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On certain series which generalise the Neumann and Kapteyn series (**)

Introduction

This note deals with the proof of certain series of a purely *formal* nature which generalise the Neumann and Kapteyn series in the theory of Bessel functions. The methods employed include the elementary manipulation of series and the summation in closed form of some of the simpler hypergeometric series of unit argument.

Lemma 1. *If* $C(\mu)$ *is an arbitrary function of* μ *and if*

$$X_\nu = \sum_{r=0}^{\infty} \frac{(-1)^r C(\nu/2+r)}{r!\ \Gamma(\nu+r+1)} \tag{1.1}$$

then we have the formal result

$$C(\nu/2) = \sum_{k=0}^{\infty} \frac{(\nu+2k)\Gamma(\nu+k)}{k!} X_{\nu+2k} \,. \tag{1.2}$$

Proof. Re-arrange the series

$$S = \sum_{k=0}^{\infty} \frac{(\nu+2k)\Gamma(\nu+k)}{k!} X_{\nu+2k} \tag{1.3}$$ as $$S = \sum_{m=0}^{\infty} A_m C(\nu/2+m) \,. \tag{1.4}$$

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For $m = 0, 1, 2, \ldots$, it is found after some manipulation that the coefficient A_m takes the form

$$(1.5) \qquad -\frac{\nu\Gamma(\nu)}{\Gamma(\nu+m+1)\,m!}\,{}_3F_2(\nu,\ \nu/2+1,\ -m;\ \nu/2,\ \nu+m+1;\ 1)\,.$$

This terminating Clausen function is well-poised and an application of Dixon's Theorem ([1], p. 21) shows that A_m vanishes for all values of the index of summation m except $m = 0$ and that $A_0 = 1$. The lemma is thus established.

Lemma 2. *If $C(\mu)$ is an arbitrary function of μ and if*

$$(2.1) \qquad Y_\nu = \sum_{r=0}^{\infty} \frac{(-1)^r C(\nu/2+r)\,\nu^{\nu+2r}}{r!\ \Gamma(\nu+r+1)}$$

then we have the formal result

$$(2.2) \qquad C(\nu/2) = \nu^2 \sum_{k=0}^{\infty} \frac{\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}\,k!} Y_{\nu+2k}\,.$$

Proof. This Lemma is established in a rather similar fashion to Lemma 1. Re-arrange the series

$$(2.3) \qquad K = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}\,k!} Y_{\nu+2k} \qquad \text{as} \qquad (2.4) \qquad K = \sum_{m=0}^{\infty} B_m C(\nu/2+m)\,.$$

After some manipulation, it will be seen that $B_0 = \nu^{-2}$ and, for $m = 1, 2, 3, \ldots$,

$$(2.5) \qquad B_m = \frac{(-1)^m \Gamma(\nu)\,\nu^{2m-1}}{m!\ \Gamma(\nu+1+m)}\,{}_{2m+1}F_{2m}\left(\begin{matrix} -m,\ \nu\ , & \nu/2+1,\ \ldots,\ \nu/2+1; \\ \nu+m+1, & \nu/2,\ \ldots,\ \nu/2\quad ; \end{matrix}\ 1\right).$$

Repeated trials with progressively increasing values of $m = 1, 2, 3, \ldots$, indicate that the hypergeometric polynomial on the right of (2.5) vanishes for all integer values of $m > 0$. We now observe that the expression for B_m is independent of the form of $C(\mu)$. Put $C(\mu) = (z^2/4)^\mu$ and note that, with this value of $C(\mu)$,

$$(2.6) \qquad Y_\nu = J(\nu z)\,.$$

It is well-known from the theory of Kapteyn series that, formally at least,

$$(z/2)^{\nu} = \nu^2 \sum_{k=0}^{\infty} \frac{\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}\, k!} J_{\nu+2k}[(\nu+2k)\, z] \tag{2.7}$$

([2], p. 571) so that B_m vanishes for all values of $m = 1, 2, 3, \ldots$, and the lemma is proved.

Lemma 3. (This is a generalisation of Lemma 1.) *If* $C(\mu)$ *is an arbitrary function of* μ *and if*

$$X_{\nu}^{h} = \sum_{r=0}^{\infty} \frac{(-1)^r\, C(\frac{\nu}{h+1}+r)}{r!\ \Gamma(\nu+1+hr)} \tag{3.1}$$

then we have the formal result

$$C(\nu/(h+1)) = \sum_{k=0}^{\infty} \frac{(\nu+k)\,\Gamma(\nu+[h+1]\,k)}{k!} X_{\nu+(h+1)k}^{h}\,. \tag{3.2}$$

Proof. Re-arrange the series

$$S^* = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+k)\,(\nu+[h+1]\,k)}{k!} X_{\nu+(h+1)k}^{h} \tag{3.3}$$

as

$$S^* = \sum_{m=0}^{\infty} A_m^*\, C\left(\frac{\nu}{h+1}+m\right). \tag{3.4}$$

For $m = 1, 2, 3, \ldots$, the coefficient A_m^* may be written in the form

$$A_m^* = -\sum_{r=0}^{\infty} \frac{(-1)^r\,(\nu+[h+1]\,r)\,\Gamma(\nu+r)}{\Gamma(\nu+1+hm+r)\ r!\ (m-r)!}\,. \tag{3.5}$$

This series may be expressed as a Clausen function. However, no known hypergeometric summation theorem can be used to tackle this function since it is well-poised only when $h = 1$ and this case has already been considered in the proof of Lemma 1. We now show by other means that the expression (3.5) vanishes unless $m=0$

(3.6)
$$A_m^* = -\sum_{r=0}^{\infty} \frac{(-1)^r(\nu-[h+1]\nu+[h+1]\nu+[h+1]r)\Gamma(\nu+r)}{\Gamma(\nu+1+hm+r)\ r!\ (m-r)!}$$

$$= h\nu \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(\nu+r)}{\Gamma(\nu+1+hm+r)\ r!\ (m-r)!}$$

$$-(h+1)\sum_{r=0}^{\infty} \frac{(-1)^r(\nu+r)\Gamma(\nu+r)}{\Gamma(\nu+1+hm+r)\ r!\ (m-r)!}.$$

This result may be re-written in the form

(3.7)
$$\frac{h\nu\Gamma(\nu)}{\Gamma(\nu+1+hm)\,m!}\ {}_2F_1\left(\begin{matrix}\nu,\ -m\ ;\\ \nu+1+hm;\end{matrix}\ 1\right)$$

$$-(h+1)\frac{\Gamma(\nu+1)}{\Gamma(\nu+1+hm)\,m!}\ {}_2F_1\left(\begin{matrix}\nu+1,\ -m;\\ \nu+1+hm;\end{matrix}\ 1\right)$$

and Vandermonde's Theorem ([2], 243) then gives

(3.8)
$$A_m^* = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+hm)\,m!}[h(hm+1)_m/(\nu+1+hm)_m$$

$$-(h+1)(hm)_m/(\nu+1+hm)_m]$$

$$= \frac{\Gamma(\nu+1)(hm+1)_{m-1}}{\Gamma(\nu+1+hm+m)\,m!}[h(hm+m)-(h+1)\,hm]=0\ .$$

As usual, the Pochhammer symbol $(a)_m$ is given by

(3.9)
$$(a)_m = a(a+1)(a+2)\dots(a+m-1) = \Gamma(a+m) = \Gamma(a+m)/\Gamma(a) \qquad (a)_0 = 1\ .$$

It is clear that the coefficient A_0^* is equal to unity, so that the lemma is proved.

Conclusion. Lemma 1 may be obtained by putting $h=1$ in Lemma 3, and if we put $C(\mu)=(z^2/4)^\mu$ in Lemma 1 and Lemma 2 respectively, series expansions fundamental to the theory of Neumann and Kapteyn series are recovered. It must be stressed that the results given in this note are purely formal as they stand, and the convergence of any series arising from specific forms of the coefficient $C(\mu)$ must be investigated separately in each case. Such special cases will be the subject of detailed study in future papers.

References

[1] H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd., Chichester, U.K., 1976.

[2] L. J. Slater, *Generalised hypergeometric functions*, Cambridge University Press, 1966.

[3] G. N. Watson, *Theory of Bessel functions*, Second edition, Cambridge University Press, 1948.

Summary

Formal series of a general character are introduced which extend the Neumann and Kapteyn series which occur in the theory of Bessel functions.

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