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On the rectilinear congruence of Lorentz manifold $[R^3, (+, +, -)]$ establishing a conformal mapping between its focal surfaces (**)

1 - The basic equations of the problem

Let R^3 be the 3-dimensional real manifold provided with the special pseudometric d such that

(1.1) $$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 .$$

The pair $[R^3, (+, +, -)] = (R^3, d)$ will be called *Lorentz manifold*. This manifold has been considered by L. Bianchi [1] and more recently by Maeda and Otsuki [3] and by Greek Mathematicians [7], [5]. It is know [5], that there are generally two types of 2-dimensional submanifolds of this manifold, denoted by I^+, I^-, with respect to whether the expression $EG\text{-}F^2$ is positive, negative respectively, (E, F and G are the fundamental quantities of the first order).

Let Q be a hyperbolic rectilinear congruence of (R^3, d) referred to the rectangular coordinate system $0xyz$ and defined by the equation

(1.2) $$\boldsymbol{R}(u, v) = \boldsymbol{r}(u, v) + t\boldsymbol{a}_0(u, v) \qquad t \in R$$

where $\boldsymbol{r} = \boldsymbol{r}(u, v)$ is the equation of its middle surface and $\boldsymbol{a}_0 = \boldsymbol{a}_0(u, v)$ is the unit vector giving the direction of the rays of the congruence. Moreover, let the parametric surfaces $u = \text{const}$, $v = \text{const}$ of the congruence be its developable surfaces.

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Let e, f, g be the coefficients of the linear element of the spherical (Gauss) representation of Q, obtained by drawing radii of the unit sphere S, parallel to the rays of Q. Moreover, let ω be the angle of the focal planes which pass through the current ray $l(u, v)$ of Q.

It is known [5] that there exist two surfaces S_1 and S_2 of the type I^+ or I^- of the manifold (R^3, d), such that the straight lines of Q are tangent to both of them. These surfaces are called *focal surfaces of the congruence* and given by the equations $[2]_2$

$$\boldsymbol{R}_1 = \boldsymbol{r}(u, v) + \rho(u, v)\,\boldsymbol{a}_0(u, v) \qquad \boldsymbol{R}_2 = \boldsymbol{r}(u, v) - \rho(u, v)\,\boldsymbol{a}_0(u, v) \tag{1.3}$$

respectively, where 2ρ is the *focal distance* of any current ray of Q satisfying the following differential equation [4]

$$\rho_{uv} + a\rho_u + b\rho_v + \rho(a_u + b_v + \varepsilon f) = 0 \qquad \varepsilon = \pm 1 \tag{1.4}$$

where a, b are the Christoffel's symbols of the second kind with respect to the linear element of the unit sphere S and are given by relations

$$a = (ge_v - fg_u)/2h^2 \qquad b = (eg_u - fe_v)/2h^2 \qquad h^2 = |eg| - f^2 . \tag{1.5}$$

The Gauss' equation of the spherical representation of Q can be written [5]

$$\begin{aligned} &\omega_{uv} + (d\sqrt{|e/g|}\,\omega_u + c\sqrt{|g/e|}\,\omega_v)\cos\omega + [\frac{\partial}{\partial u}(d\sqrt{|e/g|}) \\ &\qquad + \frac{\partial}{\partial v}(c\sqrt{|g/e|}) - \sqrt{|eg|}]\sin\omega = 0 \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} &c = (-fe_v + 2ef_u - ee_v)/2h^2 \qquad d = (-fg_u + 2gf_v - gg_u)/2h^2 \\ &\cos\omega = \frac{f}{\sqrt{|eg|}} \qquad \sin\omega = \frac{h}{\sqrt{|eg|}} \qquad \omega_u = -h(\frac{c}{e} + \frac{a}{g}) \qquad \omega_v = -h(\frac{b}{e} + \frac{d}{g}) . \end{aligned} \tag{1.7}$$

Let E_i, F_i and G_i $(i = 1, 2)$ be the fundamental quantities of first order of the focal surfaces given by (1.3) at two corresponding points. After a simple calculation we get

$$(1.8)\qquad \begin{array}{lll} E_1 = 4(\rho_u + b\rho)^2 & F_1 = -4a\rho(\rho_u + b\rho) & G_1 = 4\rho^2(a^2 + g) \\ E_2 = 4\rho^2(b^2 + e) & F_2 = -4b\rho(\rho_v + a\rho) & G_2 = 4(\rho_v + a\rho)^2 . \end{array}$$

Similarly, for the second order fundamental quantities we have

$$(1.9)\qquad \begin{array}{lll} L_1 = 2\sqrt{|e|}\,(\rho_u + b\rho)\sin\omega & M_1 = 0 & N_1 = -2\rho\sqrt{|g|}\,(\omega_v + b\sqrt{|g/e|}\sin\omega) \\ L_2 = -2\rho\sqrt{|e|}\,(\omega_u + a\sqrt{|e/g|}\sin\omega) & M_2 = 0 & N_2 = 2\sqrt{|g|}\,(\rho_v + a\rho)\sin\omega . \end{array}$$

In order that the lines of Q establish a conformal mapping between its focal surfaces, it is necessary and sufficient that

$$(1.10)\qquad \frac{E_1}{E_2} = \frac{F_1}{F_2} = \frac{G_1}{G_2} .$$

This also means that the mapping is a similarity «in the infinitesimal» and that the isotropic curves on these surfaces correspond each other.

The system (1.10) by virtue of (1.8), (1.9) takes the form

$$b(\rho_u + b\rho)(\rho_v + a\rho) = a\rho^2(b^2 + e) \qquad a(\rho_u + b\rho)(\rho_v + a\rho) = b\rho^2(a^2 + g)$$

which is equivalent to the system

$$(1.11)\qquad a(\rho_u + b\rho)(\rho_v + a\rho) = b\rho^2(a^2 + g) \qquad a^2 e = b^2 g .$$

The above analysis leads us to

Theorem 1.1. *If the functions* $e = e(u, v)$, $f = f(u, v)$, $g = g(u, v)$, $\rho = \rho(u, v)$ *are of class* C^2 *and satisfy the system of equations* (1.4), (1.6), (1.11), *then there exists a hyperbolic rectilinear congruence of the Lorentz manifold* $[R^3, (+, +, -)]$, *referred to its developable surfaces* $u = \text{const}$, $v = \text{const}$ *as parametric ones, the coefficients of the linear element of the spherical representation of which are the functions* e, f, g. *The straight lines of such a rectilinear congruence establish a conformal mapping between its focal surfaces.*

2 - Determination of the rectilinear congruences described by Theorem 1.1

Assume that the coefficients e, f, g of the linear element of the spherical representation of Q satisfy the following relations

$$(2.1) \qquad a=b \qquad e=g \qquad f=k \qquad k\neq 0 \, .$$

The first of these equations by means of (1.5) and the second of (2.1) takes the form $(e+k)(e_v-e_u)=0$, from which we get $e_u-e_v=0$. The solution of this partial differential equation is $e=g=\varphi(t)$ where φ is an arbitrary function, of class C^1, of the variable $t=u+v$.

Now, the equations (1.5), (1.7) by means of (2.1) take the form

$$(2.2) \qquad \begin{aligned} &\cos\omega=\frac{k}{\varphi} \qquad \sin\omega=\frac{\sqrt{\varphi^2-k^2}}{\varphi} \qquad \omega_u=\omega_v=\frac{k\varphi'}{\varphi\sqrt{\varphi^2-k^2}} \\ &a=b=\frac{\varphi'}{2(\varphi+k)} \qquad c=d=-\frac{\varphi'}{2(\varphi-k)} \qquad \varphi(t)\neq\pm k \, . \end{aligned}$$

The second equation (1.11) by means of (2.1) reduces to an identity and the first equation (1.11) reduces to

$$(2.3) \qquad \rho_u\rho_v+a\rho(\rho_u+\rho_v)-e\rho^2=0 \, .$$

From equations (2.2) we get $a_u=b_v=\frac{1}{2}\cdot\frac{\varphi''(\varphi+k)-\varphi'^2}{(\varphi+k)^2}$.

Hence equations (1.4) and (1.6) reduce to

$$(2.4) \qquad \rho_{uv}+\frac{1}{2}\frac{\varphi'(t)}{\varphi+k}(\rho_u+\rho_v)+\rho[\frac{\varphi''(\varphi+k)-\varphi'^2}{(\varphi+k)^2}+\varepsilon k]=0$$

$$(2.5) \qquad A(\varphi)\,\varphi''+B(\varphi)\,\varphi'+C(\varphi)\,\varphi'^2+D(\varphi)=0$$

respectively, where

$$A(\varphi)=\frac{k(\varphi-k)}{\sqrt{\varphi^2-k^2}}-\sqrt{\varphi^2-k^2} \qquad B(\varphi)=-\frac{k(2\varphi^2-k^2)}{\varphi(\varphi+k)\sqrt{\varphi^2-k^2}}$$

$$C(\varphi)=\frac{k^2}{\varphi\sqrt{\varphi^2-k^2}}+\frac{\sqrt{\varphi^2-k^2}}{\varphi-k} \qquad D(\varphi)=\varphi(\varphi-k)\sqrt{\varphi^2-k^2}\, .$$

Suppose that $\rho = \rho(u+v) = \rho(t)$; then equations (2.3), (2.4) take the form

$$\rho'^2 + \frac{\varphi'}{\varphi + k}\rho\rho' - \varphi\rho^2 = 0 \tag{2.6}$$

$$\rho'' + \frac{\varphi'}{\varphi + k}\rho' + \rho\left[\frac{\varphi''(\varphi+k) - \varphi'^2}{(\varphi+k)^2} + \varepsilon k\right] = 0 \ . \tag{2.7}$$

From the above analysis we obtain

Theorem 2.1. *For every real number k and for any real valued functions $\rho = \rho(t)$, $\varphi = \varphi(t)$ of class C^2 which are solutions of the system of equations* (2.5), (2.6), (2.7), *there exists a hyperbolic rectilinear congruence, the fundamental quantities of first order of the spherical representation of which are $e = g = \varphi(t)$, $f = k$ and the semidistance $\rho = \rho(t)$ is a common solution of* (2.6) *and* (2.7). *The straight lines of such a rectilinear congruence establish a conformal mapping between its focal surfaces.*

3 - Determination of the rectilinear congruences when the parametric net is orthogonal

Suppose that the curvilinear net (u, v) is orthogonal; therefore we have

$$f = k = 0 \ . \tag{3.1}$$

Hence the equations (2.5), (2.6), (2.7) reduce to

$$\varphi'' - \frac{1}{\varphi}\varphi'^2 - \varphi^2 = 0 \tag{3.2}$$

$$\rho'^2 + \frac{\varphi'}{\varphi}\rho\rho' - \varphi\rho^2 = 0 \tag{3.3}$$

$$\rho'' + \frac{\varphi'}{\varphi}\cdot\rho' + \left(\frac{\varphi'}{\varphi}\right)'\cdot\rho = 0 \tag{3.4}$$

respectively.

A first integral of the equation (3.4) is

$$\rho' + (\ln|\varphi|)'\rho = c_1 \qquad c_1 \in R,\ \varphi(t) \neq 0 \quad \forall t \in R$$

and using the classical formula we are led to the solution

$$\rho = \varphi^{-1}(c_2 + c_1 \textstyle\int \varphi \mathrm{d}t) \qquad c_2 \in R\ . \tag{3.5}$$

The equation (3.3) may be written

$$\rho'[\rho' + (\ln|\varphi|)'\rho] - \varphi\rho^2 = 0$$

which by virtue of (3.5) reduces to $c_1\rho' - \varphi\rho^2 = 0$, the solution of which is

$$\rho = c_1(c_3 - \textstyle\int \varphi \mathrm{d}t)^{-1} \qquad c_3 \in R\ . \tag{3.6}$$

A function satisfying equation (3.2) is $\varphi(t) = -2\mathrm{ch}^{-2} t$. Hence equation (3.5), (3.6) may be written

$$\rho = -\frac{\mathrm{ch}^2 t}{2}(c_2 - 2c_1 \mathrm{th}\, t) \qquad \rho = c_1(c_3 + 2\mathrm{th}\, t)^{-1}\ . \tag{3.7}$$

According to the Theorem 2.1. the solution ρ is given by any of the two equations (3.7) under the *condition*

$$6c_1 \mathrm{th}^2 t + 2(c_1 c_3 - c_2)\mathrm{th}\, t - (2c_1 + c_2 c_3) = 0\ . \tag{3.8}$$

So we have found

$$e = g = -2\mathrm{ch}^{-2} t \qquad f = k = 0 \qquad \rho(t) = c_1(c_3 + 2\mathrm{th}\, t)^{-1} \qquad t = u + v\ . \tag{3.9}$$

The above analysis leads us to

Theorem 3.1. *To every triple of real numbers c_i ($i = 1, 2, 3$) satisfying condition* (3.8) *corresponds a rectilinear congruence, whose fundamental quantities of first order and the semidistance are given by* (3.9). *The lines of such a rectilinear congruence establish a conformal mapping between its focal surfaces.*

The fundamental quantities of first and second order, of the focal surfaces of the congruence are:

$$E_1 = G_2 = 4(\rho' + \rho \operatorname{th} t)^2 \qquad G_1 = E_2 = 4\rho^2(3\operatorname{th}^2 t - 2)$$

$$F_1 = F_2 = -4\rho(\rho' + \rho \operatorname{th} t) \operatorname{th} t \qquad M_1 = M_2 = 0$$

$$L_1 = N_2 = 2\sqrt{2}\,(\varphi' + \rho \operatorname{th} t) \operatorname{ch}^{-1} t \qquad L_2 = N_1 = -2\sqrt{2}\,p \operatorname{ch}^{-1} t \operatorname{th} t$$

where the function ρ is given by (3.11).

From the above results we are led to the following theorem

Theorem 3.2. *The focal surfaces of each rectilinear congruence given by the Theorem* 3.1. *are the same surface in two different positions in* $[R^3, (+, +, -)]$.

This surface is of type I^-, [1], since one easily gets

$$E_1 G_1 - F_1^2 = -32\rho^2(\rho + \rho \operatorname{th} t)^2 \operatorname{ch}^{-2} t\,.$$

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Summary

This paper contains three paragraphs. In the first one we consider a hyperbolic rectilinear congruence of the Lorentz manifold $[R^3, (+, +, -)]$, *the linear element of the spherical representation (Gauss) of which is known. We give the system of partial differential equations in corresponding to the problem of determining the rectilinear congruences of the Lorentz manifold* (R^3, d) *where* $ds^2 = dx^2 + dy^2 - dz^2$, *the straight lines of which establish a conformal mapping between its focal surfaces at corresponding points. In the second and third paragraphs we determine these congruences under the assumptions* (2.1) *and* (3.1) *respectively.*

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