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Intersection of spheres and isometries (**)

1 – Several generalizations of Mazur-Ulam theorem have been obtained recently: see e.g. [5], [10] or [14]. Other authors proved that in special cases already the preservation of one distance may force a map T between two normed spaces X and Y to be an affine map (for related results see also [13] for real pre-hilbert spaces and [7] for compact metric spaces). In particular, this is true when $X = Y = R^n$ (see [3]) but not when $X = Y = l^2$ (see again [3]), or for $X = R^2$ and $Y = R^6$ (see [6]; see also [1] for other cases, and $[9]_2$ for injective maps).

If T preserves two distances d and $2d$ (for some $d > 0$), then the same conclusion holds if $X = Y$ is a real pre-hilbert space (see [12]); for injective maps, more general situations have been considered in $[9]_1$.

In [4] a generalization of the last result was proved, which runs more or less as follows.

Assume the real normed space Y to be strictly convex (a natural assumption in this context: see [2], [5]). Then a map $T: X \to Y$ satisfying certain assumptions is affine if X possess the following *property*

(i) *If a is in X, $\|a\| < 1$, then there exists b in X with* $\|a + b\| = \|a - b\| = 1$.

On the other hand, the following *property* was proved (not in a trivial way!) to hold in any normed space (see [14], Lemma 2.1).

(ii) *Given $t > 0$ and $x \in X$ with $\|x\| \leq 2t$, there exist y and z in X such that*

$$\|y\| = \|z\| = t \quad \text{and} \quad x = y + z\ .$$

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Of course, (i) implies (ii): in fact, if $\|x\| = 2t$ then (ii) is trivial (take $y = z = x/2$). Otherwise, set $a = x/2t$ (so $\|a\| < 1$): if $\|a - b\| = \|a + b\| = 1$, then $x = y + z$ with $y = t(a + b)$ and $z = t(a - b)$.

In the next section, we will show how (i) trivially holds in any space of dimension at least two. Also, we shall study the dependence of b from a in some spaces.

2 – Let $a \in X$; $\|a\| < 1$. Take any $a' \in X$ and consider the two-dimensional subspace Y generated by a and a'. Let $S = \{y \in Y: \|y - a\| = 1\}$. The function $f: x \mapsto \|x - a\| - \|x + a\|$ vanishes at same point $b \in S$ (which is a connected set): in fact, $f((1 + 1/\|a\|)a) = -2\|a\| < 0 < -|1 - 2\|a\|| + 1 = f((1 - 1/\|a\|)a)$. Such a point b is the point required in (i).

If b is a point as in (i) (thus also $-b$ is), then we have $\|b\| \leq (\|a + b\| + \|a - b\|)/2 = 1$. Equality cannot be excluded: consider e.g. in R_2^{∞}, $a = (t, 0)$; $0 < |t| < 1$; $b = (s, 1)$; $|s| \leq 1 - |t|$. Note that, for any fixed a, we cannot have more than one pair $(b, -b)$ in Y if the space is two dimensional and stictly convex.

The *modulus of convexity* of a space is defined in the following way, for $0 \leq t \leq 2$

$$\delta(t) = \inf\{1 - \|x + y\|/2: \|x\| = 1;\ \|y\| = 1;\ \|x - y\| = t\}\ .$$

Recall (see [11]) that δ is a continuous function in [0, 2).

The space is said to be (UNS) = *uniformly non-square* (respectively: (UC) = *uniformly convex*) if $\delta(t) > 0$ for some $t < 2$ (respectively: $\delta(t) > 0$ for any $t > 0$).

Choose a, b as in (i). By setting $x = a + b$, $y = b - a$, we see that $\|x - y\| = 2\|a\| = t$ implies $\|(x + y)/2\| = \|b\| \leq 1 - \delta(t)$. Note that for any space $1 - \delta(t) \geq (1 - t^2/4)^{1/2}$ because of Nordlander inequality.

Now let (a_n) be a sequence in X such that $\|a_n\| = 1 - t_n$ with $\lim_{n\to\infty} t_n = 0$; let (b_n) be a corresponding sequence obtained through (i). In general, we obtain $\|b_n\| \leq 1 - \delta(2 - 2t_n) \to 1 - \lim_{t\to 0^+} \delta(2 - t)$. If X is (UNS), then $\overline{\lim_{n\to\infty}} \|b_n\| < 1$; but when X is (UC), then $\lim_{t\to 0^+} \delta(2 - t) = \delta(2) = 1$ (see [11]), thus $\lim_{n\to\infty} \|b_n\| = 0$. Of course, strict convexity of Y is enough to imply such result (when Y is two-dimensional). But we shall prove this in another way.

Statement. *Let X be a two-dimensional, strictly convex normed space* *and denote by S its unit sphere. Given $x \in S$ and $t > 0$ set $x_t = x/(1+t)$, then choose $y_t \in Y$ such that $\|x_t \pm y_t\| = 1$. Then we have $\lim_{t \to 0} \|y_t\| = 0$.*

Proof. Take in S a point y such that $\|x + ky\| \geqslant \|x\|$ for all k in R (see [8]). For any $t > 0$, there exist A_t and B_t in R such that $y_t = A_t x + B_t y$, which implies

$$|1/(1+t) \pm A_t| \leqslant \|x/(1+t) \pm (A_t x + B_t y)\| = 1 .$$

The net $\{A_t\}$ is bounded, thus the same is true for $\{B_t\}$. Take a null sequence t_n such that both $\{A_{t_n}\}$ and $\{B_{t_n}\}$ converge, say to A and B respectively. We obtain $\lim_{n \to \infty} |1/(1+t_n) \pm A_{t_n}| = |1 \pm A| \leqslant 1$. This implies $A = 0$. We get from here

$$\lim_{n \to \infty} \|x/(1+t_n) \pm (A_{t_n} x + B_{t_n} y)\| = \|x \pm By\| = 1 = \inf\{\|x + ky\|;\ k \in R\} .$$

But when X is strictly convex, this implies $B = 0$. So $\lim_{n \to \infty} \|y_{t_n}\| = 0$; but a simple geometric argument shows that $\|y_t\|$ is decreasing with t, thus $\lim_{t \to 0} \|y_t\| = 0$.

References

[1] A. D. ALEXSANDROV, *Mappings of families of sets*, Soviet Math. Dokl. **11** (1970), 116-120.

[2] J. A. BAKER, *Isometries in normed spaces*, Amer. Math. Monthly 78 (1971), 655-658.

[3] F. S. BECKMAN and D. A. QUARLES JR., *On isometries in euclidean spaces*, Proc. Amer. Math. Soc. 4 (1953), 810-815.

[4] W. BENZ, *Isometrien in normierten räumen*, Aequationes Math. 29 (1985), 204-209.

[5] A. P. BOSZNÁY, *On a theorem of Mazur and Ulam*, Period. Math. Hungar. **16** (1985), 7-13.

[6] B. V. DEKSTER, *Non-isometric distance 1 preserving mapping $E^2 \to E^6$*, Arch. Math. 45 (1985), 282-283.

[7] T. HU and W. A. KIRK, *On local isometries and isometries in metric spaces*, Colloq. Math. 44 (1981), 53-57.

[8] R. C. JAMES, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265-292.

[9] A. V. KUZ'MINYKH: $[\bullet]_1$ *Characteristic property of isometric and similar mappings*, Siberian Math. J. 23 (1982), 380-385; $[\bullet]_2$ *Minimal conditions determining isometries and similarities*, Siberian Math. J. **26** (1985), 250-253.

[10] P. MANKIEWICZ, *On isometries in linear metric spaces*, Studia Math. **55** (1976), 163-173.

[11] W. O. RAY, *Some contributions to the geometric theory of Banach spaces with applications to nonlinear operators*, Thesis, Univ. of Iowa, 1977.

[12] M. SCHRÖDER, *Eine Ergänzung zum Satz von Beckman und Quarles*, Aequationes Math. **19** (1979), 89-92.

[13] A. TISSIER, *A right-angle preserving mapping, solution of advanced problem* 485, Amer. Math. Monthly **92** (1985), 291-292.

[14] A. VOGT, *Maps which preserve equality of distance*, Studia Math. **45** (1973), 43-48.

Abstract

In this note we discuss a condition used recently by Benz, in order to give an extension of Mazur-Ulam theorem.

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