g_{ij} is a Riemann metric. Consequently we obtain

Theorem 4.2. *If a Rizza manifold satisfies $f_{ij} + f_{ji} = 0$, it is an almost Hermitian manifold.*

We put

(4.4) $\Omega_1 = f_{ij}\, dx^i \wedge dx^j$ (4.5) $\Omega_2 = f_{ij}\, dx^i \wedge \delta y^j$

where

$$\delta y^j = dy^j + G^j{}_m dx^m. \tag{4.6}$$

Both Ω_1 and Ω_2 are globally defined 2-forms on $T(M)$. For these 2-forms, we show

Theorem 4.3. *Let M be a Rizza manifold, whose Rizza structure is given by (f, g), and Ω_1 be the 2-form defined by* (4.4). *In order that $d\Omega_1 = 0$ holds, it is necessary and sufficient that (f, g) is an almost Kaehlerian structure of M.*

Proof. Since $d\Omega_1 = \partial_k f_{ij} dx^k \wedge dx^i \wedge dx^j + \dot{\partial}_k f_{ij} dy^k \wedge dx^i \wedge dx^j$, the condition that $d\Omega_1 = 0$ can be written as

(1) $\partial_k f_{ij} + \partial_i f_{jk} + \partial_j f_{ki} - \partial_k f_{ji} - \partial_i f_{kj} - \partial_j f_{ik} = 0$

(2) $\dot{\partial}_k f_{ij} - \dot{\partial}_k f_{ji} = 0.$

The condition (2) can be rewritten as $C_{kim} f^m{}_j = C_{kjm} f^m{}_i$. So, we have $C_{kjm} f^m{}_r y^r = 0$. The Rizza's condition (4) of Theorem 4.1 leads us to $f_{ij} + f_{ji} = 0$. In accordance with Theorem 4.2, it follows that g_{ij} is a Riemann metric. That is, (f, g) is an almost Hermitian structure. In this case, the condition (1) is equivalent to $f_{ijk} = 0$. Hence (f, g) is an almost Kaehlerian structure. Conversely, if (f, g) is an almost Kaehlerian structure, then we have $\Omega_1 = \omega$ and $d\Omega_1 = 0$.

Theorem 4.4. *Let M be a Rizza manifold, whose Rizza structure is given by* (f, g), *and* Ω_2 *be the* 2*-form defined by* (4.5). *In order that* $d\Omega_2 = 0$ *holds, it is necessary and sufficient that* (f, g) *is a Kaehlerian structure of M.*

Proof. The 2-form Ω_2 is written as

$$\Omega_2 = f_{im} G^m_j dx^i \wedge dx^j + f_{ij} dx^i \wedge dy^j.$$

So we have

$$d\Omega_2 = \partial_k(f_{im} G^m{}_j)\, dx^k \wedge dx^i \wedge dx^j$$

$$+ \{\dot{\partial}_k(f_{im} G^m{}_j) - \partial_j f_{ik}\}\, dy^k \wedge dx^i \wedge dx^j + \dot{\partial}_k f_{ij}\, dy^k \wedge dx^i \wedge dy^j.$$

Thus the condition $d\Omega_2 = 0$ is written as

(1) $\partial_k(f_{im} G^m{}_j) + \partial_i(f_{jm} G^m{}_k) + \partial_j(f_{km} G^m{}_i)$

$$- \partial_k(f_{jm} G^m{}_i) - \partial_i(f_{km} G^m{}_j) - \partial_j(f_{im} G^m{}_k) = 0$$

(2) $\dot{\partial}_k(f_{im} G^m{}_j - f_{jm} G^m{}_i) - \partial_j f_{ik} + \partial_i f_{jk} = 0$

(3) $\dot{\partial}_k f_{ij} - \dot{\partial}_j f_{ik} = 0.$

The condition (3) implies $C_{kim} f^m{}_j = C_{jim} f^m{}_k$, from which $C_{kim} f^m{}_r y^r = 0$. By the same method used in the proof of Theorem 4.3, we have that g_{ij} is a *Riemann metric* and (f, g) defines an almost Hermitian structure. Then we find $f_{ij} = -f_{ji}$, $G^i{}_k = \{{}^{\,i}_{mk}\} y^m$, $\overset{*}{\Gamma}{}^i{}_{jk} = \{{}^{\,i}_{jk}\}$. Now the condition (2) can be rewritten as $\overset{\circ}{\nabla}_i f_{jk} = \overset{\circ}{\nabla}_j f_{ik}$. Then we see

$$\overset{\circ}{\nabla}_i f_{jk} = \overset{\circ}{\nabla}_j f_{ik} = -\overset{\circ}{\nabla}_j f_{ki} = -\overset{\circ}{\nabla}_k f_{ji} = \overset{\circ}{\nabla}_k f_{ij} = \overset{\circ}{\nabla}_i f_{kj} = -\overset{\circ}{\nabla}_i f_{jk}.$$

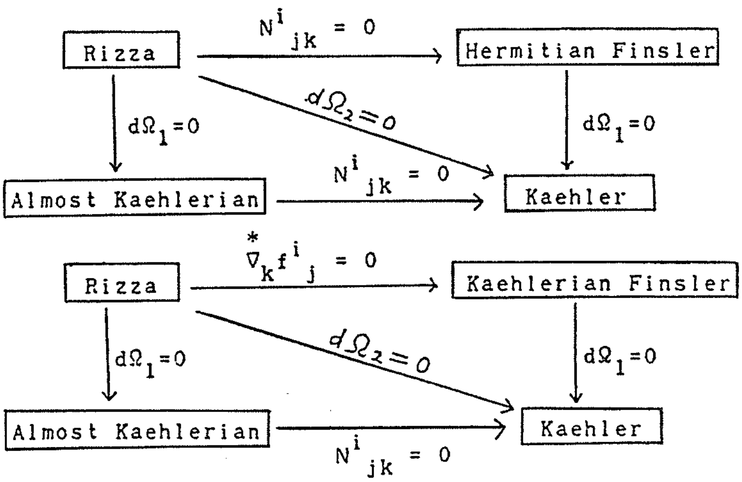
That is, we obtain $\overset{\circ}{\nabla}_i f_{jk} = 0$. Thus (f, g) defines a Kaehlerian structure. In this case, the condition (1) is satisfied identically. In effect the left hand side of (1) becomes

$$\partial_k(f_{im}\{{}^{\,m}_{rj}\} + f_{mj}\{{}^{\,m}_{ri}\})\, y^r + \partial_i(f_{jm}\{{}^{\,m}_{rk}\} + f_{mk}\{{}^{\,m}_{rj}\})\, y^r + \partial_j(f_{km}\{{}^{\,m}_{ri}\} + f_{mi}\{{}^{\,m}_{rk}\})\, y^r$$

$$= y^r \partial_r f_{ijk} = 0.$$

Conversely, if (f, g) is a Kaehlerian structure, the condition (1), (2) and (3) are all satisfied evidently.

Gathering these results, we have the following diagrams:



As well as the Cartan's Finsler connection $(\overset{*}{\Gamma}{}^i_{jk}, G^i_j, C^i_{jk})$, we know the Berwald's Finsler connection $(G^i{}_{jk}, G^i{}_j, 0)$, where $G^i{}_{jk} = \dot{\partial}_k G^i{}_j$. Let us represent $\overset{G}{\nabla}_k$ the h-covariant derivative with respect to the Berwald's connection. If G^i_{jk} are functions of position alone, namely, $\dot{\partial}_h G^i{}_{jk} = 0$ holds, then the Finsler space is said to be a Berwald space [**16**].

Now, let us suppose that (f, g) is a Kaehlerian Finsler structure. Then we have

$$\overset{*}{\nabla}_k f^i{}_j = \partial_k f^i{}_j + \overset{*}{\Gamma}{}^i{}_{mk} f^m{}_j - f^i{}_m \overset{*}{\Gamma}{}^m{}_{jk} = 0 .$$

From the relation $\overset{*}{\Gamma}{}^i{}_{km} y^m = G^i_k$, we have

$$y^m \partial_m f^i{}_j + G^i{}_m f^m{}_j - f^i{}_m G^m{}_j = 0 .$$

Differentiating partially with respect to y^k, we have

$$\overset{G}{\nabla}_k f^i_j = \partial_k f^i{}_j + G^i{}_{mk} f^m{}_j - f^i{}_m G^m_{jk} = 0 . \tag{4.7}$$

Let $H_h{}^i{}_{jk}$ be the h-curvature tensor of the Berwald connection, that is

(4.8)
$$H_h{}^i{}_{jk} = \delta_k G^i{}_{hj} + G^i{}_{rk} G^r{}_{hj} - \delta_j G^i{}_{hk} - G^i{}_{rj} G^r{}_{hk}$$

where we put $\delta_k = \partial_k - G^m{}_k \dot{\partial}_m$.

Applying the Ricci identity for $\overset{G}{\nabla}_k$ to f^i_j [**16**], we have

(4.9)
$$H_r{}^h{}_{jk} f^r{}_i = f^h{}_r H_i{}^r{}_{jk}.$$

On the other hand, if an m-dimensional Finsler space M $(m>3)$ satisfies $H_i{}^h{}_{jk} = K(g_{ij}\delta^h{}_k - g_{ik}\delta^h{}_j)$, then M is called a Finsler space of constant curvature [**16**]. Concerning this case, Fukui [2] has proved

Theorem 4.5 (Fukui). *Let M be a $2n$-dimensional Kaehlerian Finsler manifold. If M is a Finsler space of constant curvature and $n \geqslant 2$, then the h-curvature tensor of the Berwald connection vanishes.*

Proof. In this case, (4.9) can be rewritten as

$$K(g_{rj}\delta^h{}_k - g_{rk}\delta^h{}_j) f^r{}_i = f^h{}_r K(g_{ij}\delta^r{}_k - g_{ik}\delta^r{}_j).$$

Now we suppose $K \neq 0$, then we have

$$f_{ji}\delta^h{}_k - f_{ki}\delta^h{}_j = g_{ij} f^h{}_k - g_{ik} f^h{}_j.$$

Contracting this equation with respect to h and j, we find $(1-2n) f_{ki} = f_{ik}$. From the relation (4) in Theorem 3.1, we find $(1-2n) f_{ki} = -f_{ki} - 2C_{ikm} f^m{}_r y^r$. Since $n \geqslant 2$, we find $f_{ij} = \dfrac{1}{n-1} C_{ijm} f^m{}_r y^r$. Again, from the relation (4) in Theorem 3.1, we have $C_{ijm} f^m{}_r y^r = 0$. Thus we have $f_{ij} = 0$. This is a contradiction. Consequently we obtain $K = 0$.

Remark. If the metric is a Riemannian one, the h-curvature tensor $H_i{}^h{}_{jk}$ coincides with the Riemann-Christoffel's curvature tensor, and this theorem reduces to the well-known Bochner's theorem: *If a $2n$-dimensional Kaehler manifold is of constant curvature and $n \geqslant 2$, then it is of zero curvature.*

Now we consider the case $n = 1$, that is, the manifold M is 2-dimensional. In this case, the equation $(1-2n) f_{ki} = f_{ik}$, which appeared in the middle of the proof of the Theorem 4.5, becomes $f_{ik}+f_{ki}=0$. Then, in accordance with Theorem 4.2, M is a Riemann manifold. On the other hand, in a 2-dimensional Riemann manifold, the relation $R_i{}^h{}_{jk}=K(g_{ij}\delta^h_k-g_{ik}\delta^h{}_j)$ holds identically. Hence we obtain

Theorem 4.6. *A 2-dimensional Kaehlerian Finsler manifold satisfies* $H_i{}^h{}_{jk}=K(g_{ij}\delta^h{}_k-g_{ik}\delta^h{}_j)$ *if and only if it is a Riemann manifold or it satisfies* $H_i{}^h{}_{jk}=0$.

By the way, in a $2n$-dimensional Finsler manifold, $G_i{}^h{}_{jk}=\dot{\partial}_i G^h{}_{jk}$ and $G_{ij}=G_i{}^m{}_{jm}$ are tensor fields, and satisfy $y^m G_{mi}=0$ and $G_{ij}=G_{ji}$. Moreover the tensor field

$$D_i{}^h{}_{jk}=G_i{}^h{}_{jk}-\frac{1}{2n+1}\{y^h\dot{\partial}_k G_{ij}+\delta^h{}_i G_{jk}+\delta^h{}_j G_{ki}+\delta^h{}_k G_{ij}\}$$

is known as *Douglas tensor* [15]. With respect to this, Fukui [2] has shown

Theorem 4.7 (Fukui). *If a Kaehlerian Finsler manifold has vanishing Douglas tensor, then it is a Berwald space.*

Proof. From the assumption we have

$$G_i{}^h{}_{jk}=\frac{1}{2n+1}(y^h\dot{\partial}_k G_{ij}+\delta^h{}_i G_{jk}+\delta^h{}_j G_{ki}+\delta^h{}_k G_{ij}),$$

and $\overset{*}{\nabla}_k f^i{}_j=0$. Hence, from (4.7), we have also

$$\partial_k f^i{}_j+G^i{}_{mk}f^m{}_j-f^i{}_m G^m{}_{jk}=0.$$

Differentiating this with respect to y^h, we find $G_m{}^i{}_{kh}f^m{}_j=f^i{}_m G_j{}^m{}_{kh}$. Thus we have

$$f^i{}_m(y^m\dot{\partial}_h G_{jk}+\delta^m{}_j G_{kh}+\delta^m{}_k G_{hj}+\delta^m{}_h G_{jk})$$
$$=(y^i\dot{\partial}_h G_{mk}+\delta^i{}_h G_{mk}+\delta^i{}_m G_{kh}+\delta^i{}_k G_{hm})f^m{}_j.$$

Contracting this with respect to i and h, we have

$$f^r{}_m y^m \dot{\partial}_r G_{jk} + f^r{}_k G_{rj} = 2n f^r{}_j G_{rk}. \tag{4.10}$$

Transvecting (4.10) with y^j, we find $G_{kr} f^r{}_m y^m = 0$.

Differentiating this with respect to y^j, we find $f^r{}_m y^m \dot{\partial}_r G_{jk} = -G_{kr} f^r{}_j$. That is, $f^r{}_m y^m \dot{\partial}_r G_{jk} = -G_{kr} f^r{}_j = -G_{jr} f^r{}_k$. By virtue of (4.10), we have $G_{kr} f^r{}_j = 0$, that is, $G_{kj} = 0$. Thus we obtain $G_i{}^h{}_{jk} = 0$. Consequently the manifold is a Berwald space.

Besides, with respect to the holomorphic sectional curvature in a Kaehlerian Finsler manifold, some results have been obtained by Dragomir-Ianus [**1**], Fukui [**2**] and Royden [**22**]. Especially, Royden has studied the subject from the standpoint of the theory of functions of several complex variables.

5 - The induced Moór metric

Let M be a Rizza manifold determined by $(f^i{}_j(x),\ g_{ij}(x, y))$. If we put $\tilde{g}_{ij} = \frac{1}{2\pi} \int_0^{2\pi} g_{ij}(x, \phi_\theta y)\, d\theta$, then we find that

$$\tilde{g}_{ij} = \frac{1}{2}(g_{ij}(x, y) + g_{pq}(x, y) f^p{}_i(x) f^q{}_j(x)) \tag{5.1}$$

holds [**6**]. It is obvious that $\tilde{g}_{ij}$ is a generalized Finsler metric and satisfies that $\tilde{g}_{ij} = \tilde{g}_{ji}$, $\tilde{g}_{ij}(x, y)$ is $(0)p$-homogeneous for y^i and $\tilde{g}_{ij}(x, y)\xi^i \xi^j$ is positive definite. Moreover we can see easily

$$\tilde{g}_{pq}(x, y) f^p{}_i(x) f^q{}_j(x) = \tilde{g}_{ij}(x, y). \tag{5.2}$$

Differentiating the both sides of the equation (3) in Theorem 3.1 with respect to y^k, we have

$$g_{jk} = \dot{\partial}_k g_{pq} f^p{}_j f^q{}_r y^r + g_{pq} f^p{}_j f^q{}_k. \tag{5.3}$$

Since $g_{jk} = g_{kj}$, we have $\dot{\partial}_k g_{pq} f^p{}_j f^q{}_r y^r = \dot{\partial}_j g_{pq} f^p{}_k f^q{}_r y^r$, from which

$$y^r \dot{\partial}_k \tilde{g}_{rj} = \frac{1}{2}(\dot{\partial}_k g_{rj} + \dot{\partial}_k g_{pq} f^p{}_r f^q{}_j) y^r = \frac{1}{2} \dot{\partial}_k g_{pq} f^p{}_j f^q{}_r y^r = \frac{1}{2} \dot{\partial}_j g_{pq} f^p{}_k f^q{}_r y^r = y^r \dot{\partial}_j \tilde{g}_{rk}.$$

Hence we find

$$\dot{\partial}_k \tilde{g}_{pq} y^p y^q = 0. \tag{5.4}$$

By (5.3), we get also $g_{jr} y^r = g_{pq} f^p{}_j f^q{}_r y^r = (2\tilde{g}_{jr} - g_{jr}) y^r$. Hence we have $g_{jm} y^m = \tilde{g}_{jm} y^m$. Differentiating the last equation with respect to y^k, we obtain

$$g_{jk} = \dot{\partial}_k \tilde{g}_{jm} y^m + \tilde{g}_{jk}. \tag{5.5}$$

Since $g_{jk} \xi^j \xi^k$ is positive definite, so is $(\dot{\partial}_k \tilde{g}_{jm} y^m + \tilde{g}_{jk}) \xi^j \xi^k$.

By the way, if a generalized Finsler metric $\tilde{g}_{ij}(x, y)$ satisfies the condition that $\tilde{g}_{ij} = \tilde{g}_{ji}$, $\tilde{g}_{ij} \xi^i \xi^j$ is positive definite and $\tilde{g}_{ij}(x, y)$ is $(0)p$-homogeneous for y, then $\tilde{g}_{ij}(x, y)$ is called a *Moór metric* [**19**].

Now, with respect to a Rizza structure $(f^i{}_j(x), g_{ij}(x, y))$, we have obtained that $\tilde{g}_{ij}(x, y)$, which is defined by (5.1), is a Moór metric and satisfies (5.2), (5.4) and the condition that $(\tilde{g}_{jk} + \dot{\partial}_k \tilde{g}_{jm} y^m) \xi^j \xi^k$ is positive definite.

Conversely, let M be a manifold admitting an almost complex structure $f^i{}_j(x)$ and a Moór metric $\tilde{g}_{ij}(x, y)$ satisfying (5.2), (5.4) and the condition that $(\tilde{g}_{jk} + \dot{\partial}_k \tilde{g}_{jm} y^m) \xi^j \xi^k$ is positive definite. In this case, we define a new tensor g_{ij} by (5.5), that is,

$$g_{ij}(x, y) = \tilde{g}_{ij}(x, y) + \dot{\partial}_j \tilde{g}_{im}(x, y) y^m.$$

Putting $L^2(x, y) = \tilde{g}_{pq}(x, y) y^p y^q$, we find, by virtue of (5.4) and the homogeneity condition of $\tilde{g}_{ij}(x, y)$ for y, that $\dot{\partial}_j L^2 = 2\tilde{g}_{jp} y^p$ holds. And we obtain $\dot{\partial}_i \dot{\partial}_j (\frac{1}{2} L^2) = g_{ij}$. From our assumption, $g_{ij}(x, y) \xi^i \xi^j$ is positive definite, $g_{ij} = g_{ji}$ holds and $g_{ij}(x, y)$ is $(0)p$-homogeneous for y. So, $g_{ij}(x, y)$ is a Finsler metric. Moreover, it is easily seen that $g_{ij} y^j = \tilde{g}_{ij} y^j$. On the other hand, from the assumption (5.2), we have $\tilde{g}_{im} f^m{}_j = -\tilde{g}_{jm} f^m{}_i$, from which $\tilde{g}_{ij} f^i{}_m y^m y^j = -\tilde{g}_{im} f^i{}_j y^m y^j$. That is, we have $\tilde{g}_{ij} f^i{}_m y^m y^j = 0$. Hence we obtain $g_{ij} f^i{}_m y^m y^j = 0$. Thus, applying Theorem 3.1, we find that $f^i{}_j(x)$ and $g_{ij}(x, y)$ construct a Rizza structure. Consequently we obtain

Theorem 5.1. *Let M be a manifold admitting an almost complex structure $f^i{}_j(x)$. In order that M admits a Finsler metric which defines a Rizza structure together with $f^i{}_j(x)$, it is necessary and sufficient that M admits a*

Moór metric $\tilde{g}_{ij}(x, y)$ satisfying the conditions

(1) $\tilde{g}_{jk}(x, y) = \tilde{g}_{pq}(x, y) f^p{}_j(x) f^q{}_k(x)$ (2) $\dot{\partial}_k \tilde{g}_{pq}(x, y) y^p y^q = 0$

(3) $(\tilde{g}_{jk}(x, y) + \dot{\partial}_k \tilde{g}_{jm}(x, y) y^m) \xi^j \xi^k$ is positive definite.

Remark. In accordance with Miron [18], the condition for a generalized Finsler metric such that

(i) $\dot{\partial}_k \tilde{g}_{pq}(x, y) y^p y^q = 0$ (ii) $\det|\delta^i{}_j + \tilde{g}^{im} \dot{\partial}_j \tilde{g}_{km} y^k| \neq 0$

is called the *regularity condition*. In the present case, the condition (2) in Theorem 5.1 coincides with (i) of the regularity condition. However, the condition (3) is a little stronger than (ii). Because, if (3) is satisfied, $\det|\tilde{g}_{ij} + \dot{\partial}_j \tilde{g}_{ik} y^k| \neq 0$. Of course $(\tilde{g}_{ij})^{-1}$ exists. Hence (ii) is satisfied. But the converse is not always true.

6 - $(f, \tilde{g}, N)$-structures

In this section we consider an m-dimensional manifold M equipped with a non-linear connection $N^i{}_j(x, y)$. To give a non-linear connection implies that a horizontal distribution is assigned globally in the tangent bundle $T(M)$. It is needless to say that the transformation rule of $N^i{}_j(x, y)$ is given by

(6.1) $$\bar{N}^a{}_c(\bar{x}, \bar{y}) \frac{\partial \bar{x}^c}{\partial x^i} + \frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^m} y^m = \frac{\partial \bar{x}^a}{\partial x^m} N^m{}_i(x, y).$$

The quantity $G^i{}_j$ treated in 4 was a kind of non-linear connection.

Now we put

(6.2) $$X_i = \partial/\partial x^i - N^m{}_i \partial/\partial y^m \qquad Y_i = \partial/\partial y^i.$$

Then $\{X_i, Y_i\}$ is a local $2m$-frame field in $T(M)$, which we call the *N-frame*. And $\{X_i\}$ is a basis of the horizontal distribution in $T(M)$ and $\{Y_i\}$ is a basis of the vertical distribution in $T(M)$. The transformation rule of $\{X_i, Y_i\}$ are given by

(6.3) $$X_i = \frac{\partial \bar{x}^a}{\partial x^i} \bar{X}_a \qquad Y_i = \frac{\partial \bar{x}^a}{\partial x^i} \bar{Y}_a.$$

Now we assume moreover that $m=2n$ and that M admits an almost complex structure $f^i{}_j(x)$ and a Moór metric $\tilde{g}_{ij}(x, y)$ satisfying the condition

$$\tilde{g}_{pq}(x, y) f^p{}_i(x) f^q{}_j(x) = \tilde{g}_{ij}(x, y). \tag{6.4}$$

We will say, in this case, that M admits an $(f, \tilde{g}, N)$-*structure*. Here we do not assume that $\tilde{g}_{ij}$ is the induced Moór metric from a Rizza structure.

Now we can define globally on $T(M)$ a (1, 1)-tensor field F such that

$$F(X_i) = f^m{}_i X_m \qquad F(Y_i) = f^m{}_i Y_m. \tag{6.5}$$

It is apparent that F is an *almost complex structure on* $T(M)$ [5]. Moreover we can define an inner product $\langle , \rangle$ such that

$$\langle X_i, X_j\rangle = \tilde{g}_{ij} \qquad \langle X_i, Y_j\rangle = 0 \qquad \langle Y_i, Y_j\rangle = \tilde{g}_{ij}. \tag{6.6}$$

Then the inner product gives $T(M)$ a globally defined *Riemann metric* $\tilde{G}$ [5]. The components of F and $\tilde{G}$ with respect to the N-frame are written as

$$F = \begin{pmatrix} f^i{}_j(x) & 0 \\ 0 & f^i{}_j(x) \end{pmatrix} \qquad \tilde{G} = \begin{pmatrix} \tilde{g}_{ij}(x, y) & 0 \\ 0 & \tilde{g}_{ij}(x, y) \end{pmatrix}. \tag{6.7}$$

In addition we have ${}^tF\tilde{G}F = \tilde{G}$. Therefore $(F, \tilde{G})$ defines an almost Hermitian structure on $T(M)$. Thus we obtain

Theorem 6.1. *If a manifold M admits an $(f, \tilde{g}, N)$-structure, its tangent bundle $T(M)$ admits an almost Hermitian structure.*

Remark. Theorem 6.1 can be rewritten more precisely as follows: *If a manifold M admits an $(f, \tilde{g}, N)$-structure, then the tangent bundle $T(M)$ admits a $D(U(n))$-structure (as a G-structure). Here $U(n)$ is a real unitary group of order $2n$ and* $D(U(n)) = \{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \mid A \in U(n)\}$ [7], [10], [11], [13].

Next we consider such tensor fields F_1, F_2, F_3 on $T(M)$ that the components with respect to the N-frame are given by

$$F_1 = \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 0 & -E_{2n} \\ E_{2n} & 0 \end{pmatrix}. \tag{6.8}$$

Straightforward calculation leads us to

$$F_1^2 = F_2^2 = F_3^2 = -E_{4n}$$

$$F_1 F_2 = -F_2 F_1 = F_3, \qquad F_2 F_3 = -F_3 F_2 = F_1, \qquad F_3 F_1 = -F_1 F_3 = F_2$$

$${}^tF_1 \tilde{G} F_1 = {}^tF_2 \tilde{G} F_2 = {}^tF_3 \tilde{G} F_3 = \tilde{G}\,.$$

Hence we obtain

Theorem 6.2. *If a manifold admits an $(f, \tilde{g}, N)$-structure, then its tangent bundle admits an almost quaternion metric structure.*

Now, in a manifold equipped with an $(f, \tilde{g}, N)$-structure, let us put

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{g}^{im}(X_k \tilde{g}_{jm} + X_j \tilde{g}_{km} - X_m \tilde{g}_{jk})\,. \tag{6.9}$$

Then $\tilde{\Gamma}^i_{jk}$ is symmetric with j and k and satisfies the transformation rule of a linear connection. So, we represent by $\tilde{\nabla}_k$ the h-covariant derivative with respect to $(\tilde{\Gamma}^i_{jk}, N^i{}_j)$ [3], [13], [18]. Then direct calculation leads us to

$$\tilde{\nabla}_k \tilde{g}_{ij} = 0\,. \tag{6.10}$$

Remark. This connection $\tilde{\Gamma}^i_{jk}$ closely resembles to the *Miron-Cartan connection* defined by Miron [18] and named by Hashiguchi [3]. The only difference between them is the choice of non-linear connections.

Now we put

$$\tilde{f}_{ij}(x, y) = \tilde{g}_{im}(x, y) f^m{}_j(x)\,. \tag{6.11}$$

By virtue of (6.4) we have

$$\tilde{f}_{ij} = -\tilde{f}_{ji} \qquad \tilde{f}_{im} f^m{}_j = -\tilde{g}_{ij}\,. \tag{6.12}$$

Putting

$$\tilde{F}_{ijk} = X_i \tilde{f}_{jk} + X_j \tilde{f}_{ki} + X_k \tilde{f}_{ij} \tag{6.13}$$

we have

$$\tilde{F}_{ijk} = \tilde{\nabla}_i \tilde{f}_{jk} + \tilde{\nabla}_j \tilde{f}_{ki} + \tilde{\nabla}_k \tilde{f}_{ij}. \tag{6.14}$$

So, $\tilde{F}_{ijk}$ is a tensor field of Finsler type.

Next, the relation $\tilde{\Gamma}^i_{jk}(x, y) = \tilde{\Gamma}^i_{kj}(x, y)$ leads us to

$$N^h{}_{ij} = \tilde{\nabla}_r f^h{}_i f^r{}_j - \tilde{\nabla}_r f^h{}_j f^r{}_i + f^h{}_r \tilde{\nabla}_i f^r{}_j - f^h{}_r \tilde{\nabla}_j f^r{}_i. \tag{6.15}$$

Moreover, let us put $\tilde{N}_{hij} = \tilde{g}_{hm} N^m{}_{ij}$, then we have

$$\tilde{N}_{hij} = \tilde{\nabla}_r \tilde{f}_{hi} f^r{}_j - \tilde{\nabla}_r \tilde{f}_{hj} f^r{}_i + \tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j - \tilde{f}_{hr} \tilde{\nabla}_j f^r{}_i.$$

From $(6.12)_2$, we have $\tilde{\nabla}_i \tilde{f}_{hr} f^r{}_j = -\tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j$. Using this, we calculate

$$\begin{aligned} &\tilde{F}_{ihr} f^r{}_j - \tilde{F}_{jhr} f^r{}_i \\ &= -\tilde{\nabla}_r \tilde{f}_{hi} f^r{}_j + \tilde{\nabla}_r \tilde{f}_{hj} f^r{}_i - \tilde{f}_{hr} \tilde{\nabla}_i f^r{}_j + \tilde{f}_{hr} \tilde{\nabla}_j f^r{}_i - \tilde{\nabla}_h \tilde{f}_{ir} f^r{}_j + \tilde{\nabla}_h \tilde{f}_{jr} f^r{}_i \\ &= -\tilde{N}_{hij} + \tilde{\nabla}_h \tilde{f}_{ri} f^r{}_j - \tilde{f}_{jr} \tilde{\nabla}_h f^r{}_i = -\tilde{N}_{hij} + 2\tilde{\nabla}_h \tilde{f}_{ri} f^r{}_j. \end{aligned}$$

That is, we have

$$\tilde{F}_{ihr} f^r{}_j - \tilde{F}_{jhr} f^r{}_i = -\tilde{g}_{hm} N^m{}_{ij} + 2\tilde{\nabla}_h \tilde{f}_{ri} f^r{}_j. \tag{6.16}$$

Hence, if $\tilde{F}_{ijk} = 0$ and $N^h{}_{ij} = 0$ hold, then $\tilde{\nabla}_h f^i{}_j = 0$ holds true.

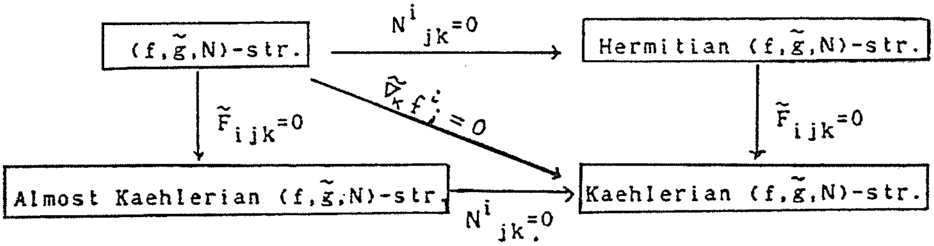
Conversely, if $\tilde{\nabla}_k f^i{}_j = 0$ holds, according to (6.10), (6.14) and (6.15), $\tilde{F}_{ijk} = 0$ and $N^h{}_{ij} = 0$ hold true. Consequently we obtain

Theorem 6.3. *In order that $\tilde{\nabla}_k f^i{}_j = 0$ holds good in a manifold equipped with an $(f, \tilde{g}, N)$-structure, it is necessary and sufficient that $N^h{}_{ij} = 0$ and $\tilde{F}_{ijk} = 0$ hold good.*

Of course, this theorem is a generalization of Eckmann's famous theorem.

Now, an $(f, \tilde{g}, N)$-structure whose almost complex structure f is integrable is said to be an *Hermitian $(f, \tilde{g}, N)$-structure*. An $(f, \tilde{g}, N)$-structure satisfying $\tilde{F}_{ijk} = 0$ is said to be an *almost Kaehlerian $(f, \tilde{g}, N)$*-structure and an $(f, \tilde{g}, N)$-structure satisfying $\tilde{\nabla}_k f^i{}_j = 0$ is said to be a *Kaehlerian $(f, \tilde{g}, N)$-structure.*

Now we obtain the following diagram



Now, let us assume that a manifold M admits a Rizza structure $(f,\ g)$, and let $\tilde{g}_{ij}$ be the induced Moór metric from $(f,\ g)$. Let $G^i{}_j$ be the non-linear connection defined in 4. Then $(f^i{}_j,\ \tilde{g}_{ij},\ G^i{}_j)$ determines an $(f,\ \tilde{g},\ N)$-structure, which we call an $(f,\ \tilde{g},\ N)$-structure derived from a Rizza structure. If the $(f,\ \tilde{g},\ N)$-structure is a Kaehlerian $(f,\ \tilde{g},\ N)$-structure, then the original Rizza structure $(f,\ g)$ is said to be a *quasi Kaehlerian Finsler structure*. The condition for a Rizza structure $(f,\ g)$ to be a quasi Kaehlerian Finsler structure is given by $\tilde{\nabla}_k f^i{}_j=0$, and it is equivalent to $\tilde{F}_{ijk}=0$ and $N^i{}_{jk}=0$.

Well, g is a Finsler metric and $\tilde{g}_{ij}=\frac{1}{2}(g_{ij}+g_{pq}f^p{}_i f^q{}_j)$ holds. Then $\tilde{f}_{ij}=\frac{1}{2}(g_{im}f^m{}_j-g_{jm}f^m{}_i)$. So, we have

$$\tilde{f}_{ij}=\frac{1}{2}(f_{ij}-f_{ji})\,. \tag{6.17}$$

On the other hand, $X_k=\delta_k$ holds in our case. Hence

$$\tilde{F}_{ijk}=\frac{1}{2}(\delta_i f_{jk}+\delta_j f_{ki}+\delta_k f_{ij}-\delta_i f_{kj}-\delta_j f_{ik}-\delta_k f_{ji})\,.$$

Putting

$$F_{ijk}=\frac{1}{2}(\overset{*}{\nabla}_i f_{jk}+\overset{*}{\nabla}_j f_{kj}+\overset{*}{\nabla}_k f_{ij}-\overset{*}{\nabla}_i f_{kj}-\overset{*}{\nabla}_j f_{ik}-\overset{*}{\nabla}_k f_{ji}) \tag{6.18}$$

we have $\tilde{F}_{ijk}=F_{ijk}$. Thus we obtain

Theorem 6.4. *A Rizza manifold is a quasi Kaehlerian Finsler manifold if and only if the equations* $F_{ijk}=0$ *and* $N^i{}_{jk}=0$ *are satisfied.*

Moreover we can show

Theorem 6.5. *If a Rizza manifold is a Kaehlerian Finsler manifold, it is a quasi Kaehlerian Finsler manifold.*

Proof. By virtue of (6.18), our assumption $\overset{*}{\nabla}_k f^i{}_j = 0$ leads us to $F_{ijk} = 0$. On the other hand, by virtue of Theorem 4.1, $\overset{*}{\nabla}_k f^i{}_j = 0$ shows us that $N^i{}_{jk} = 0$. Consequently, the theorem is proved.

7 - A generalization of Yano-Westlake's theorem

Concerning almost Hermitian manifolds, the following is known as *Yano-Westlake's Theorem* [26], [27]:

Theorem. *A necessary and sufficient condition that a 2n-dimensional almost Hermitian manifold be locally conformal to an almost Kaehlerian manifold is that,*

$$f_{ijk} = \frac{1}{2(n-1)}(f_{ij} f_k + f_{jk} f_i + f_{ki} f_j) \qquad \textit{for } n > 2$$

and

$$\partial_j f_i = \partial_i f_j \qquad \textit{for } n = 2$$

where $f_{ijk} = \partial_i f_{jk} + \partial_j f_{ki} + \partial_k f_{ij}$ *and* $f_k = f_{krs} f^r{}_m g^{ms}$. *The equations* $f_{ijk} = 0$ *are satisfied identically for* $n = 1$. *Hence a 2-dimensional almost Hermitian manifold is always almost Kaehlerian.*

In the paper [13], a manifold has been said to admit a $(\tilde{g},\ N)$-*structure*, when and only when it admits a Moór metric $\tilde{g}$ and a non-linear connection N. At the same time, the structure $(e^{2\sigma(x)}\tilde{g},\ N)$ has been called a *conformal change* of the $(\tilde{g},\ N)$-structure where $\sigma(x)$ is a scalar field on the manifold. Some tensor fields which are invariant under the conformal changes have been shown explicitly.

Now we consider a manifold M admitting an $(f,\ \tilde{g},\ N)$-structure, and we call the structure $(f,\ e^{2\sigma(x)}\tilde{g},\ N)$ a conformal change of the $(f,\ \tilde{g},\ N)$-structure. If we put $\tilde{g}^*_{ij}(x,\ y) = e^{2\sigma(x)}\tilde{g}_{ij}(x,\ y)$, the tensor $\tilde{F}^*_{ijk}$ for $\tilde{g}^*_{ij}$ is written as

$$\tilde{F}^*_{ijk} = X_i(\tilde{g}^*_{jm} f^m{}_k) + X_j(\tilde{g}^*_{km} f^m{}_i) + X_k(\tilde{g}^*_{im} f^m{}_j)\,.$$

Calculating this, we obtain

$$\tilde{F}^{*}_{ijk} = e^{2\sigma(x)}\{\tilde{F}_{ijk} + 2(\partial_i\sigma\tilde{f}_{jk} + \partial_j\sigma\tilde{f}_{ki} + \partial_k\sigma\tilde{f}_{ij})\}. \tag{7.1}$$

Putting

$$\tilde{F}_k = \tilde{F}_{krs} f^r{}_m \tilde{g}^{ms} \tag{7.2}$$

and using (6.12), we see

$$\tilde{F}^{*}_k = e^{-2\sigma(x)}\tilde{F}^{*}_{krs} f^r{}_m \tilde{g}^{ms} = \tilde{F}_k + 4(n-1)\,\partial_k\sigma.$$

Hence, for $n>1$, we have

$$\partial_k\sigma = \frac{1}{4(n-1)}(\tilde{F}^{*}_k - \tilde{F}_k). \tag{7.3}$$

Thus we have

$$\tilde{F}^{*}_{ijk} = e^{2\sigma(x)}\{\tilde{F}_{ijk} + \frac{1}{2(n-1)}(\tilde{F}^{*}_i - \tilde{F}_i)\tilde{f}_{jk} + (\tilde{F}^{*}_j - \tilde{F}_j)\tilde{f}_{ki} + (\tilde{F}^{*}_k - \tilde{F}_k)\tilde{f}_{ij}\}.$$

Here we put

$$\tilde{Q}_{ijk} = \tilde{F}_{ijk} - \frac{1}{2(n-1)}(\tilde{F}_i\tilde{f}_{jk} + \tilde{F}_j\tilde{f}_{ki} + \tilde{F}_k\tilde{f}_{ij}) \tag{7.4}$$

$$\tilde{Q}^i_{jk} = \tilde{g}^{im}\tilde{Q}_{mjk}. \tag{7.5}$$

Paying attention to the relation $\tilde{f}^{*}{}_{jk} = e^{2\sigma(x)}\tilde{f}_{jk}$, we have $\tilde{Q}^{*i}_{jk} = \tilde{Q}^i{}_{jk}$. Hence we obtain

Theorem 7.1. *In a 2n-dimensional manifold admitting an $(f, \tilde{g}, N)$-structure, the tensor field*

$$\tilde{Q}^i{}_{jk} = \tilde{g}^{im}\{\tilde{F}_{mjk} - \frac{1}{2(n-1)}(\tilde{F}_m\tilde{f}_{jk} + \tilde{F}_j\tilde{f}_{km} + \tilde{F}_k\tilde{f}_{mj})\}$$

is invariant under the conformal changes of the $(f, \tilde{g}, N)$-structure where the dimension of the manifold is greater than 2.

Now, let us assume that $(f^i{}_j, \tilde{g}^*{}_{ij}, N^i{}_j)$ be a conformal change of the given $(f, \tilde{g}, N)$-structure and be an almost Kaehlerian $(f, \tilde{g}, N)$-structure. The assumption is represented by $\tilde{g}^*_{ij} = e^{2\sigma(x)}\tilde{g}_{ij}$ and $\tilde{F}^*{}_{ijk} = 0$. Then we have $\tilde{F}_k{}^* = 0$ and

$\partial_k \sigma = -\dfrac{1}{4(n-1)}\tilde{F}_k$. Thus the relations

$$\dot{\partial}_j \tilde{F}_k = 0 \qquad \partial_j \tilde{F}_k = \partial_k \tilde{F}_j \tag{7.6}$$

must hold identically. Moreover $\tilde{F}^*{}_{ijk} = 0$ means $\tilde{Q}^{*i}{}_{jk} = 0$, and so $\tilde{Q}^i{}_{jk} = 0$. That is, the relation

$$\tilde{F}_{ijk} = \frac{1}{2(n-1)}(\tilde{F}_i \tilde{f}_{jk} + \tilde{F}_j \tilde{f}_{ki} + \tilde{F}_k \tilde{f}_{ij}) \tag{7.7}$$

must hold identically.

Conversely, let us suppose that a manifold M admits a $(f, \tilde{g}, N)$-structure satisfying the condition (7.6) and (7.7). Then (7.6) implies that, fo *any point* $p \in M$, there exists such a suitabLe local coordinate neighlurhood (U, x^i) that $p \in U$ and U admits a local scalar field $\sigma(x)$ satisfying $\partial_k \sigma = -\dfrac{1}{4(n-1)}\tilde{F}_k$. In each U, the equation (7.7) is also satisfied identically. Hence, by virtue of (7.1), the structure $(f^i{}_j, \tilde{g}^*{}_{ij}, N^i{}_j)$ satisfies $\tilde{F}^*{}_{ijk} = 0$ in the neighbourhood U where $\tilde{g}^* = e^{2\sigma(x)}\tilde{g}$. That is, the $(f, \tilde{g}, N)$-structure is locally conformal to an almost Kaehlerian $(f, \tilde{g}, N)$-structure. Consequently we obtain

Theorem 7.2. *Let M^{2n} $(n \geqq 2)$ be a $2n$-dimensional manifold admitting an $(f, \tilde{g}, N)$-structure. A necessary and sufficient condition that M^{2n} be locally conformal to an almost Kaehlerian $(f, \tilde{g}, N)$-structure is that*

$$\tilde{Q}^i{}_{jk} = 0 \qquad \dot{\partial}_j \tilde{F}_k = 0 \qquad \partial_j \tilde{F}_k = \partial_k \tilde{F}_j$$

hold good.

With respect to the case where $n = 1$, it is clear that any 2-dimensional

$(f, \tilde{g}, N)$-manifold always satisfies $\tilde{F}_{ijk}=0$. Hence the following is true: *An $(f, \tilde{g}, N)$-structure on a 2-dimensional manifold is always an almost Kaehlerian $(f, \tilde{g}, N)$-structure.*

References

[1] S. DRAGOMIR and S. IANUS, *On the holomorphic sectional curvatures of Kaehlerian Finsler spaces*, Tensor (N.S.) **39** (1982).

[2] M. FUKUI, *Complex Finsler manifolds* (to appear).

[3] M. HASHIGUCHI, *On generalized Finsler spaces*, An. Ştiinţ. Univ. «Al. I. Cuza» (Iaşi), Seçt. Mat. **30** (1984).

[4] E. HEIL, *A relation between Finslerian and Hermitian metrics*, Tensor (N.S.) **16** (1965).

[5] Y. ICHIJYŌ, *Almost complex structures of tangent bundles and Finsler metrics*, J. Math. Kyoto Univ. **6** (1967).

[6] Y. ICHIJYŌ, *Almost Hermitian Finsler manifolds*, Tensor (N.S.), **37** (1982).

[7] Y. ICHIJYŌ, *On almost Finsler structures*, Proc. Romanian-Japanese Coll. Finsler Geom., Braşov (1984).

[8] Y. ICHIJYŌ, *On some G-structures defined on tangent bundles*, Tensor **42** (1985).

[9] Y. ICHIJYŌ, *Almost Hermitian Finsler manifolds and non-linear connections*, Conf. Sem. Mat., Univ. Bari **214** (1986).

[10] Y. ICHIJYŌ, *On the Finsler group and an almost symplectic structure on a tangent bundle*, J. Math. Kyoto Univ. **28** (1988).

[11] Y. ICHIJYŌ, *The $D(O(n))$-structures in tangent bundles* (to appear).

[12] Y. ICHIJYŌ, *Remarks on Rizza manifolds* (to appear).

[13] Y. ICHIJYŌ, *Conformal curvature tensors of a generalized Finsler metric* (to appear).

[14] S. KOBAYASHI, *Negative vector bundles and complex Finsler structures*, Nagoya Math. J. **57** (1975).

[15] M. MATSUMOTO, *Projective changes of Finsler metrics and projectively flat Finsler spaces*, Tensor **34** (1980).

[16] M. MATSUMOTO, *Foundation of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Ohtsu-shi, Japan, 1986.

[17] Y. MATSUSHIMA, *Differentiable manifolds*, Marcel Dekker, New York, 1972.

[18] R. MIRON, *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto Univ. **23** (1983).

[19] A. MOÓR, *Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume*, Acta Sci. Math. Szeged **17** (1956).

[20] G. B. RIZZA, *Strutture di Finsler sulle varietà complesse*, Atti Acad. Naz. Lincei Rend. **33** (1962).

[21] G. B. RIZZA, *Strutture di Finsler di tipo quasi Hermitiano*, Riv. Mat. Univ. Parma **4** (1963).

[22] H. L. ROYDEN, *Complex Finsler metrics*, Proc. AMS-IMS-SIAM Joint Sum. Res. Conf. on Complex differential geometry and non-linear differential equations, Brunswick, Maine, 1984; Contemp. Math. **49** (1986).

[23] H. RUND, *The differential geometry of Finsler spaces*, Springer, Berlin, 1959.

[24] H. RUND, *Generalized metrics on complex manifolds*, Math. Nachr. **37** (1967).

[25] A. E. TAYLOR, *Introduction to functional analysis*, Wiley & Sons, New York, 1959.

[26] W. J. WESTLAKE, *Conformal Kähler manifolds*, Proc. Cambridge Philos. Soc., **50** (1954).

[27] K. YANO, *Geometria conforme in varietà quasi Hermitiane*, Atti Accad. Naz. Lincei Rend. **16** (1954).

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