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A. CARBONE and G. MARINO (*)

Fixed points and almost fixed points of nonexpansive maps in Banach spaces (**)

Introduction

Let N be a normed linear space over the real field R. Set, for x, y in N

$$\tau(x,\ y) = \lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}\ .$$

The following properties of τ are well known (see [3] and can be easily proved):

(i) $\tau(x,\ ax+by) = a\|x\| + b\tau(x,\ y)$ for $a \in R$ $b \geq 0$ x and y in N.

(ii) $\tau(x,\ y+z) \leq \tau(x,\ y) + \tau(x,\ z)$ for x, y, z in N.

(iii) $-\tau(x,\ -y) \leq \tau(x,\ y) = \tau(-x,\ -y)$ for x, y in N.

Equality holds in (iii) for any pair x, y if and only if N is smooth, while $\tau(x,\ y) = -\tau(x,\ -y) = \dfrac{(x,\ y)}{\|x\|}$ if N is a Hilbert space. Now set, for any y, z in N

$$LS(z,\ y) = \{x \in N;\ \tau(z-x,\ y-z) < 0\} \qquad LS'(z,\ y) = \{x \in N;\ \tau(z-x,\ z-y) > 0\}.$$

(*) Indirizzo degli AA.: Dipartimento di Matematica, Università della Calabria, I-87036 Arcavacata di Rende (CS).

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Set, also, for any z, y in N,

$$G(z,\ y)=\{x\in N;\ \|x-y\|\leq\|x-z\|\}\ .$$

These sets have been studied first in the Hilbert space setting, then in general: note that if N is Hilbert, then we have

$$G(z,\ y)\subset LS(z,\ y)=LS'(z,\ y)\quad \text{whenever } y\neq z\ . \tag{1}$$

By using the above defined sets, several results concerning fixed points theory have been given: see e.g. the survey paper [**6**]. We recall two such results. The set X will be assumed to be nonempty throughout.

Theorem 0.1 [**1**] Let N be a Hilbert space; let X be a closed, convex subset of N, and $T\colon X\to X$ a nonexpansive mapping. If a point $z\in X$ exists such that $LS(z,\ Tz)\cap X$ is nonempty and bounded, then T has a fixed point.

Theorem 0.2 ([**2**], Theorem 2.1) Let N be a Banach space whose bounded closed convex subsets have the fixed point property for nonexpansive self-mappings. Let X be a closed convex subset of N, and suppose $T\colon X\to X$ is a nonexpansive mapping. If there exists $z\in X$ such that $G(z,\ Tz)\cap X$ is bounded, then T has a fixed point in X.

Of course, Theorem 0.2 is an extension of Theorem 0.1. Note that for X bounded, also the converse Theorem 0.1 is true (cf. [**6**], p. 249).

In the present paper, we give extensions of Theorems 0.1 and 0.2. Also, we give a characterization of strict convexity of a space, by using the sets $LS(z,\ y)$ and $G(z,\ y)$.

After the present introduction, the paper contains two sections: **1** contains the results (as well as some remarks). All the proofs are in **2**.

1 - Results

We begin with a proposition, concerning the geometry of normed spaces.

Theorem 1.1. *For a normed space N, the following are equivalent:*

(a) *N is strictly convex.*

(b) *For any z, y in N, $z \neq y$, we have* $G(z, y) \subset LS(z, y)$.

(c) *For any z, y in N, $z \neq y$, we have* $G(z, y) \subset LS'(z, y)$.

Remark 1. The following example shows that the inclusion in (b) or (c) is in general strict, also in nice spaces. Let $N = R^2$ with the euclìean norm. If $z = (0, 1)$ and $y = (1, 0)$, then $G(z, y)$ and $LS(z, y)$ are as in the following picture

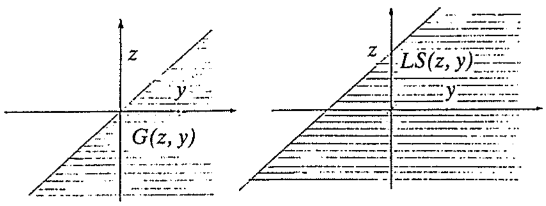

Fig. 1

Remark 2. When X is not strictly convex, the set $G(z, y)$ can be «remarkably» larger than $LS'(z, y)$ (which contains $LS(z, y)$). This can be seen the following example. Let $N = R^2$ endowed with the l_1 norm; $z = (0, 1)$, $y = (1, 0)$. It is straightforward to verify that the sets $G(z, y)$, $LS(z, y)$ and $LS'(z, y)$ are as in the following picture

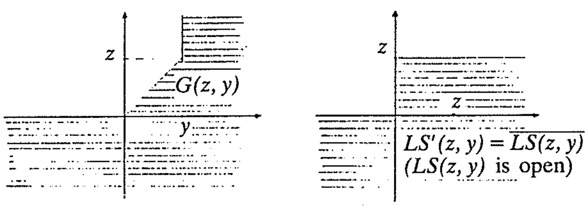

Fig. 2

Remark 3. In any space N, for any pair z, y we have

$$G(z, y) \subset \widetilde{LS}(z, y) = \{x \in N;\ \tau(z - x,\ y - z) \leq 0\}\ .$$

This inclusion is proved at the beginning of the proof of Theorem 1.1, (a) ⇒ (b) (see 3). In the above example, we have

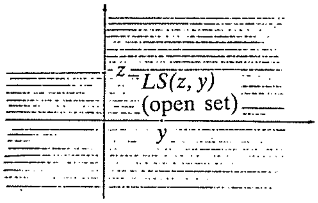

Fig. 3

Now we shall state some results concerning fixed points.

Theorem 1.2. *Let* $T: X \to X$ *be a nonexpansive mapping, where* X *is a closed, convex subset of* N. *If there exists a point* $z \in X$ *such that* $LS(z, Tz) \cap X$ *is bounded, then a bounded sequence* $\{x_n\}$ *exists such that* $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Recall that $T: X \to X$ is said to possess the «almost fixed points property» (AFPP) if $\inf\{\|x - Tx\|;\ x \in X\} = 0$.

Note that any nonexpansive mapping $T: X \to X$ has (AFPP) if X is bounded closed and convex (this can be seen if we assume $\theta \in X \subset \{x \in X;\ \|x\| \leq 1\}$ by considering — for r sufficiently close to 1 - the fixed points of the contraction mappings $rT: X \to X$).

Also, the following is known: if $T: X \to X$ is nonexpansive and a bounded sequence $\{x_n\}$ in X exists such that $\|x_n - Tx_n\| \to 0$, then T has a fixed points if one of the following holds:

(d) N is uniformly convex and complete.

(e) N is reflexive (or also: K is weakly compact) and it satisfies Opial's condition, i.e.: if a sequence $\{s_n\}$ converges weakly to s, then $\lim_{n\to\infty} \inf \|s_n - x\| > \lim_{n\to\infty} \inf \|s_n - s\|$ for all $x \in N \setminus \{s\}$.

These (and more general) results are proved in [4] (p. 247).

Thus we obtain the following Corollary, which extends Theorem 0.1.

Corollary 1.1. *Let all the assumptions of Theorem* 1.2 *hold. Then* T *has a fixed point in* N *satisfies* (d) *or* (e).

Remark 4. By using our Theorem 1.1, the first part of Corollary 1.1 follows also from Theorem 0.2.

It is known (see [5]) that in reflexive spaces, a closed convex subsets has (AFPP) for all nonexpansive mappings if and only if it is «linearly bounded» (i.e., it has bounded intersection with all lines in N). Therefore we can state the following

Corollary 1.2. *Let N be reflexive and X a bounded, convex subset of X. If $LS(z, Tz) \cap X$ is bounded for all nonexpansive mappings $T: X \to X$, then X is linearly bounded.*

Finally, we state another fixed point theorem, which extends Theorem 0.2 in case N is not assumed to be strictly convex. Note that the proof we shall give is essentially the same as given in [2] for Theorem 0.2.

Theorem 1.3. *Let N be such that its closed, bounded, convex subsets have the fixed point property for nonexpansive self-mappings. Let $T: X \to X$ be a nonexpansive mapping, with X closed convex. If there exists $z \in X$ such that $LS(z, Tz) \cap X$ is bounded, then T has a fixed point in X.*

Remark 5. Let N be the space R^2 endowed with the l_1 norm. Let $X = \{(a, b) \in R^2;\ b \geq \max(a-2;\ 0)\}$ and T a nonexpansive mapping such that $T((0, 1)) = (1, 0)$. The existence of at least a fixed point is assured by Theorem 1.3 (by using $z = (0, 1)$; Theorem 0.2 does not imply the same conclusion (at least for $z = (0, 1)$).

2 - Proofs

In this section we shall prove the three theorems stated in **1**.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b). Assume that N is strictly convex. Let $z, y \in N$; $z \neq y$. We want to prove the inclusion $G(z, y) \subset LS(z, y)$.

Let $x \notin LS(z, y)$, i.e., $\tau(z-x, y-z) \geq 0$. Then we have (by using (i) and (ii)) $\|z-x\| = \tau(z-x, z-x) = \tau(z-x, y-x) - \tau(z-x, y-z) \leq \|y-x\|$ (up to now, the strict convexity of N has not been used).

Also, $\|z-x\|=\|y-x\|$ implies

$$\tau(z-x,\ y-z)=0 \tag{2}$$

and

$$\tau(z-x,\ y-x)=\|y-x\|\,. \tag{3}$$

Since N is strictly convex, the function of t: $\|z-x+t(y-z)\|$ must be strictly convex since $z-x\neq y-x$. But we have instead from (2), since $\|z-x\|=\|y-x\|$, that it would have the constant value of its minimum for $t\in[0,\ 1]$: this contradiction proves that $\|z-x\|<\|y-x\|$, thus $x\notin G(z,\ y)$.

(b) $\Rightarrow$ (c). It follows immediately from the definition, by using (iii).

(c) $\Rightarrow$ (a). Assume that N is not strictly convex. Thus, there is on the unit sphere of N a segment joining two points y and z. Now set $w=(y+z)/2$. We have $\theta\in G(w,\ y)$, since $\|w\|=\|y\|=1$, while $\theta\notin LS'(w,\ y)$: in fact, $y-w=(y-z)/2$ and the function $\|w+t(y-w)\|-\|w\|$ has the value 0 for $|t|\leq 1$, so $-\tau(w,\ w-y)=\tau(w,\ y-w)=0$. This proves that (c) implies (a), thus concluding the proof of the theorem.

Proof of Theorem 1.2. If $LS(z,\ Tz)\cap X=\phi$, then clearly z is fixed for T (since otherwise $Tz\in LS(z,\ Tz)$).

Assume now $LS(z,\ Tz)\cap X$ nonempty and bounded. For every $\lambda\in(0,\ 1)$ define the contraction mapping $T_\lambda : X\to X$ by

$$T_\lambda x=(1-\lambda)z+\lambda Tx\,. \tag{4}$$

T_λ has a fixed point x_λ. Thus

$$z-x_\lambda=\lambda(z-Tx_\lambda) \tag{5}$$

$$T_\lambda z-Tz=(1-\lambda)(z-Tz)\,. \tag{6}$$

Also, we obtain

$$\begin{aligned}&\tau(z-x_\lambda,\ z-x_\lambda)+\tau(z-x_\lambda,\ Tz-z)\\ &=\tau(z-x_\lambda,\ Tz-x_\lambda)\leq\tau(z-x_\lambda,\ Tz-T_\lambda z)+\tau(z-x_\lambda,\ T_\lambda z-x_\lambda)\end{aligned}$$

$$\|x_\lambda - z\| = \tau(z - x_\lambda,\ z - x_\lambda) \leq \tau(z - x_\lambda,\ (1-\lambda)(Tz - z)) + \tau(z - x_\lambda,\ T_\lambda z - x_\lambda) - \tau(z - x_\lambda,\ Tz - z)$$
$$= (1-\lambda)\,\tau(z - x_\lambda,\ Tz - z) - \tau(z - x_\lambda,\ Tz - z) + \tau(z - x_\lambda,\ T_\lambda z - x_\lambda)\ .$$

But $\|T_\lambda z - x_\lambda\| = \|T_\lambda z - T_\lambda x_\lambda\| \leq \lambda \|z - x_\lambda\|$, thus

$$\|x_\lambda - z\| \leq -\lambda\tau(z - x_\lambda,\ Tz - z) + \lambda\|z - x_\lambda\|$$

and then

$$(1-\lambda)\|x_\lambda - z\| \leq -\lambda\tau(z - x_\lambda,\ Tz - z)\ . \tag{7}$$

Moreover, if there exists λ such that $z = x_\lambda$, we obtain by (5) that z is fixed for T and we are done. On the other hand, if $z \neq x_\lambda$ for all $\lambda \in (0,\ 1)$, then (7) implies $x_\lambda \in LS(z,\ Tz) \cap X$. Furthermore, $Tx_\lambda \in LS(z,\ Tz) \cap X$ too; indeed, we have from (5) $-\tau(z - Tx_\lambda,\ Tz - z) = -(1/\lambda)\,\tau(z - x_\lambda,\ Tz - z) > 0$. Hence, by setting $\delta = \sup\{\|z - y\|;\ y \in LS(z,\ Tz) \cap X\} < \infty$, we obtain

$$\|x_\lambda - Tx_\lambda\| = \|(1-\lambda)z + \lambda Tx_\lambda - Tx_\lambda\| = (1-\lambda)\|z - Tx_\lambda\| \leq (1-\lambda)\,\delta \xrightarrow[\lambda\to 1]{} 0\ .$$

Then there exists in X a bounded sequence $\{x_{\lambda_n}\}$ such that $\lim_{n\to\infty}(x_{\lambda_n} - Tx_{\lambda_n}) = 0$, which proves the Theorem.

Proof of Theorem 1.3. Assume $z \neq Tz$, otherwise we have noting to prove. We must show that a bounded, closed and convex set exists, which is invariant under T.

Take z as in the assumption and set $R/4 = \sup\{\|x - Tz\|,\ x \in LS(z,\ Tz) \cap X\}$; than set $K = \{x \in X;\ \|Tz - x\| \leq R\}$ ($K \neq \phi$ since $Tz \in K$). We want to prove that $T(K) \subset K$. We consider two cases:

(α) Assume $x \in K \cap LS(z,\ Tz)$. Since

$$-\tau(z - (z + Tz)/2,\ Tz - z) = 1/2\|z - Tz\| > 0\ ,$$

we have

$$(z + Tz)/2 \in LS(z,\ Tz) \cap X\ .$$

By definition of R, we obtain

$$\frac{R}{4} \geq \left\| \frac{z+Tz}{2} - Tz \right\| = \frac{\|z-Tz\|}{2}$$

hence
$$\|x-z\| \leq \|x-Tz\| + \|Tz-z\| \leq \frac{R}{4} + \frac{R}{2} < R \, .$$

Since T is nonexpansive, we have $\|Tx-Tz\| \leq R$, hence $Tx \in K$.

(β) Assume $x \in K$; $x \notin LS(z, \ Tz) \cap X$. Then $\|x-z\| < \|Tz-x\|$, (see proof. of Theorem 1.1, (a) $\Rightarrow$ (b)), hence

$$\|Tx-Tz\| \leq \|x-z\| < \|Tz-x\| < R$$

so, also in this case, $Tx \in K$.

Thus, T maps K into K, which completes the proof.

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Abstract

In this paper we generalize two fixed point theorems for nonexpansive mappings on convex, closed (not necessarily bounded) subset of a Banach space. Also, we characterize strict convexity of the space by using some geometric sets playing a role in these theorems.

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