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On a hyperbolic model in magnetofluid dynamics with heat conduction (**)

1 - Introduction

As well known the classical constitutive theory of Fourier and Navier-Stokes gives rise to a parabolic model for magnetofluid dynamics [7], so that no finite wave speeds are permitted. It is remarkable to point out that even in the case of an adiabatic inviscid motion the magnetofluid dynamics governing system is hyperbolic only if the electrical conductivity is infinite.

The aim of this paper is to propose a hyperbolic model taking into account the main dissipative effects arising during the motion of a fluid through an external magnetic field. The paper is organized as follows.

In 2 we recall the classical model of magnetofluid dynamics [7]. In 3 we stress out that the entropy equation can be viewed like a supplementary law for the F.N.S. governing system. Then, within the context of irreversible thermodynamics, by means of the procedure given in [10] we are able to obtain a first order hyperbolic system of governing equations which can be reduced also to a symmetric form.

In 4 we characterize the main features of wave propagation compatible with the proposed model. Among others we recover as particular cases some results already obtained within the classical theory (e.g. for an inviscid rigid conductor and an adiabatic motion of an inviscid fluid).

Moreover we are able to point out the influence on wave propagation of the different dissipative effects present in the model.

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2 - Classical Fourier and Navier-Stokes constitutive theory

Let us consider the motion of a viscous fluid through an external magnetic field described by the following system of equations [7]

$$\partial_t \rho + \partial_k(\rho u_k) = 0 \qquad \partial_t(\rho u_i) + \partial_k[\rho u_i u_k + \mathrm{t}_{ik} + (H^2/8\pi)\,\delta_{ik} - (1/4\pi)\,H_i H_k] = 0$$

$$\partial_t \varepsilon + \partial_k\{\varepsilon u_k + [\mathrm{t}_{kj} + (H^2/8\pi)\,\delta_{kj} - (1/4\pi)\,H_k H_j]\,u_j + q_k + (c/4\pi\sigma_0)\,\varepsilon_{kij} J_i H_j\} = 0 \tag{2.1}$$

$$\partial_t H_i + \partial_k[u_k H_i - H_k u_i + (c/\sigma_0)\,\varepsilon_{ikm} J_m] = 0 \qquad \partial_k(\varepsilon_{ikj} H_j) = (4\pi/c)\,J_i$$

where the conditions

$$\operatorname{div} H = 0 \qquad \operatorname{div} J = 0$$

and the relation

$$J = \sigma_0[E + (1/c)(u \wedge H)] = (c/4\pi)\ \mathrm{rot}\, H$$

hold.

In (2.1) ρ is the mass density, u the fluid velocity, $\mathrm{t} = pI - \sigma$ the stress tensor, σ the viscous tensor, p the fluid pressure, $\varepsilon = \rho\mathrm{e} + \rho(u^2/2) + (H^2/8\pi)$ represents the total energy where e is the internal energy, q the heat flux, H the magnetic field, J the conduction current, E the electric field, c the light velocity in the vacuum, $\partial_t = (\partial/\partial t)$, $\partial_k = (\partial/\partial x_k)$ where t and x_k denote respectively time coordinate and space coordinates. Moreover $\wedge$ denotes the exterior product, ε_{ijk} the Levi-Civita symbol, I the unit matrix.

The system of equations (2.1) must be supplemented by constitutive laws. In particular on the usual approach (Fourier and Navier-Stokes equations) the heat flux q and the stress-viscous tensor σ are related linearly to the temperature gradient and to the velocity gradient respectively as follows

$$\nabla\theta = -\varkappa q$$

$$(1/2)\,[\nabla \otimes u + u \otimes \nabla] - (1/3)\,I\ \operatorname{div} u = \alpha\sigma^{\mathrm{D}} \qquad \operatorname{div} u = \beta\ \mathrm{tr}\,\sigma \tag{2.2}$$

when $(1/\varkappa) = \chi$ is the thermal conductivity, $(1/2\alpha) = \eta$ and $(1/3\beta) = \zeta$ are the viscosity coefficients, θ is the absolute temperature, $\nabla_i = \partial_i$, $v \otimes w = v_i w_k$, $\sigma^{\mathrm{D}} = \sigma - (1/3) I \,\mathrm{tr}\, \sigma$ is the deviator stress of σ. Since the following entropy inequality

$$(2.3) \qquad \partial_t(\rho S) + \partial_k[\rho u_k S + (q_k/\theta)]$$
$$= -(1/\theta^2)\, q_k\, \partial_k \theta + (1/\theta)\, \sigma^{\mathrm{D}}_{ik}\, \partial_i u_k + (1/3\theta)\, \mathrm{tr}\, \sigma\, \partial_k u_k + (1/\theta)\,(J^2/\sigma_0) \geqslant 0$$

must be satisfied for any thermodynamics process, the II principle of thermodynamics requires that $\varkappa$, α, β are positive (in (2.3) S is the entropy density).

It is simple matter to see that, because of the constitutive equations assumed for q, σ^{D}, $\mathrm{tr}\, \sigma$, J the system of equations (2.1) and (2.2) is not hyperbolic. Consequently the model given by (2.1) and (2.2) does not allow for finite wave speeds. To avoid this paradox we make use of an approach proposed by T. Ruggeri in [**10**].

3 - The hyperbolic model via «generators»

The system of governing equations (2.1) and (2.2) can be written under the conservative form

$$(3.1) \qquad \partial\alpha F^{\alpha}(U) = f(U) \qquad \text{where:}$$

$\alpha = 0, 1, 2, 3 \qquad x^0 = t \qquad F^0 = [\rho,\ \rho u_i,\ \varepsilon,\ H_i,\ 0_3,\ 0_5,\ 0_1,\ 0_3]^T$ (1)

$F^k = [\rho u_k,\ \rho u_i u_k + \mathrm{t}_{ik} + (H^2/8\pi)\, \delta_{ik} - (1/4\pi)\, H_i H_k$

$\varepsilon u_k + (\mathrm{t}_{kj} + (H^2/8\pi)\, \delta_{kj} - (1/4\pi)\, H_k H_j)\, u_j + q_k + (c/4\pi\sigma_0)\, \varepsilon_{kij} J_i H_j$

$u_k H_i - H_k u_i + (c/\sigma_0)\, \varepsilon_{ikm} J_m,\ \theta\delta_{ik} \qquad (1/2)(u_j \delta_{ik} + u_i \delta_{jk}) - (1/3)\, \delta_{ij} u_k,\ u_k,\ \varepsilon_{ikj} H_j]^T$

$f = [0_1,\ 0_3,\ 0_1,\ 0_3,\ -\varkappa q_i,\ \alpha\sigma^{\mathrm{D}}_{ij},\ \beta\, \mathrm{tr}\, \sigma,\ (4\pi/c)\, J_i]^T$

$U = [\rho,\ u,\ e,\ H,\ q,\ \sigma^{\mathrm{D}},\ \mathrm{tr}\, \sigma,\ J]^T$

(1) The superscript T means for transportation.

In (3.1) the field U must obey to the constraints

$$\partial_k B^k(U) = 0 \qquad \text{where} \qquad B^k = [H_k,\ J_k]^T . \tag{3.2}$$

The entropy equation (2.3) may be considered like a supplementary conservation law for the system (3.1), endowed by (3.2), of the following form

$$\partial_\alpha h^\alpha(U) = g(U) \leqslant 0 \tag{3.3}$$

where $\quad h^0 = -\rho S,\ h^k = -[\rho u_k S + (q_k/\theta)]$

$$g = -(\varkappa/\theta^2)\, q^2 - (\alpha/\theta)\, \sigma^{\mathrm{D}} : \sigma^{\mathrm{D}} - (\beta/3\theta)\,(\mathrm{tr}\,\sigma)^2 - (1/\theta)\,(J^2/\sigma_0) \leqslant 0$$ (2).

According to the theoretical framework concerning with first order quasilinear conservative systems compatible with a supplementary conservation law developed in [**10**], [**11**] (3) the consistency of (3.1), (3.2) and (3.3) leads to the existence of a field U' (main field), a vector K and four scalars h'^α such that the following relations hold

$$U' \cdot \mathrm{d}F^0 = \mathrm{d}h^0 \qquad U' \cdot \mathrm{d}F^k = \mathrm{d}h^k + K \cdot \mathrm{d}B^k \qquad U' \cdot f = g \leqslant 0 \tag{3.4}$$

$$h'^0 = U' \cdot F^0 - h^0 \qquad h'^k = U' \cdot F^k - K \cdot B^k - h^k \tag{3.5}$$ (4).

From (3.4) we obtain

$$U' = [(1/\theta)(G - (u^2/2)),\ (u/\theta),\ -(1/\theta)(H/4\pi\theta)$$
$$(q/\theta^2),\ -(\sigma^{\mathrm{D}}/\theta),\ -(\mathrm{tr}\,\sigma/3\theta),\ -(c/4\pi\sigma_0\,\theta)\,J]^T$$

$$K = [-(1/4\pi\theta)(H \cdot u),\ 0] .$$

(2) We denote with $\psi : \psi' = \mathrm{tr}(\psi \otimes \psi')$.

(3) An exaustive list of references on this subject may be found in $[2]_2$, [4], [5].

(4) Moreover if there are not constraints, that is $B^k = 0$, the general theory [**10**], [**11**] is recovered. Furthermore the quantities U', h'^α, f, k are called «generators» (see the note on page 6). We observe that in the hyperbolic case the (3.4) are studied in $[2]_1$.

So that (3.5) specialize to

$$h'^0 = (p/\theta) + (H^2/8\pi\theta) \qquad h'^k = (1/\theta)\,\mathbf{t}_{ik}\,u_i + (1/\theta)\,q_k + (H^2/8\pi\theta)\,u_k + (c/4\pi\sigma_0\,\theta)\,\varepsilon_{kij}\,J_i H_j\ .$$

($G = \mathfrak{e} + (p/\rho) - \theta S$ denotes the chemical potential).

Now, within the framework of the extended thermodynamics, making use of the method of approach proposed in [**10**] we aim to obtain a hyperbolic model to describe magnetofluid dynamics.

According to the basic assumption of the irreversible thermodynamics [**9**], we introduce a new entropy density of non-equilibrium

$$S_N = S_N(\rho,\ e,\ q,\ \sigma^{\mathrm{D}},\ \mathrm{tr}\,\sigma,\ J)\ . \tag{3.6}$$

That is equivalent to consider q, σ^{D}, $\mathrm{tr}\,\sigma$ as effective field variables which satisfy suitable field equations to be determined.

In (3.6) J has been included among the non-equilibrium variables. In fact when we consider states which are far from the equilibrium a variation of the magnetic field induces in a conductor an electric field giving rise to a conduction current.

Therefore the electric field forces produce work so that a suitable Gibbs relation as well as a suitable entropy density must be determined. Such a situation does not take place at the equilibrium where the stationary electric field vanishes inside on the conductor (see [**7**], p. 70).

In this section we determined the generators for the system (2.1) (2.2). Now, following the procedure used in [**10**], we modify the thermodynamics quantities involved in the generators in order to obtain a hyperbolic model.

In our case the chemical potential only must be considered. Within the extended thermodynamics framework we introduce the non-equilibrium chemical potential

$$G_N = G_N(\rho,\ e,\ q,\ \sigma^{\mathrm{D}},\ \mathrm{tr}\,\sigma,\ J)\ .$$

As in [**10**], we assume that the non-equilibrium generators have the same form as the corresponding Fourier and Navier-Stokes quantities (hypothesis of invariance of the generators).

Thus our analysis will be carried on as follows.

Once the modified generators U'_N, h'^{α}_N, f_N, K_N have determined, then the system

$$\partial_\alpha F_N^\alpha(U_N) = f_N(U_N) \tag{3.7}$$

and the related supplementary law

$$\partial_\alpha h_N^\alpha(U_N) = g_N(U_N) \tag{3.8}$$

will be completely characterized. Furthemore we will obtain necessary and sufficient conditions for the convexity of h_N^0 which automatically guarantee the hyperbolicity of the system (3.7).

In fact if it is possible to choose $F_N^0 = U_N^0$ and h_N^0 is a convex function of F_N^0 then there is global invertibility for the mapping $U'_N = U'_N(U_N)$ (see [**1**]).

Consequently since

$$F_N^0 = (\partial h_N'^0/\partial U_N') \qquad F_N^k = (\partial h_N'^k/\partial U_N') + B_N^k(\partial K_N/\partial U_N') \text{ (}^5\text{)} \tag{3.9}$$

system (3.7) can be written

$$\begin{aligned}(\partial^2 h_N'^0/\partial U_N'\,\partial U_N')\,\partial_t U_N' + [(\partial^2 h_N'^k/\partial U_N'\,\partial U_N')\\ + B_N^k(\partial^2 K_N/\partial U_N'\,\partial U_N')]\,\partial_k U_N' = f_N(U_N)\,.\end{aligned} \tag{3.10}$$

As $h_N'^0$ is the Legendre conjugate function of h_N^0, then $(\partial^2 h_N'^0/\partial U_N'\,\partial U_N')$ is positive definite, and (3.10) is a symmetric hyperbolic and conservative system.

Since we have $f_N = f$, then the entropy source g does not vary

$$g_N = U_N' \cdot f_N = g\ .$$

Moreover, as $h_N'^\alpha = h'^\alpha$, $K_N \cdot B_N^k = K \cdot B$, the calculation of F_N^α, h_N^α leads to [**10**]

$$\begin{aligned}&F^0\,\mathrm{d}(U_N' - U') + (F_N^0 - F^0)\,\mathrm{d}U_N' = 0 &\qquad F_N^k = F^k + u_k(F_N^0 - F^0)\\ &h_N^0 = h^0 + U_N'(F_N^0 - F^0) + (U_N' - U')F_N^0 &\qquad h_N^k = h^k + u_k(h_N^0 - h^0)\ .\end{aligned} \tag{3.11}$$

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(5) If U_N', $h_N'^\alpha$, f_N, K_N are given then the system (3.7) and the supplementary law (3.8) are completely determined by (3.9) and (3.5), whereupon the quantities U_N', $h_N'^\alpha$, f_N, K_N can be considered like «generators» of (3.7) and (3.8).

The vector F_N^0 to be determined has the form

$$F_N^0 = [\rho,\ \rho u_i,\ \varepsilon,\ H_i,\ \rho X_i,\ -\rho Y_{ij}^{\mathrm{D}},\ -3\rho Z,\ -(4\pi\sigma_0\rho/c)\Lambda_i]^T$$

where X, Y^{D}, Z, Λ are quantities to be found.

From $(3.11)_1$ we have

$$\mathrm{d}\mathcal{S} = X \cdot \mathrm{d}q^* + Y^{\mathrm{D}} : \mathrm{d}\sigma^{\mathrm{D}*} + Z\ \mathrm{d}\,\mathrm{tr}\,\sigma^* + \Lambda \cdot \mathrm{d}J^* \tag{3.12}$$

where

$$\mathcal{S} = (G - G_N/\theta) \qquad q^* = (q/\theta^2) \qquad \sigma^{*\mathrm{D}} = (\sigma^{\mathrm{D}}/\theta) \qquad \mathrm{tr}\,\sigma^* = (\mathrm{tr}\,\sigma/\theta) \qquad J^* = (J/\theta)\ .$$

Moreover from (3.12) we obtain

$$X = (\partial\mathcal{S}/\partial q^*), \quad Y^{\mathrm{D}} = (\partial\mathcal{S}/\partial\sigma^{*\mathrm{D}}), \quad Z = (\partial\mathcal{S}/\partial\,\mathrm{tr}\,\sigma^*), \quad \Lambda = (\partial\mathcal{S}/\partial J^*)\ . \tag{3.13}$$

So that the knowledge of the function $\mathcal{S}$ determines X, Y^{D}, Z, Λ. From $(3.11)_{2,3}$ follows

$$F_N^k = F^k + [0_1,\ 0_3,\ 0_1,\ 0_3,\ \rho u_k X_i,\ -\rho u_k Y_{ij}^{\mathrm{D}},\ -3\rho u_k Z,\ -(4\pi\rho\sigma_0/c)\, u_k \Lambda_i]^T$$

$$h_N^0 = -\rho S - \rho\mathcal{S} + \rho q^* \cdot X + \rho\sigma^{\mathrm{D}*} : Y^{\mathrm{D}} + \rho\ \mathrm{tr}\,\sigma^* Z + \rho J^* \cdot \Lambda\ .$$

Requiring (3.8) to represent the entropy balance equation we identify h_N^0 with $(-\rho S_N)$ so that we get

$$S_N = S + \mathcal{S} - (q^* \cdot X + \sigma^{\mathrm{D}*} : Y^{\mathrm{D}} + \mathrm{tr}\,\sigma^* Z + J^* \cdot \Lambda) \tag{3.14}$$

together with the Gibbs relation

$$\mathrm{d}S_N = \mathrm{d}S - (q^* \cdot \mathrm{d}X + \sigma^{\mathrm{D}*} : \mathrm{d}Y^{\mathrm{D}} + \mathrm{tr}\,\sigma^*\, \mathrm{d}Z + J^* \cdot \mathrm{d}\Lambda)\ .$$

Finally from $(3.11)_4$ we have

$$h_N^k = -[\rho u_k S_N + (q_k/\theta)]\ .$$

Consequently the system (3.7) specializes to

$$
(3.15)\quad
\begin{aligned}
&\partial_t \rho + \partial_k(\rho u_k) = 0\\
&\partial_t(\rho u_i) + \partial_k[\rho u_i u_k + t_{ik} + (H^2/8\pi)\,\delta_{ik} - (1/4\pi)\,H_i H_k] = 0\\
&\partial_t \varepsilon + \partial_k\{\varepsilon u_k + q_k + [t_{ik} + (H^2/8\pi)\,\delta_{ik} - (1/4\pi) H_i H_k]\,u_i + (c/4\pi\sigma_0)\,\varepsilon_{kij} J_i H_j\} = 0\\
&\partial_t H_i + \partial_k[u_k H_i - H_k u_i + (c/\sigma_0)\,\varepsilon_{ikm} J_m] = 0\\
&\partial_t(\rho X_i) + \partial_k(\rho u_k X_i + \theta\delta_{ik}) = -\varkappa q_i\\
&\partial_t(\rho Y^{D}_{ij}) + \partial_k[\rho u_k Y^{D}_{ij} - (1/2)\,(u_j\delta_{ik} + u_i\delta_{jk}) + (1/3)\,\delta_{ij} u_k] = -\alpha\sigma^{D}_{ij}\\
&\partial_t(\rho Z) + \partial_k[\rho u_k Z - (1/3)\,u_k] = -(\beta/3)\,\mathrm{tr}\,\sigma\\
&\partial_t(\rho \Lambda_i) + \partial_k[\rho u_k \Lambda_i - (c/4\pi\sigma_0)\,\varepsilon_{ikj} H_j] = -(1/\sigma_0)\,J_i\,.
\end{aligned}
$$

Of course such a system automatically satisfies the entropy balance

$$
\begin{aligned}
&\partial_t(\rho S_N) + \partial_k[\rho u_k S_N + (q_k/\theta)]\\
&= (\varkappa/\theta^2)\,q^2 + (\alpha/\theta)\,\sigma^{D}:\sigma^{D} + (\beta/3\theta)\,(\mathrm{tr}\,\sigma)^2 + (1/\theta)\,(J^2/\sigma_0) \geqslant 0\,.
\end{aligned}
$$

The convexity of the function $h^0_N = h^0_N(F^0_N)$ requires the following condition to be fulfilled for any non vanishing variation δF^0_N.

$$
(3.16)\quad
\begin{aligned}
\delta U'_N \cdot \delta F^0_N &= \delta\{(1/\theta)\,[G - (u^2/2)]\}\,\delta\rho + \delta(u/\theta)\,\delta(\rho u)\\
&\quad - \delta(1/\theta)\,\delta[\rho e + \rho(u^2/2)] + (1/4\pi\theta)\,(\delta H)^2\\
&\quad + \rho(\delta X\cdot\delta q^* + \delta Y^{D}:\delta\sigma^{D*} + \delta Z\,\delta\,\mathrm{tr}\,\sigma^* + \delta\Lambda\cdot\delta J^*) > 0\,.
\end{aligned}
$$

It can be seen that if G at the equilibrium is a convex function of p and θ (see [12]), then (3.16) holds if and only if $\mathcal{S}$ is a convex function of q^*, σ^{D*}, $\mathrm{tr}\,\sigma^*$, J^*.

As a consequence this leads to the acquisition of the maximum of entropy at the equilibrium.

In fact at the equilibrium we have $\mathcal{S} = 0$, therefore the condition of convexity of $\mathcal{S}$ is written

$$(3.17) \qquad -\mathcal{S} + q^* \cdot X + \sigma^{D*} : Y^D + \operatorname{tr} \sigma^* Z + J^* \cdot \Lambda > 0$$

and from (3.14), (3.17) gives $S_N < S$.

An interesting special case of the system (3.15) is obtained if we choose

$$(3.18) \qquad \mathcal{S} = (1/2)(\alpha_0 q^* \cdot q^* + \alpha_1 \sigma^{D*} : \sigma^{D*} + \alpha_2 \operatorname{tr} \sigma^* \operatorname{tr} \sigma^* + \alpha_3 J^* \cdot J^*)$$

with α_0, α_1, α_2, α_3 positive constants.

On account of (3.13) from (3.15), we deduce

$$(3.19) \qquad \begin{aligned} &\rho\alpha_0 \, d/dt(q/\theta^2) + \nabla\theta = -\varkappa q \\ &\rho\alpha_3 \, d/dt(J/\theta) - (c/4\pi\sigma_0) \operatorname{rot} H = -(1/\sigma_0) J \, . \end{aligned}$$

The equations $(3.19)_{1,2}$ are of Maxwell-Cattaneo like with variable relaxation times.

The system obtained through the procedure sketched above is hyperbolic, conservative and also it can be written in symmetric form. Therefore this model exhibits as main features:

(i) real and finite velocity of propagation of the discontinuity waves;

(ii) compatibility with propagation of shock waves;

(iii) a locally well-position of the Cauchy problem if the initial data belong to a Sobolev space H^s, with $s > 4$ [3];

(iv) the determination of the constitutive functions is reduced to that of functions $\mathcal{S}$.

4 - Discontinuity waves for the adiabatic motion of an inviscid fluid

Here we will point out the main features of wave propagation compatible with the hyperbolic model for magnetofluid dynamics obtained through the analysis worked out in the previous sections.

It is noteworthy to remark that the classic governing system of the magneto-fluid dynamics (in the case of an adiabatic motion of an inviscid fluid) is hyperbolic only if $\sigma_0 \to \infty$, whereas the mathematical model (3.15) leads to finite wave speeds also in the case of finite electrical conductivity.

In what follows we choose the function $\mathcal{J}$ characterizing the new evolution equations in (3.15) of the form (3.18).

Let $\phi(X, t) = 0$ represent the wave front equation. We assume that the field quantities are continuous across the wave surface while their first order derivatives suffer a jump denoted by

$$\delta(\cdot) = \partial/\partial\phi = 0^{+}(\cdot) - \partial/\partial\phi = 0^{-}(\cdot) \ .$$

Moreover we define [2]$_3$, [6]

$$\lambda = -(\partial_t \phi/|\nabla\phi|) \qquad n = (\nabla\phi/|\nabla\phi|) \ .$$

Here the system ruling the weak discontinuity waves specializes to

$$(4.1) \qquad \begin{aligned} &-\omega\delta\rho + \rho\delta u_n = 0 \\ &-\omega\delta u + (1/\rho)\, a^2 n\delta\rho + (1/4\pi\rho)\, n(H\cdot\delta H) - (H_n/4\pi\rho)\,\delta H = 0 \\ &-\omega\delta H + H\delta u_n - H_n\delta u + (c/\sigma_0)\, n\wedge\delta J = 0 \\ &-\omega\delta J - (c\theta_0/4\pi\sigma_0\rho\alpha_3)\, n\wedge\delta H = 0 \ . \end{aligned}$$

In (4.1) θ_0 represents the constant value of the (absolute) temperature. Thus the wave speeds are

$$(4.2) \qquad \omega = 0 \quad \text{with multiplicity } m = 4$$

$$(4.3) \qquad \omega = \pm\sqrt{(c^2\theta_0/4\pi\sigma_0^2\rho\alpha_3) + (H_n^2/4\pi\rho)} \ .$$

It is simple matter to show that the corresponding waves of (4.3) are exceptional.

Moreover if $\sigma_0 \to \infty$, then (4.3) reduce to the Alfvén speeds characterizing the transverse waves.

The remaining velocities are given by

$$(4.4) \qquad \varphi = \pm\{(1/2)(a^2 + \beta_1^2 + h^2 \pm \sqrt{\Delta})\}^{(1/2)}$$

where

$$\beta_1^2 = (c^2\theta_0/4\pi\sigma_0^2\rho\alpha_3) + (H_n^2/4\pi\rho) \qquad h^2 = (H_T^2/4\pi\rho)$$

$$\Delta = (a^2 - \beta_1^2 - h^2)^2 + 4a^2h^2 > 0 \; .$$

If $\sigma_0 \to \infty$ from (4.4) the well known fast and slow waves of magnetofluid dynamics are recovered [2]$_3$, [6].

Now we aim to point out the role played in the evolution of waves by the dissipative effects involved in the model we are studying herein. Let us consider a plane wave propagating into a constant state with the velocity

$$\lambda = \omega + u_n \tag{4.5}$$

where

$$\omega = \{(1/2)(a^2 + \beta_1^2 + h^2 + \sqrt{\Delta})\}^{(1/2)} \qquad \Delta = (a^2 - \beta_1^2 - h^2)^2 + 4a^2h^2 > 0 \; .$$

In (4.5) λ is an eigenvalue of the matrix $\mathcal{A}_n$ given by

$$\mathcal{A}_n = \begin{bmatrix} u_n & \rho n & 0 & 0 \\ (a^2/\rho)\, n & u_n I & (1/4\pi\rho)(n \otimes H - H_n I) & 0 \\ 0 & (H \otimes n - H_n I) & u_n I & (c/\sigma_0)\, \varepsilon_{ijk} n_j \\ 0 & 0 & -(c\theta_0/4\pi\sigma_0\rho\alpha_3)\, \varepsilon_{ijk} n_j & u_n I \end{bmatrix}$$

The left and right eigenvectors of $\mathcal{A}_n$ corresponding to λ are

$$\ell = \frac{a^2(\omega^2 - \beta_1^2)}{2\omega^2\sqrt{\Delta}} \Big[1,\; \frac{\rho\omega}{a^2(\omega^2 - \beta_1^2)}\,((\omega^2 - \beta_1^2)\, n - (H_n/4\pi\rho)\, H_T),$$
$$\frac{\omega^2}{4\pi a^2(\omega^2 - \beta_1^2)}\, H_T,\; -\frac{c\omega}{4\pi\sigma_0\, a^2(\omega^2 - \beta_1^2)}\, n \wedge H_T\Big]$$

$$d = \Big[1,\; \frac{\omega}{\rho(\omega^2 - \beta_1^2)}\,((\omega^2 - \beta_1^2)\, n - (H_n/4\pi\rho)\, H_T),\; \frac{\omega^2}{\rho(\omega^2 - \beta_1^2)}\, H_T,$$
$$-\frac{c\theta_0\,\omega}{4\pi\sigma_0\rho^2\, a_3(\omega^2 - \beta_1^2)}\, n \wedge H_T\Big]^T .$$

Moreover ℓ and d satisfy the normalization condition ($\ell \cdot d = 1$). According to the general theory of non-linear wave propagation (see [2]$_3$, [6]) we have

$$\delta U = \Pi\, d_0 \qquad \text{with} \qquad U = [\rho,\ u,\ H,\ J]^T\,.$$

The wave amplitude Π satisfies the following Bernoulli equation

$$\frac{\mathrm{d}\Pi}{\mathrm{d}t} + A\Pi^2 + B\Pi = 0\,. \tag{4.6}$$

In (4.6) $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \Lambda_i \partial_i$ where

$$\Lambda = \lambda n + (\partial\lambda/\partial n) - \{n \cdot (\partial\lambda/\partial n)\}\, n = \omega n + u - \frac{a^2 H_n}{4\pi\rho\omega\sqrt{\Delta}}\, H_T$$

represents the radial velocity. Moreover A and B are given by

$$A = (\nabla\lambda \cdot d)_0 = \frac{1}{2\rho\omega\sqrt{\Delta}\,(\omega^2 - \beta_1^2)}\, \{\rho p_{\rho\rho}(\omega^2 - \beta_1^2)^2 + h^2\omega^4 + 2\sqrt{\Delta}\,\omega^2(\omega^2 - \beta_1^2)\}$$

$$B = -\{\nabla(\ell \cdot f) \cdot d\}_0 = (1/\tau^2) \frac{c^2 h^2}{8\pi^2 \sigma_0 \rho \sqrt{\Delta}\,(\omega^2 - \beta_1^2)}$$

where $\tau = (\rho\alpha_3\sigma_0/\theta_0)$ is the relaxation time of the electrical conductivity equation $(3.15)_8$.

Integration of (4.6) leads to

$$\Pi(t) = \frac{B\Pi(0)}{A\Pi(0)\{\exp(Bt) - 1\} + B\, \exp(Bt)}\,. \tag{4.7}$$

The wave amplitude becomes unbounded if there exists a «critical time» t_c such that the denominator in (4.7) vanishes.

From (4.7) it is easy to see that

$$t_c = (1/B)\, \lg \left| \frac{A(0)}{B + A(0)} \right|\,. \tag{4.8}$$

As well known at $t = t_c$ a discontinuity wave may evolve into a shock wave. Bearing in mind that $B > 0$, inspection of (4.8) shows that a real t_c exists if the following conditions hold [8]

$$(4.9)\qquad \begin{array}{ll} \text{(i)}\ \ A>0 & \Pi(0)<-(B/A)<0 \\ \text{(ii)}\ \ A<0 & \Pi(0)>-(B/A)>0\,. \end{array}$$

Assuming $p_{\varrho\varrho}>0$ [2]$_3$ so that $A>0$, hereafter we will consider only the case (i). However it is easily seen that in the case (ii) the analysis developed further would lead to similar results as in the case (i).

From the kinematical conditions of compatibility the initial amplitude $\Pi(0)$ is given by the initial jump of mass density

$$(4.10)\qquad \Pi(0)=-\frac{[\partial_t\rho](0)}{\lambda}\,.$$

Thus on account of $(4.9)_1$ in order that a t_c exists the following relation must be fulfilled

$$[\partial_t\rho](0)>\lambda(B/A)$$

$$=(1/\tau^2)\frac{c^2h^2\omega\lambda}{4\pi^2\sigma_0}\,\frac{1}{(1/\alpha_3^2)(\rho p_{\rho\rho}\alpha_3\Omega_3^2+h^2\alpha_3\Omega_2^2+2\Omega_1\Omega_2\Omega_3)}$$

$$=\frac{\theta_0^2c^2h^2}{4\pi^2\sigma_0^3\rho^2}\,\frac{\Omega_2+\sqrt{\alpha_3\Omega_2}\,u_n}{(\rho p_{\rho\rho}\alpha_3\Omega_3^2+h^2\alpha_3\Omega_2^2+2\Omega_1\Omega_2\Omega_3)}$$

where

$$\Omega_1=\{(\alpha_3a^2-(c^2\theta_0/4\pi\sigma_0^2\rho)-\alpha_3(H^2/4\pi\rho))^2+\alpha_3^2a^2(H_T^2/\pi\rho)\}^{(1/2)}$$

$$\Omega_2=(1/2)\{\alpha_3a^2+(c^2\theta_0/4\pi\sigma_0^2\rho)+\alpha_3(H^2/4\pi\rho)+\Omega_1\}$$

$$\Omega_3=\Omega_2-\{(c^2\theta_0/4\pi\sigma_0^2\rho)+\alpha_3(H_n^2/4\pi\rho)\}\,.$$

Moreover t_c takes the form

$$t_c=\tau^2\frac{8\pi^2\sigma_0\rho}{c^2h^2}\frac{\Omega_1\Omega_3}{\alpha_3^2}\lg\frac{[\partial_t\rho](0)}{[\partial_t\rho](0)-\lambda(B/A)}$$

$$=\frac{8\pi^2\sigma_0^3\rho^3}{\theta_0^2c^2h^2}\Omega_1\Omega_3\lg\frac{[\partial_t\rho](0)}{[\partial_t\rho](0)-\lambda(B/A)}\,.$$

When $\tau \to 0$, that is $\alpha_3 \to 0$, then the quantity $\lambda(B/A) \to \infty$ so that a large initial discontinuity is needed for the existence of t_c and for the evolution of the discontinuities. Meanwhile $t_c \to 0$ and consequently the discontinuity wave may evolve into a shock wave in a short time. In words we can say that when the hyperbolic model approachs the parabolic one ($\alpha_3 \to 0$) then a stronger condition on the initial jump is required for the growth of the discontinuities.

Furthermore when $\sigma_0 \to \infty$ then the condition $(4.9)_1$ is written

$$[\partial_t \rho](0) > 0 .$$

Therefore we find the condition obtained in $[2]_3$ for the existence of a shock wave in magnetofluid dynamics. In this case the critical time is

$$t_c = \frac{2\rho\lambda'}{[\partial_t \rho](0)} \frac{R_1 R_3 \sqrt{R_2}}{(\rho p_{\rho\rho} R_3^2 + h^2 R_2^2 + 2R_1 R_2 R_3)},$$

where

$$R_1 = \{(a^2 - (H^2/4\pi\rho))^2 + a^2(H_T^2/\pi\rho)\}^{(1/2)}$$

$$R_2 = (1/2)\{a^2 + (H^2/4\pi\rho) + R_1\} \qquad R_3 = R_2 - (H_n^2/4\pi\rho)$$

and λ' is the fast wave speeds.

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Summary

As is well known the classical system of magnetofluid dynamics [7] *with finite conducibility and viscosity effects is not hyperbolic. To avoid such a paradox the procedure given in* [**10**] *is applied within the framework of the irreversible thermodynamics. The propagation of weak discontinuities compatible with the hyperbolic and conservative system so obtained is studied. Some results already known are found as particular cases (if the electrical conductivity is infinite). The influence on wave propagation of the different dissipative effects is studied.*

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