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On two new characterizations of the Fourier transform for distributions (**)

1 - Introduction

Pandey and Pathak [6] introduced the distributional Fourier transform $F(n)$ of the generalized function $f \in F'(I)$, $I = (-\pi, \pi)$ as

(1.1) $$F(n) = \langle f(t),\ \psi_n(t) \rangle \qquad (n = 0,\ 1,\ 2,\ \dots)$$

$\psi_n(t)$ be the eigenfunctions defined by

(1.2) $$\dots\ \psi_n(t) = \begin{cases} 1/\sqrt{(2\pi)} & \text{for } n = 0 \\ \cos nt/\sqrt{\pi} & \text{for } n = 2k \\ \sin nt/\sqrt{\pi} & \text{for } n = 2k-1 \qquad (k = 1,\ 2,\ \dots) \end{cases}$$

and, of course, $\psi_n(t)$ be the eigen functions of the Sturm-Liouville problem

(1.3) $$\frac{d^2 y}{dt^2} + \lambda y = 0 \qquad y(-\pi) = y(\pi) \qquad y'(-\pi) = y'(\pi)$$

and eigenvalues be

(1.4) $$\dots\ \lambda_n = n^2 .$$

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$F'(I)$ is the dual of testing function space $F(I)$ defined in **2** of this paper.

In the present paper we give two new characterizations of the Fourier transform for distributions by the help of dilatations U_n and the exponential shifts T^{-p} introduced earlier by E. Gesztalyi [4]. It is interesting to note here that Gesztelyi considered two transformations viz., dilatations U_n and exponential shifts T^{-p} which are defined for ordinary functions f, complex number p and positive integer n by

$$U_n f(t) = nf(nt) \tag{1.5}$$

$$T^{-p} f(t) = e^{-pt} f(t) . \tag{1.6}$$

Gesztelyi shows that whenever the sequence $\{u_n f\}$ converges (in the sense of Mikusinski-convergence [5]) the limit is necessary a complex number. Also he proved that if f is a function which has Laplace transform at p, then the sequence of functions $\{U_n T^{-p} f(t)\}$ converges (in the Mikusinski sense) as $n \to \infty$ to the classical Laplace transform of f at p. He then defined the Laplace transform of a Mikusinski operator x as the limit (whenever it exist in the sense of Mikusinski convergence) of the sequence $\{U_n T^{-p} x\}$ and shows that his definition generalizes the previous formulation of the Laplace transform of Mikusinski operator of G. Doetsch [3] and V. A. Ditkin [1], [2].

In 1975, D.B. Price [7] working on the same line shows that whenever f is a distribution such that the sequence $\{U_n f\}$ converges as $n \to \infty$ to a distribution h, then h must be a linear combination of the delta distribution and the distribution p.v. $1/t$. Moreover, if $\{U_n T^{-p} f\}$ converges for two complex numbers with different real parts then the limit must be a constant multiple of delta distribution. Price [7] also defined the Laplace transform of a distribution f using sequence of the form $\{U_n T^{-p} f\}$ and showed that the new definition is equivalent to the Schwartz's extension of the transform to distributions. He also introduced spaces B and B_0 and their duals B' and B_0'. In fact, B_0 is a subspace of $B(R^n)$ (or, where R^n is understood, by B, the space of all complex valued functions of an n-dimensional real variable $t = (t_1, t_2, \ldots, t_n)$ which possesses continuous and bounded partial derivatives of all orders) consisting of those functions in B each of whose derivatives, approaches to zero as $|t| \to \infty$. Whereas B_0' (the dual of B_0) is a subspace of $\mathscr{D}'$ and a distribution f in B_0' is completely determined by its values on $\mathscr{D}$. Price [7] also showed that each distribution f on B_0' has a unique extension $\hat{f}$ in B' and that the sequence $\{U_j f\}$ converges to $\langle f, 1 \rangle \delta$ whenever f is in B_0'.

Recently working on the same lines the present author Sinha [8] has introduced new characterizations of the Mellin, Stieltjes, K, Hankel, Maijer-Laplace and finite Sturm-Liouville transform for distributions.

2 - The testing function space $F(I)$ and its dual $F'(I)$

The testing function space $F(I)$, $I=(-\pi,\ \pi)$ consists of complex valued C^∞ functions $\phi(t)$ defined over I satisfying the conditions

$$\gamma_k(\phi)=\sup_{-\pi<t<\pi}|D^{2k}\phi(t)|<\infty \qquad (k=0,\ 1,\ 2,\ \ldots)\ . \tag{2.1}$$

The topology over $F(I)$ is generated by the sequence of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ ([9], p. 8) and the concept of convergence and completeness over $F(I)$ is defined in the usual way. $F'(I)$ denotes the dual of $F(I)$.

3 - Two new characterizations of the Fourier transform

In this section we give two new characterizations of the Fourier transform for one-dimensional distributions.

We will say that *a distribution f is Fourier transformable* if there is an open interval $(\alpha,\ \beta)$ such that whenever $p=-\dfrac{1}{t}\log\psi_n(t)$ $(n=0,\ 1,\ 2,\ \ldots,\ t\neq 0)$, a complex number, $\operatorname{Re}p\in(\alpha,\ \beta)$, $T^{-p}f$ is a distribution in B_0' where B_0' is the dual of B_0, a subspace of $\mathscr{D}'$ as defined in [7] and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problem (1.3) with eigenvalue (1.4).

If $(\alpha,\ \beta)$ is the largest such open interval then the set $\Omega=\{p\colon \operatorname{Re}p\in(\alpha,\ \beta)\}$ is called *the domain of definition* of the Fourier transform for f.

If f is a Fourier transformable distribution where the transform has domain of definition Ω, then for $p\in\Omega$, we define the Fourier transform $\mathscr{F}[f](p)$ of f at p by

$$\mathscr{F}[f](p)=\frac{1}{\phi(0)}\lim_{j\to\infty}\langle U_jT^{-p}f,\ \phi\rangle \tag{3.1}$$

where ϕ is a test function in $\mathscr{D}$ with $\phi(0)\neq 0$.

We have another characterization also as

$$(3.2) \qquad \mathscr{F}[f](p) = \langle T^{-p} f,\ 1\rangle$$

where $p = -\frac{1}{t}\log\psi_n(t)$, $t \neq 0$, $n = 0, 1, 2, \ldots$, $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problem (1.3) with eigenvalue (1.4). From (3.2) we see that $\mathscr{F}[f](p)$ is a complex valued function of complex variable p with domain Ω.

4 - Linearity of $\mathscr{F}$

The mapping $\mathscr{F}$ is linear. For, if f and g are distributions that are Fourier transformable at p and a and b are complex numbers then $(af + bg)$ is Fourier transformable at p and

$$(4.1) \qquad \mathscr{F}[af + bg](p) = \langle T^{-p}(af + bg),\ 1\rangle$$

$$= \{a\langle T^{-p} f,\ 1\rangle + b\langle T^{-p} g,\ 1\rangle\} = a\,\mathscr{F}[f](p) + b\,\mathscr{F}[g](p)\,.$$

5 - Analicity of $\mathscr{F}$

Theorem 5.1. *If f is a distribution that is Fourier transformable in Ω then $\mathscr{F}[f](p)$ is analytic function of p in Ω and*

$$(5.1) \qquad \frac{\mathrm{d}}{\mathrm{d}p}\,\mathscr{F}[f](p) = \mathscr{F}[-tf(t)](p)\,.$$

Proof. Here $\Omega = \{p\colon \operatorname{Re} p \in (\alpha, \beta)\}$ and also $p = -\frac{1}{t}\log\psi_n(t)$ $(t \neq 0,\ n = 0, 1, 2, \ldots)$ and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville Problem (1.3) with eigenvalue (1.4).

We choose $p_0 \in \Omega$ and $\varepsilon \in (0, 1)$ such that $\varepsilon < \min\{\operatorname{Re} p_0 - \alpha,\ \beta - \operatorname{Re} p_0\}$. If $\lambda(t) = \mathrm{e}^{\varepsilon t} + \mathrm{e}^{-\varepsilon t}$ then $\frac{1}{\lambda}\varepsilon S \subset B_0$ and $(\lambda T^{-p_0} f) \in B_0'$. Also whenever $|p - p_0| < \varepsilon$ we have

$$\frac{\mathscr{F}[f](p)-\mathscr{F}[f](p_0)}{p-p_0}=\langle\frac{e^{-pt}-e^{-p_0t}}{p-p_0}f(t),\ 1(t)\rangle=\langle\lambda(t)\,e^{-p_0t}f(t),\ \frac{1}{\lambda(t)}[\frac{e^{-(p-p_0)t}-1}{p-p_0}]\rangle$$

$$=\langle\lambda(t)\,e^{-p_0t}f(t),\ \frac{-t}{\lambda(t)}+\frac{(p-p_0)\,t^2}{\lambda(t)}\sum_{j=2}^{\infty}\frac{[-(p-p_0)\,t]^{j-2}}{j!}\rangle\,.$$

Here each derivative is

$$\frac{t^2}{\lambda(t)}\sum_{j=2}^{\infty}\frac{[-(p-p_0)\,t]^{j-2}}{j!}$$

is bounded in absolute value away from zero by the corresponding derivative of $\frac{t^2}{\lambda(t)}\exp|(p-p_0)\,t|$ and is therefore in S (defined in [7]). Thus as $p\to p_0$

$$\frac{1}{\lambda(t)}\,|\frac{e^{-(p-p_0)t}-1}{p-p_0}|\quad\text{converges in } B_0 \text{ to}\quad\frac{-t}{\lambda(t)}\quad\text{and we have}$$

$$\frac{d}{dp_0}\,\mathscr{F}[f](p_0)=\lim_{p\to p_0}\frac{\mathscr{F}[f](p)-\mathscr{F}[f](p_0)}{p-p_0}=\langle\lambda(t)\,T^{-p_0}f(t),\ \frac{-t}{\lambda(t)}\rangle$$

$$=\langle T^{-p_0}[-tf(t)],\ 1(t)\rangle=\mathscr{F}[-tf(t)](p)\,.$$

6 - Treatment of the convolution of two distributions

Statement. *If f and g are Fourier transformable distributions such that the domain of their respective transform have intersection Ω, then $(f*g)$ is Fourier transformable in Ω and for every $p\in\Omega$ we have*

$$\mathscr{F}[f*g](p)=\mathscr{F}[f](p)*\mathscr{F}[g](p)\,.$$

Proof. Here $\Omega=\{p\colon \alpha<\operatorname{Re}p<\beta\}$. If $p=-\frac{1}{t}\log\psi_n(t)\in\Omega$ is a complex number ($t\neq 0$, $n=0, 1, 2, \ldots$) and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problems (1.3) with eigenvalue (1.4), then $T^{-p}f$ and $T^{-p}g$ are both in B_0'. So by the Lemma 2.3 (p. 20) of [7] we have $T^{-p}f*T^{-p}g=T^{-p}(f*g)$ is in B_0'. Thus $(f*g)$ is Fourier transformable at p, from (3.2) and the definition of the convolution we get

$$\mathscr{F}[f*g](p) = \langle T^{-p}(f*g),\ 1\rangle = \langle T^{-p}f * T^{-p}g,\ 1\rangle = \langle T^{-p}f(t) \otimes T^{-p}g(\tau),\ 1(t+\tau)\rangle$$
$$= \langle T^{-p}f(t) \otimes T^{-p}g(\tau),\ 1(t)\,1(\tau)\rangle = \langle T^{-p}f,\ 1\rangle\,\langle T^{-p}g,\ 1\rangle = \mathscr{F}[f](p)\ \mathscr{F}[g](p)\ .$$

7 - Inversion and uniqueness theorms for Fourier transform

In the present section we will state a Theorem 7.1 which includes both inversion and uniqueness theorems as its corollary.

In what follows we will have as an independent variable at several times and the real variable t and the real and immaginary parts of the complex variable $p = -\dfrac{1}{t}\log\psi_n(t)$ $(t \neq 0,\ n = 0, 1, 2, \ldots)$ and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problem (1.3) with eigenvalue (1.4). For this reason we will sometime indicate the particular independent variable η for a space or an operator by a subscript e.g., $\langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta$ where $f(\eta) \in B'_{0_\eta}$ and ω is a parameter.

Theorem 7.1. *If f is a distribution in B'_{0_t}, then*

$$f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t}\langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta\, d\omega \tag{7.1.1}$$

where the limit is taken in $\mathscr{D}'_t$.

Proof. The integral in (7.1-1) is well defined because $\langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta$ is a continuous function of ω. Let $\phi \in \mathscr{D}_t$ and r be a positive real number. Then from the standard theorems on integration of distributions and test functions with respect to parameters, we have

$$\langle \int_{-r}^{r} e^{i\omega t}\langle f(\tau),\ e^{-i\omega\eta}\rangle_\eta\, d\omega,\ \phi(t)\rangle_t = \int_{-r}^{r}\langle e^{-i\omega t}\langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta,\ \phi(t)\rangle_t\, d\omega$$
$$= \int_{-r}^{r}\langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta\,\langle e^{i\omega t},\ \phi(t)\rangle_t\, d\omega = \int_{-r}^{r}\langle f(\eta),\ \langle e^{i\omega(t-\eta)},\ \phi(t)\rangle_t\rangle_\eta\, d\omega$$
$$= \langle f(\eta),\ \int_{-r}^{r}\langle e^{i\omega(t-\eta)},\ \phi(t)\rangle_t\, d\omega\rangle_\eta = \langle f(\eta),\ \int_{-r}^{r} e^{-i\omega\eta}\int_{-\infty}^{\infty} e^{i\omega t}\phi(t)\, dt\, d\omega\rangle_\eta$$
$$= \langle f(\eta),\ \int_{-r}^{r} e^{i\xi\eta}\tilde{\phi}(\xi)\, d\xi\rangle_\eta$$

where $\xi = -\omega$ and $\bar{\phi}(\xi)$ is the Fourier transform of $\phi(t)$. Obviously as $r \to \infty$ $\int_{-r}^{r} e^{i\omega\eta}\bar{\phi}(\xi)\, d\xi \to 2\pi\phi(\eta)$ uniformly with respect to η and similarly

$$\frac{d^k}{d\eta^k}\Big[\int_{-r}^{r} e^{i\xi\eta}\bar{\phi}(\xi)\, d\xi\Big] = \int_{-r}^{r} (i\xi)^k e^{i\xi\eta}\bar{\phi}(\xi)\, d\xi \to 2\pi\phi^{(k)}(\eta) \qquad \text{uniformly}$$

so the limit in B_η of $\int_{-r}^{r} e^{i\xi\eta}\bar{\phi}(\xi)\, d\xi$ as $r \to \infty$ is $2\pi\phi(\eta)$ and we have

$$\Big\langle \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta\, d\omega,\ \phi(t)\Big\rangle$$

$$= \frac{1}{2\pi}\Big\langle f(\eta),\ \lim_{t\to\infty} \int_{-r}^{r} \langle e^{i\omega(t-\eta)},\ \phi(t)\rangle_t\, d\omega\Big\rangle_\eta = \frac{1}{2\pi}\langle f(\eta),\ 2\pi\phi(\eta)\rangle_\eta = \langle f(t),\ \phi(t)\rangle_t\,.$$

Thus as distributions

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t} \langle f(\eta),\ e^{-i\omega\eta}\rangle_\eta\, d\omega\,.$$

There are three corollaries of the above theorem.

Corollary 1. *If σ is a real number such that $e^{-\sigma t}f(t)$ is in B'_{0_t}, then as distributions*

(7.1.2) $$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} e^{pt} \langle e^{-p\eta} f(\eta),\ 1(\eta)\rangle_\eta\, dp\,.$$

Proof. If $e^{-\sigma t}f(t) \in B'_{0_t}$ then as long as

$$\mathrm{Re}\, p = \sigma,\ e^{-pt}f(t) \in B'_{0_t},\ e^{-\sigma t}f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t}\langle e^{-\sigma\eta} f(\eta),\ e^{-i\omega\eta}\rangle_\eta\, d\omega\,.$$

Thus we have

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{\sigma t} e^{i\omega t}\langle e^{-\sigma\eta} f(\eta),\ e^{-i\omega\eta}\rangle_\eta\, d\omega = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{\sigma-ir}^{\sigma+ir} e^{pt}\langle e^{-p\eta} f(\eta),\ 1(\eta)\rangle_\eta\, dp\,.$$

Corollary 2. (Inversion theorem). *If f is Fourier transformable in* $\Omega = \{p\colon \alpha < \operatorname{Re} p < \beta\}$, $p = -\frac{1}{t} \log \psi_n(t)$ $(t \neq 0,\ n = 0,\ 1,\ 2,\ \ldots)$ *and* $\psi_n(t)$ *is the eigenfunctions* (1.2) *of the Sturm-Liouville problem* (1.3) *with eigenvalue* (1.4), *then as long as* $\alpha < \sigma < \beta$

$$f(t) \lim_{r \to \infty} \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} e^{pt}\, \mathscr{F}[f](p)\, dp \tag{7.1.3}$$

where the limit is taken in $\mathscr{D}'_t$.

Corollary 3. (Uniqueness theorem). *If f and g are Fourier transformable distributions such that*

$$\mathscr{F}[f](p) = \mathscr{F}[g](p) \tag{7.1.4}$$

on some vertical line in the common domain of the transform of f and g, then $f = g$ *as distributions.*

8 - A sufficient condition that an analytic function $F(p)$ be the Fourier transform of a distribution $f(t)$ and characterization of the distribution

Statement. *If* $F(p)$ *is an analytic function for* $p = -\frac{1}{t} \log \psi_n(t)$ $(n = 0,\ 1,\ 2,\ \ldots)$ *in* $\Omega = \{\sigma + i\omega\colon \alpha < \sigma < \beta\}$ *and is bounded in* Ω *by a polynomial in* ω *(or in* $|p|$*) then*

$$F(p) = \mathscr{F}[f](p) \tag{8.1}$$

where the distribution $f(t)$ *is defined by*

$$f(t) = \frac{1}{2\pi i} \lim_{r \to \infty} \int_{\sigma - ir}^{\sigma + ir} e^{pt} F(p)\, dp \tag{8.2}$$

for any fixed value of σ *such that* $\sigma \in (\alpha,\ \beta)$, $t \neq 0$ *real number and* $\psi_n(t)$ *be the eigenfunction* (1.2) *of the Sturm-Liouville problem* (1.3) *with eigenvalue* (1.4).

Proof. We shall prove the above statement in the following four steps: Step 1: f is a distribution. Step 2: f is independent of the value of σ. Step 3: $e^{-\sigma t}f(t) \in B'_{0_t}$ as long as $\alpha < \sigma < \beta$. Step 4: $F(p) = \mathscr{F}[f](p) = \langle T^{-p}f,\ 1\rangle$ for every p in Ω.

Proofs of Steps 1, 2 and 3 are the same as given in [7], the only difference here is that we take $p = -\frac{1}{t}\log\psi_n(t)$.

We now proceed to give an outline of the proof of the last Step 4 as follows: in order to prove $F(p) = \mathscr{F}[f](p) = \langle T^{-p}f,\ 1\rangle$ for every $p \in \Omega$ here $p = -\frac{1}{t}\log\psi_n(t)$ $(t \neq 0,\ n = 0, 1, 2, \dots)$ and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problems (1.3) with the eigenvalue (1.4). As p is a complex number so we take $p = \sigma + i\eta$ where $\alpha < \sigma < \beta$ and suppose that ϕ is a function in $\mathscr{D}_t$ with $\phi(0) = 1$ and such that the support of ϕ is contained in $(-1, 1)$. Then from [7] result if f is in B'_0, then $\lim_{j\to\infty} U_j f = \langle f,\ 1\rangle$, where the limit is taken in $\mathscr{D}'$, we see that

$$\langle e^{-pt}f(t),\ 1(t)\rangle = \lim_{j\to\infty} \langle U_j e^{-pt}f(t),\ \phi(t)\rangle \tag{8.3}$$

and starting from here we get the following results

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\sigma + i\omega)\,\langle e^{i(\omega\eta)},\ \phi(\tfrac{t}{j})\rangle\, d\omega \tag{8.4}$$

$$= \lim_{j\to\infty} \frac{(-1)^k}{j^{k-1}} \int_{-j}^{j} e^{-i\eta t}\, \frac{\phi^{(k)}(\frac{t}{j})}{j}\, F^{-1}[G(\omega)]\,(t)\, dt\,.$$

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F_2(\sigma + i\omega)\,\langle e^{i(\omega-\eta)t},\ \phi(\tfrac{t}{j})\rangle_t\, d\omega \tag{8.5}$$

$$= \lim_{j\to\infty} \int_{-\infty}^{\infty} e^{-i\eta t}\phi(\tfrac{t}{j})\, F^{-1}[F_2(\sigma + i\omega)]\,(t)\, dt$$

and hence we have

$$\langle T^{-p}f,\ 1\rangle = F(p) = \mathscr{F}[f](p) \qquad \alpha < \operatorname{Re} p < \beta\,. \tag{8.6}$$

9 - Some standard operation-transform formulae for the distributional Fourier transform

In this section we will introduce some standard transform formulae using the characterizations of the transform given in (3.1).

Let f be a Fourier transformable distribution whose transform has domain of definition $\Omega = \{p\colon \alpha < \mathrm{Re}\, p < \beta\}$, here $p = -\frac{1}{t} \log \psi_n(t)$ $(n = 0, 1, 2, \ldots, t \neq 0)$ and $\psi_n(t)$ be the eigenfunctions (1.2) of the Sturm-Liouville problem (1.3) with eigenvalue (1.4). Then $f^{(1)}$ is also Fourier transformable in Ω. To compute the transform of $f^{(1)}$, let ϕ be a function in $\mathscr{D}$ such that $\phi(0) = 1$, $\phi'(0) \neq 0$ and let $j > 0$ is an integer. Then if $p \in \Omega$ we have

$$\langle U_j T^{-p} f^{(1)}, \ \phi \rangle = \langle f^{(1)}(t), \ \mathrm{e}^{-pt} \phi(\frac{t}{j}) \rangle = \langle f(t), \ p\mathrm{e}^{-pt} \phi(\frac{t}{j}) - \frac{1}{j} \mathrm{e}^{-pt} \phi^{(1)}(\frac{t}{j}) \rangle$$

$$= p \langle U_j T^{-p} f, \ \phi \rangle - \frac{1}{j} \langle U_j T^{-p} f, \ \phi^{(1)} \rangle \ .$$

As $j \to \infty$, the term $\frac{1}{j} \langle U_j T^{-p} f, \ \phi^{(1)} \rangle$ converges to zero and so from (3.1) we have

$$\mathscr{F}[f^{(1)}](p) = \lim_{j \to \infty} p \langle U_j T^{-p}, \ \phi \rangle = p\, \mathscr{F}[f](p) \ . \tag{9.1}$$

By induction we can prove that for every positive integer k

$$\mathscr{F}[f^{(k)}](p) = p\, \mathscr{F}[f^{(k-1)}](p) = p^k \mathscr{F}[f](p) \ . \tag{9.2}$$

Another operation transform formula can be obtained from (5.1) viz.,

$$\mathscr{F}[-tf(t)](p) = \frac{\mathrm{d}}{\mathrm{d}p} \mathscr{F}[f](p) \ .$$

This formula can be extended by the method of induction, for every positive integer k to get

$$\mathscr{F}[t^k f(t)](p) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}p^k} \mathscr{F}[f](p) \ . \tag{9.3}$$

If f is Fourier transformable in Ω, then $f(t-\eta)$ is Fourier transformable in Ω for every real number η and we have

$$\langle U_j T^{-p} f(t-\eta),\ \phi(t)\rangle = \langle f(t-\eta),\ e^{-pt}\phi(\frac{t}{j})\rangle = \langle f(t),\ e^{-p(t-\eta)}/(\frac{t+\eta}{j})\rangle$$

$$= e^{-p\eta}\langle U_j T^{-p} f(t),\ \phi(t+\eta)\rangle\ .$$

Now, $\phi(t+\eta) \in \mathscr{D}$ and as long as $\phi(\eta) \neq 0$

$$\lim_{j\to\infty} e^{-p\eta}\langle U_j T^{-p} f(t),\ \phi(t+\eta)\rangle = \frac{1}{\phi(\eta)}\ e^{-p\eta}\langle T^{-p} f,\ 1\rangle\ \langle \delta(t),\ \phi(t+\eta)\rangle\ .$$

So we have

$$\mathscr{F}[f(t-\eta)](p) = e^{-p\eta}\ \mathscr{F}[f](p)\ . \tag{9.4}$$

If q is a fixed complex number and if f is Fourier transformable in Ω then $e^{-qt}f(t)$ is Fourier transformable in $\Omega' = \{p\colon \alpha - \operatorname{Re} q < \operatorname{Re} p < \beta - \operatorname{Re} q\}$ and we have

$$\langle U_j T^{-p}[e^{-qt} f(t)],\ \phi(t)\rangle = \langle U_j T^{-(p+q)} f,\ \phi\rangle\ .$$

Thus, whenever $p \in \Omega$

$$\mathscr{F}[e^{-qt} f(t)](p) = \mathscr{F}[f](p+q)\ . \tag{9.5}$$

If k is a fixed positive integer and f is Fourier transformable in Ω, then $(U_k f)$ is Fourier transformable in $\Omega'' = \{p\colon k\alpha < \operatorname{Re} p < \beta k\}$. For $p \in \Omega''$ we have

$$\langle U_j T^{-p}\{U_k f\},\ \phi\rangle = \langle U_k f,\ e^{-pt}\phi(\frac{t}{j})\rangle = \langle f(t),\ e^{(-p/k)t}\phi(\frac{t}{jk})\rangle = \langle U_j T^{-p/k} f,\ \phi(\frac{t}{k})\rangle\ .$$

As $j \to \infty$, this converges to $\langle T^{-p/k} f,\ 1\rangle\ \langle \delta,\ \phi\rangle$. So we get the formula

$$\mathscr{F}[U_k f](p) = \mathscr{F}[f](\frac{p}{k})\ . \tag{9.6}$$

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Abstract

Two new characterizations of the Fourier transform for distributions have been developed using dilatations U_n and the exponential shifts T^p. The standard theorems on analicity, uniqueness, invertibility and some standard operation transform formulas for the distributional Fourier transform are proved.

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